A note on null hypersurfaces of indefinite Kaehler space forms

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Abstract. In this paper, we show that totally screen umbilic, screen conformal and Hopf null hypersurfaces are nonexistent in indefinite Kaehler space forms of nonzero constant holomorphic sectional curvatures. Furthermore, we prove that all totally screen umbilic null hypersurface immersions into indefinite Kaehler space forms are affinely equivalent to graph immersions.

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1 Introduction

In the book [2], the authors started the study of null submanifolds of semi-Riemannian manifolds. Their work was later updated by K.L. Duggal and B. Sahin in the book [3] and also by K.L. Duggal and D.H. Jin in the book [4]. In the above books, the authors laid a foundation for research on null geometry by constructing their structural equations, among other results. In fact, they introduced a non-degenerate screen distribution to construct a null transversal vector bundle which is non-intersecting to its null tangent bundle and developed local geometry of null curves, hypersurfaces and in general, the submanifolds of arbitrary codimension. Other pioneering works on the theory include that of D.N. Kupeli [12]–whose approach is purely intrinsic compared to that of [2, 3, 4], which is extrinsic. Since then, many researchers including but not limited to [1, 6, 7], have researched on null submanifolds and many interesting results have been obtained. Null hypersurfaces appears in general relativity as models of different types of black hole horizons (see [2, 3] for details) and their theory is fundamental to modern mathematical physics.

Chapter 6 of [2] (also see Chapter 6 of [3]) has been devoted to null submanifolds of indefinite Kaehler manifolds. It has been shown in [2, Theorem 2.5] that the indefinite Kaehler space forms of nonzero constant holomorphic sectional curvature do not admit any totally umbilic null hypersurfaces. Furthermore, it has been proved that

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any totally screen umbilic null hypersurface are indeed totally screen geodesic (see [2, Proposition 2.4] for more details). On the other hand, D.H. Jin [8] has introduced the notion of Hopf null hypersurfaces, in which he has proved that indefinite Kaehler space forms of non-zero constant holomorphic sectional curvatures do not admit any Hopf null hypersurfaces. In this paper, we extend the study on the geometry of null hypersurfaces of indefinite Kaehler manifolds. In particular, we prove that totally screen umbilic null hypersurfaces of indefinite Kaehler space forms are affinely equivalent to graph immersions. The rest of the paper is arranged as follows: In Section 2, we quote some basic notions necessary for the entire paper. Section 3 is devoted to totally umbilic null hypersurfaces, while Section 4 is on the geometry of Hopf null hypersurfaces.

2 Preliminaries

Let $\mathbb{C}^m$ be the m-dimensional complex number space and $\bar{M}$ be a Hausdorff space. An open chart on $\bar{M}$ is a pair $(\mathcal{U}, \phi)$, where $\mathcal{U}$ is an open set of $\bar{M}$ and ϕ is a homeomorphism of $\mathcal{U}$ on an open set of $\mathbb{C}^m$. A *complex structure* on $\bar{M}$ of dimension m, is a collection of open charts $(\mathcal{U}_i, \phi_i)_{i\in I}$ on $\bar{M}$ such that the following conditions are satisfied: (a) $\bar{M} = \cup_{i\in I}\mathcal{U}_i$, that is, $\{\mathcal{U}_i\}_{i\in I}$ is an open covering of $\bar{M}$. (b) For each $i, j \in I$, the mapping $\psi_j \circ \phi_i^{-1}$ is a holomorphic mapping of $\phi_i(\mathcal{U}_i \cap \mathcal{U}_j)$ onto $\phi_j(\mathcal{U}_i \cap \mathcal{U}_j)$. (c) The collection $(\mathcal{U}_i, \phi_i)_{i\in I}$ is a maximal family of open charts for which (a) and (b) hold. A Hausdorff space $\bar{M}$ endowed with a complex structure of dimension m is called a *complex manifold* (see more details in [2, Chapter 6]). Let $(z^A = x^A + iy^A)$, $A \in \{1, \dots, m\}$, $i = \sqrt{-1}$, be a complex local coordinate system on a neighbourhood $\mathcal{U}$ of $z \in \bar{M}$. Thus, $\bar{M}$ can be thought of as a particular smooth manifold of real dimension $2m$. It follows that the endomorphism $\bar{J} : T_z\bar{M} \longrightarrow T_z\bar{M}$; $\bar{J}\partial_{x^A} = \partial_{y^A}$; $\bar{J}\partial_{y^A} = -\partial_{x^A}$, does not depend on the complex local coordinate system (see [2, p. 191]). Therefore, there exists an automorphism $\bar{J}$ of the tangent bundle $T\bar{M}$ satisfying $\bar{J}^2 = -I$, where I is the identity on $T\bar{M}$. A real $2m$-dimensional manifold $\bar{M}$ endowed with the automorphism $\bar{J}$ satisfying $\bar{J}^2 = -I$, is called an *almost complex manifold*, and $\bar{J}$ is said to be an *almost complex structure* on $\bar{M}$. It is well-known [2, p. 191] that the almost complex structure $\bar{J}$ defines a complex structure on $\bar{M}$, if and only if, $N_{\bar{J}} = 0$ vanishes identically on $\bar{M}$, where $N_{\bar{J}}$ is the Nijenhuis tensor field of $\bar{J}$.

Consider a semi-Riemannian metric $\overline{g}$ of index $0 < v < 2m$, on the almost complex manifold $(\bar{M}, \bar{J})$. Then we say that the pair $(\bar{J}, \overline{g})$ is an *indefinite almost Hermitian structure* on $\bar{M}$, and $\bar{M}$ is an *indefinite almost Hermitian manifold*, if $\bar{J}_z$ is a linear isometry of the semi-Euclidean space $(T_z\bar{M}, \overline{g}_z)$, for any $z \in \bar{M}$, that is, $\overline{g}_z(\bar{J}_zX_z, \bar{J}_zY_z) = \overline{g}_z(X_z, Y_z)$. If, moreover, $\bar{J}$ defines a complex structure on $\bar{M}$, then $(\bar{J}, \overline{g})$ and $\bar{M}$ are called *indefinite Hermitian structure* and *indefinite Hermitian manifold*, respectively. It follows that the index of $\overline{g}$ is an even number $v = 2q$. Next, consider an indefinite almost Hermitian manifold $(\bar{M}, \bar{J}, \overline{g})$ and denote by $\overline{\nabla}$ the Levi-Civita connection on $\bar{M}$ with respect to $\overline{g}$. Then, according to [2, Chapter 6], $\bar{M}$ is called an *indefinite Kaehler manifold* if $\bar{J}$ is parallel with respect to $\overline{\nabla}$, that is, $(\overline{\nabla}_X\bar{J})Y = 0$, for all $X, Y \in \Gamma(T\bar{M})$. Here, and in the rest of the paper, $\Gamma(\Xi)$ denotes the set of smooth sections of the vector bundle Ξ. An *indefinite complex space form* [2, p. 191] is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c and it is denoted by $\bar{M}(c)$. The curvature tensor field of $\bar{M}(c)$ is given by the same formulae as in case of positive definite metrics, i.e.,

$$\begin{aligned}\bar{R}(X,Y)Z = (c/4)[\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(\bar{J}Y,Z)\bar{J}X \\ - \bar{g}(\bar{J}X,Z)\bar{J}Y + 2\bar{g}(X,\bar{J}Y)\bar{J}Z], \quad \forall\, X,Y,Z \in \Gamma(T\bar{M}).\end{aligned} \tag{2.1}$$

Let (M,g) be a *null hypersurface* of a semi-Riemannian manifold $(\bar{M},\bar{g})$. This implies that at each point $x \in M$, the restriction g $(= \bar{g}_x|_{T_xM})$ is *degenerate.* That is to say, there exist a non-zero vector $u \in T_xM$ such that $\bar{g}(u,v) = 0$, for any $v \in T_xM$. Precisely, in null setting, the normal bundle $TM^\perp$ of the null hypersurface M is a rank 1 vector subbundle of the tangent bundle TM. This contradicts the classical theory of non-degenerate hypersurfaces for which the normal bundle has a trivial intersection with its respective tangent bundle. Thus, the geometry of null hypersurfaces differs significantly from that of non-degenerate hypersurfaces, due to that non-trivial intersection in TM and $TM^\perp$. In the book [2, Chapter 4] (also see [3, Chapter 2]), the authors proceeded by fixing, on the null hypersurface, a geometric data formed by a null section and a *screen distribution*, denoted as $S(TM)$ (see [2, p. 78]). A screen distribution on M is considered as a nondegenerate complementary bundle of $TM^\perp$ in TM. This name is justified as in the case M is a null (lightlike) cone of a 4-dimensional semi-Riemannian manifold, integral curves of vector fields in $TM^\perp$ are null (lightlike) rays and the fibres of $S(TM)$ can be visualised as screens that are transversal to these rays. It is crucial to note that a screen distribution is not unique, but canonically isomorphic to the *nondegenerate quotient tangent bundle* $TM/TM^\perp$ [12, Definition 3.2.1, p. 46].

Hence, we have the decomposition of TM as $TM = TM^\perp \oplus_{orth} S(TM)$, where $\oplus_{orth}$ denotes an orthogonal direct sum. It is well-known [2, Theorem 1.1] that for any null section ξ of $TM^\perp$, there exists a unique null section N of $S(TM)^\perp$ such that $g(\xi,N) = 1$. It follows that there exists a *null transversal vector bundle*, $tr(TM)$, locally spanned by N and $\bar{g}(N,N) = \bar{g}(N,Z) = 0$, for any $Z \in \Gamma(S(TM))$. Let $tr(TM)$ be complementary (but not orthogonal) vector bundle to TM in $T\bar{M}$. Then, we have the following decomposition of $T\bar{M}|_M$ as $T\bar{M}|_M = S(TM) \oplus_{orth} [TM^\perp \oplus tr(TM)]$. Let P be the projection morphism of TM on to $S(TM)$. Then, the local Gauss and Weingarten equations of M and $S(TM)$ are the following (see [2, p. 82–85] for more details);

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \quad \overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.2}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \quad \nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{2.3}$$

for all $X,Y \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$ and $N \in \Gamma(tr(TM))$. Here, ∇ and ∇^* are the induced linear connections on TM and $S(TM)$, respectively, B is the local second fundamental form of M and C is the local second fundamental form on $S(TM)$. Furthermore, A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively, and τ is a differential 1-form on TM. Next, let $\theta = \bar{g}(N,\cdot)$ be a 1-form metrically equivalent to N defined on $\bar{M}$. Then, take $\eta = i^*\theta$ to be its restriction on M, where $i : M \hookrightarrow \bar{M}$ is the inclusion map. It is known that ∇^* is a metric connection on $S(TM)$, while ∇ is generally *not* a metric connection. In fact, from the fact $\overline{\nabla}\bar{g} = 0$, we get the expression of ∇g as

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \ \ \forall\, X,Y,Z \in \Gamma(TM). \tag{2.4}$$

Moreover, B is known to be independent of the choice of $S(TM)$ and satisfies

$$B(X,\xi) = 0, \ \ \forall\, X \in \Gamma(TM). \tag{2.5}$$

It has been shown that ∇g vanishes if and only if $B = 0$, i.e., when M is *totally geodesic* [3, p. 76]. In fact, from (2.4), if $\nabla g = 0$ we have $B(X,Y)\eta(Z) + B(X,Z)\eta(Y) = 0$, for all $X, Y, Z \in \Gamma(TM)$. Replacing Z with ξ in this relation, and considering (2.5), we get $B(X,Y)\eta(\xi) = 0$. That is, $B = 0$ since $\eta(\xi) = \bar{g}(\xi, N) = 1$. The fundamental forms B and C are related to their shape operators by the following equations

$$g(A^*_\xi X, Y) = B(X,Y), \quad \bar{g}(A^*_\xi X, N) = 0, \tag{2.6}$$

$$g(A_N X, PY) = C(X, PY), \quad \bar{g}(A_N X, N) = 0, \tag{2.7}$$

for all $X, Y \in \Gamma(TM)$. It follows from (2.6) and (2.7) that both A^*_ξ and A_N are screen-valued operators. Let us denote by R and $\bar{R}$ the curvature tensors of M and $\bar{M}$, respectively. Then, using the Gauss-Weingarten formulae (2.2) and (2.3), we have the following Gauss-Codazzi equations for M and $S(TM)$ (see more details in [2, 3]).

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + B(X,Z)A_N Y - B(Y,Z)A_N X + [(\nabla_X B)(Y,Z) \\ &\quad - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)]N,\end{aligned} \tag{2.8}$$

$$\begin{aligned}R(X,Y)\xi &= -\nabla^*_X A^*_\xi Y + \nabla^*_Y A^*_\xi X + A^*_\xi [X,Y] - \tau(X)A^*_\xi Y + \tau(Y)A^*_\xi X \\ &\quad + [C(Y, A^*_\xi X) - C(X, A^*_\xi Y) - 2d\tau(X,Y)]\xi,\end{aligned} \tag{2.9}$$

$$\begin{aligned}\bar{R}(X,Y)N &= -\nabla_X A_N Y + \nabla_Y A_N X + A_N[X,Y] + \tau(X)A_N Y - \tau(Y)A_N X \\ &\quad + [B(Y, A_N X) - B(X, A_N Y) - 2d\tau(X,Y)]N,\end{aligned} \tag{2.10}$$

where $d\tau(X,Y) = (1/2)[X(\tau(Y)) - Y(\tau(X)) - \tau([X,Y])]$, for all $X, Y, Z \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$ and $N \in \Gamma(tr(TM))$.

Let (M, g) be a null hypersurface of $2m$-dimensional, $m > 1$, indefinite almost Hermitian manifold, where $\bar{g}$ is a semi-Riemannian metric of index $v = 2q$, $0 < q < m$. From the fact $\bar{g}(\bar{J}X, Y) + \bar{g}(X, \bar{J}Y) = 0$, we note that $\bar{g}(\bar{J}\xi, \xi) = 0$ and thus, $\bar{J}\xi \in \Gamma(TM)$. Therefore, $\bar{J}TM^\perp$ is a distribution on M of rank 1 such that $TM^\perp \cap \bar{J}TM^\perp = \{0\}$. This enables one to choose a screen distribution $S(TM)$ such that it contains $\bar{J}TM^\perp$ as a vector subbundle. Then we consider a local section N of the null transversal vector bundle $tr(TM)$ of M with respect to $S(TM)$. It follows that $\bar{J}N$ also lies in $S(TM)$. In fact, $\bar{g}(\bar{J}N, \xi) = -\bar{g}(N, \bar{J}\xi) = 0$, and thus $\bar{J}N$ is tangent to M. As $\bar{g}(\bar{J}N, N) = 0$, it follows that the $\bar{J}N$ is a smooth vector field of $S(TM)$. From the facts ξ and N are null vector fields, we deduce that $\bar{J}\xi$ and $\bar{J}N$ are null vector fields. Moreover, $\bar{g}(\bar{J}\xi, \bar{J}N) = \bar{g}(\xi, N) = 1$. Hence, $\bar{J}TM^\perp \oplus \bar{J}tr(TM)$ is a vector subbundle of $S(TM)$ of rank 2, with hyperbolic planes as fibres. Then there exists a non-degenerate distribution D_0 on M such that $S(TM) = [\bar{J}TM^\perp \oplus \bar{J}tr(TM)] \perp D_0$. Moreover, it is easy to check that D_0 is an almost complex distribution with respect to $\bar{J}$, i.e., $\bar{J}D_0 = D_0$. Thus, we have $TM = [\bar{J}TM^\perp \oplus \bar{J}tr(TM)] \perp D_0 \perp TM^\perp$. Next we consider the local null vector fields

$$U = -\bar{J}N \quad \text{and} \quad V = -\bar{J}\xi. \tag{2.11}$$

Then any vector field on M is expressed as $X = SX + u(X)U$, where u is a 1-form locally defined on M by $u(X) = g(X, V)$. Applying $\bar{J}$ to this relation gives

$\bar{J}X = JX + u(X)N$, where J is a tensor field of type $(1,1)$ globally defined on M by $JX = \bar{J}SX$, for all $X \in \Gamma(TM)$. It follows that

$$J^2X = -X + u(X)U, \quad u(U) = 1, \ \forall\, X \in \Gamma(TM). \tag{2.12}$$

It is well-known [2, Proposition 2.1] that (J, u, U) defines an *almost contact structure* on M. However, it is *not* a contact metric structure on M. In fact, $g(JX, JY) = g(X, Y) - u(X)v(Y) - u(Y)v(X)$, for all $X, Y \in \Gamma(TM)$, where v is a 1-form locally defined on M by $v(X) = g(X, U)$, for all $X \in \Gamma(TM)$. By a direct calculation, we have $(\nabla_X J)Y = u(Y)A_N X - B(X, Y)U$, for all $X, Y \in \Gamma(TM)$. Replacing Y by ξ and U, in turn, in this relation we derive

$$\nabla_X V = JA^*_\xi X - \tau(X)V \ \text{ and } \ \nabla_X U = JA_N X + \tau(X)U, \tag{2.13}$$

for all $X \in \Gamma(TM)$. Furthermore, we have

$$B(X, U) = C(X, V), \quad \forall\, X \in \Gamma(TM). \tag{2.14}$$

Next, we give some examples of null hypersurfaces of an indefinite Kaehler manifold.

Example 2.1 (Duggal-Bejancu [5])**.** Consider $\mathbb{R}^{2(m+1)}_{2s}$ with the metric $\bar{g}(x, y) = -\sum_{i=1}^{2s} x^i y^i + \sum_{j=2s+1}^{2(m+1)} x^j y^j$, and the almost complex structure

$$\bar{J}(x^1, x^2, \ldots, x^{2m+1}, x^{2m+2}) = (-x^2, x^1, \ldots, -x^{2m+2}, x^{2m+1}).$$

Then,

1. the null cone Λ^{2m+1}_{2s-1} of $\mathbb{R}^{2(m+1)}_{2s}$ is a null hypersurface, whose normal bundle is spanned by the global null vector field $\xi = (x^1, x^2, \ldots, x^{2m+2})$;
2. the hyperplanes of $\mathbb{R}^{2(m+1)}_{2s}$ given by the equations
$$\sum_{a=3}^{2(m+1)} \varrho_a x^a - \sqrt{\varrho/2}(x^1 + x^2) = 0, \ \ \varrho = \sum_{a=3}^{2(m+1)} \varrho_a^2,$$
with $\xi = (-\sqrt{\varrho/2}, -\sqrt{\varrho/2}, \varrho_3, \ldots, \varrho_{2m+2})$ is a null hypersurface;
3. the hypersurface of $\mathbb{R}^4_2$ given by the equations
$$x^1 = u^1 \cosh u^2 + \sinh u^2, \ \ x^2 = u^3, \ \ x^3 = u^1 + u^2,$$
$$x^4 = u^1 \sinh u^2 + \cosh u^2,$$
with $\xi = (\cosh u^2, 0, 1, \sinh u^2)$ is a null hypersurface of $\mathbb{R}^4_2$.

3 Totally screen umbilic null hypersurfaces

We say that the screen distribution $S(TM)$ is *totally umbilic* [2, p. 109] if on any coordinate neighbourhood $\mathcal{U} \subset M$, there exists a smooth function λ such that

$$C(X, PY) = \lambda g(X, PY), \quad \forall\, X, Y \in \Gamma(TM_{|\mathcal{U}}). \tag{3.1}$$

In case $\lambda = 0$, we say that $S(TM)$ is *totally geodesic*. A null hypersurface whose screen distribution is totally umbilic is called a *totally screen umbilic* null hypersurface. It follows from (3.1) that on a totally screen umbilic null hypersurface, we have $C(\xi, PX) = 0$, for all $X \in \Gamma(TM_{\mathcal{U}})$. Equivalently, $A_N\xi = 0$. In the same line, a null hypersurface is called *totally umbilic* [2, p. 107] if, locally, on each $\mathcal{U}$ there exists a smooth function ρ such that

$$B(X,Y) = \rho g(X,Y), \quad \forall\, X, Y \in \Gamma(TM_{|\mathcal{U}}). \tag{3.2}$$

The case $\rho = 0$ corresponds to a *totally geodesic* null hypersurface. As an example, we have the following.

Example 3.1. Consider the null cone Λ^{2m+1}_{2s-1} of $\mathbb{R}^{2(m+1)}_{2s}$ in Example 2.1. As $\xi = (x^1, x^2, \ldots, x^{2m+2})$ is a position vector field, then $\nabla_X\xi = \overline{\nabla}_X\xi = X$, for all $X \in \Gamma(T\Lambda^{2m+1}_{2s-1})$. It follows from (2.2) and (2.3) that $A^*_\xi X = -PX$ and $\tau(X) = -\eta(X)$, for all $X \in \Gamma(T\Lambda^{2m+1}_{2s-1})$. Hence, Λ^{2m+1}_{2s-1} is totally umbilic with $\rho = -1$.

Remark 3.2. We note that $S(T\Lambda^{2m+1}_{2s-1})$ is *not totally umbilic.* In fact, as $A^*_\xi X = -PX$, we see that $B(X,V) = -u(X)$, for all $X \in \Gamma(T\Lambda^{2m+1}_{2s-1})$. Setting $X = U$ in this relation and using (2.14), we get $-1 = B(U,V) = C(V,V)$. Therefore, if $S(T\Lambda^{2m+1}_{2s-1})$ is totally umbilic, we get $-1 = \lambda g(V,V) = 0$, which is a contradiction.

In [9], the authors defined an affine immersion as follows: Let $f : M \longrightarrow \bar{M}$ be an immersion of a manifold M as a hypersurface of $\bar{M}$ and ∇ and $\overline{\nabla}$ be torsion-free connections on M and $\bar{M}$, respectively. Then f is an *affine immersion* if there exists locally a transversal vector field N along f such that $\overline{\nabla}_{f_*X} f_*Y = f_*(\nabla_X Y) + B(X,Y)N$, for all $X, Y \in \Gamma(TM)$, where f_* is the differential map of f. In the usual way, we put $\overline{\nabla}_{f_*X} N = -A_N(f_*X) + \tau(f_*X)N$. Such a definition was also used by Duggal-Bejancu [2, p. 100], and it was concluded that any null isometric immersion is an affine immersion. Suppose ∇ is a flat connection on M. Let $\phi : M \longrightarrow \mathbb{R}^{m+1}$ such that every point $x \in M$ has a neighborhood $\mathcal{U}$ on which ϕ is an affine connection preserving diffeomorphism with an open neighborhood $\mathcal{V}$ of $\phi(x)$ in $\mathbb{R}^{m+1}$. Consider $\mathbb{R}^{m+1}$ as a hyperplane of $\mathbb{R}^{m+2}$ and let N be a parallel vector field, transversal to $\mathbb{R}^{m+1}$. Then, for any differentiable function $F : M \longrightarrow \mathbb{R}$, define $f : M \longrightarrow \mathbb{R}^{m+2}$; $f(x) = \phi(x) + F(x)N$, for all $x \in M$. Thus, f is an affine immersion with $A_N = 0$, called the *graph immersion* with respect to F. Accordingly, we quote the following result.

Proposition 3.1 (Duggal-Bejancu [2], Proposition 5.2)**.** *Let (M, g) be a null hypersurface of $\mathbb{R}^{m+2}_q$ with a parallel screen distribution $S(TM)$. Then the immersion of M is affinely equivalent to the graph immersion of a certain function $F : M \longrightarrow \mathbb{R}^{m+2}_q$.*

Next, we give a characterization of all totally screen umbilic null hypersurfaces in indefinite Kaehler space forms.

Theorem 3.2. *Let (M, g) be a totally screen umbilic null hypersurface of an indefinite Kaehler space form $\bar{M}(c)$. Then, the following are all true;*

1. *M is totally screen geodesic, i.e. $\lambda = 0$;*

2. *$c=0$, i.e. $\bar{M}(c)$ is $\mathbb{R}^{2(m+1)}_{2s}$;*
3. *∇ is a flat connection on M;*
4. *the immersion of (M,g) in $\bar{M}(c)$ is affinely equivalent to the graph immersion of a certain function $F: M \longrightarrow \mathbb{R}$.*

Proof. Setting $X=\xi$, and $Y=Z=U$ in (2.8) and then take the $\bar{g}$-product with ξ leads to

$$\begin{aligned}\bar{g}(\bar{R}(\xi,U)U,\xi) &= (\nabla_\xi B)(U,U) - (\nabla_U B)(\xi,U) + \tau(\xi)B(U,U)\\ &= \xi B(U,U) - 2B(\nabla_\xi U,U) - B(A^*_\xi U,U) + \tau(\xi)B(U,U). \end{aligned} \tag{3.3}$$

From relation (2.14) and the assumption $S(TM)$ is totally umbilic, we have

$$B(U,U) = C(U,V) = \lambda g(U,V) = \lambda, \tag{3.4}$$

$$B(A^*_\xi U,U) = C(A^*_\xi U,V) = \lambda B(U,V) = \lambda C(V,V) = \lambda^2 g(V,V) = 0. \tag{3.5}$$

On the other hand, using the second relation of (2.13) and the fact that $A_N\xi = 0$ on any screen umbilic null hypersurfaces, we see that $\nabla_\xi U = \tau(\xi)U$. Thus, we have

$$B(\nabla^*_\xi U,U) = B(\nabla_\xi U,U) = \tau(\xi)B(U,U) = \tau(\xi)C(U,V) = \lambda\tau(\xi). \tag{3.6}$$

Then, putting (3.4), (3.5) and (3.6) in (3.3) leads to

$$\bar{g}(\bar{R}(\xi,U)U,\xi) = \xi\lambda - \lambda\tau(\xi). \tag{3.7}$$

Next, letting $X=\xi$ and $Y=Z=U$ in (2.1) leads to

$$\bar{R}(\xi,U)U = (3c/4)N. \tag{3.8}$$

Replacing (3.8) in (3.7) gives

$$c = (4/3)[\xi\lambda - \lambda\tau(\xi)]. \tag{3.9}$$

Furthermore, letting $X=\xi$ and $Y=PZ=U$ in 2.10 and taking the $\bar{g}$-product with respect to M, we get

$$\bar{g}(\bar{R}(\xi,U)U,N) = \lambda g(\nabla_U\xi,U) = -\lambda B(U,U) = -\lambda C(U,V) = -\lambda^2, \tag{3.10}$$

in which we have used (2.4), (2.5), (2.14) and (3.1). It follows from (3.10) and (3.8) that $\lambda^2=0$, or simply $\lambda=0$. Hence, $S(TM)$ is totally geodesic, which proves (1). Then, from (3.9), we get $c=0$, which proves (2). Note from (2.8), and the fact $A_N=0$, that $R=0$ and hence ∇ is flat. This proves (3). Finally, we see, from (2.3) that $\nabla_X PY = \nabla^*_X PY \in \Gamma(S(TM))$, for all $X,Y\in\Gamma(TM)$. This shows that $S(TM)$ is parallel and then (4) follows from Proposition 3.1, hence the proof. □

The following is a consequence of Theorem 3.2.

Corollary 3.3. *There exist no any totally screen umbilic null hypersurface of the indefinite Kaehler space form $\bar{M}(c\neq 0)$.*

A null hypersurface (M, g) of an semi-Riemannian manifold $\bar{M}$ is said to be *screen conformal* [3, Definition 2.2.1] if there exists, on the neighbourhood $\mathcal{U} \subset M$, a non-vanishing smooth function φ such that $C = \varphi B$. The conformality is said to be global if $\mathcal{U} = M$. In case φ is a constant function, then M is called *screen homothetic*. For screen conformal null hypersurfaces of indefinite Kaehler space forms, we have the following.

Theorem 3.4. *Let (M, g) be a screen conformal null hypersurface of an indefinite Kaehler space form $\bar{M}(c)$. Then $c = 0$. Moreover, either M is totally geodesic or φ is a solution of the partial differential equation $\xi\varphi - 2\varphi\tau(\xi) = 0$.*

Proof. From (2.8), (2.10) and the fact that M is screen conformal, we derive

$$\bar{g}(\bar{R}(X,Y)PZ, N) - \varphi\bar{g}(\bar{R}(X,Y)PZ, \xi) = (X\varphi)B(Y, PZ) - (Y\varphi)B(X, PZ)$$
$$2\varphi\tau(Y)B(X, PZ) - 2\varphi\tau(X)B(Y, PZ), \quad \forall\, X, Y, Z \in \Gamma(TM). \tag{3.11}$$

Then, applying (2.1) to (3.11) and then put $X = \xi$ gives

$$\begin{aligned}(c/4)[g(Y, PZ) + u(PZ)v(Y) + 2u(Y)v(PZ) - 3\varphi u(Y)u(PZ)]\\ = [\xi\varphi - 2\varphi\tau(\xi)]B(Y, PZ), \quad \forall\, X, Y, Z \in \Gamma(TM).\end{aligned} \tag{3.12}$$

Letting $Y = V$ and $PZ = U$ in (3.12) gives

$$c/2 = [\xi\varphi - 2\varphi\tau(\xi)]B(V, U). \tag{3.13}$$

On the other hand, putting $Y = U$ and $PZ = V$ gives

$$3c/4 = [\xi\varphi - 2\varphi\tau(\xi)]B(U, V). \tag{3.14}$$

From (3.13), (3.14) and the symmetry of B, we get $(1/4)c = 0$, or simply $c = 0$. Finally, as $c = 0$ we have, from (3.12), that $[\xi\varphi - 2\varphi\tau(\xi)]B(X, PZ) = 0$, from which either $B = 0$ and showing that M is totally geodesic or $\xi\varphi - 2\varphi\tau(\xi) = 0$, which completes the proof. □

The following result follows from Theorem 3.4.

Corollary 3.5. *There exist no any screen conformal null hypersurface of an indefinite Kaehler space form $\bar{M}(c \neq 0)$.*

We also have the following result.

Corollary 3.6. *Let (M, g) be a screen conformal null hypersurface of an indefinite Kaehler space form $\bar{M}(c)$, such that $\xi\varphi - 2\varphi\tau(\xi) \neq 0$. Then, the immersion of (M, g) into $\bar{M}$ is affinely equivalent to the graph immersion of a certain function $F : M \longrightarrow \mathbb{R}$.*

Proof. When $\xi\varphi - 2\varphi\tau(\xi) \neq 0$, we have seen that M is totally geodesic. As M is also screen conformal, we see that $C = 0$, i.e. M is totally screen geodesic. It then follows from (2.8) and the fact $c = 0$ that $R = 0$, i.e. ∇ is a flat connection on M. Note, also, that $\nabla_X PY = \nabla^*_X PY \in \Gamma(S(TM))$, for all $X, Y \in \Gamma(TM)$. Hence, $S(TM)$ is parallel and by Proposition 3.1, the immersion of (M, g) in $\bar{M}$ is affinely equivalent to the graph immersion of a certain function $F : M \longrightarrow \mathbb{R}$, hence the proof. □

We wind up this section by making the following observation.

Theorem 3.7. *The indefinite complex space forms $\bar{M}(c \neq 0)$ do not admit any totally umbilic, totally screen umbilic and screen conformal null hypersurfaces.*

4 Hopf null hypersurfaces

Let (M,g) be a null hypersurface of a semi-Riemannian manifold $(\bar{M},\bar{g})$. We have seen, in the previous sections, that there are two shape operators on M, that is A_N and A^*_ξ, and two local second fundamental forms B and C. From the relations (2.6) and (2.7), we notice that both A_N and A^*_ξ are screen-valued, and interrelates with their local second fundamental forms. Due to this interrelatedness, D.H. Jin [8, Definitions 5.1 and 5.8] defines *Hopf* and *quasi Hopf* null hypersurfaces of an indefinite Kaehler manifolds as follows;

Definition 4.1 (D.H. Jin [8])**.** Let (M,g) be a null hypersurface of an almost complex manifold $\bar{M}$. Then, M is called

1. Hopf if the vector field U, of (2.11), is a principal vector field with respect to A^*_ξ, i.e. $A^*_\xi U=\alpha U$, for some smooth function α;
2. quasi Hopf if the vector field U, of (2.11), is a principal vector field with respect to A_N, i.e. $A_N U=\beta U$, for some smooth function β.

It follows from Definition 4.1 that a totally umbilic null hypersurface is Hopf, with $\alpha=\rho$, while a totally screen umbilic null hypersurface is quasi Hopf, with $\beta=\lambda$. Unlike B, the local second fundamental form C is generally *non-symmetric* on $S(TM)$. In fact, by a direct calculation, we have $C(X,Y)-C(Y,X)=\eta([X,Y])$, for all $X,Y\in\Gamma(S(TM))$. It follows from this relation that C is symmetric on $S(TM)$ if and only if $S(TM)$ is an integrable distribution. A null hypersurface for which $S(TM)$ is integrable is often referred to as a *screen integrable null hypersurface*. Some obvious examples of such hypersurfaces are the totally screen geodesic and screen conformal ones. Suppose that (M,g) is a screen integrable null hypersurface. Then from (2.14) and the nondegeneracy of $S(TM)$, we have $A_N V=A^*_\xi U$. In view of this relation, we have the following;

Lemma 4.1. *If $A^*_\xi U=\alpha U$ on a screen integrable null hypersurface of an indefinite Kaehler manifold, then $A_N V=\alpha U$.*

From (2.5), we note that $A^*_\xi\xi=0$, i.e. ξ is an eigenvector of A^*_ξ whose eigenfunction is 0. In contrast, $A_N\xi\neq 0$ even on a screen integrable null hypersurface. Thus, we may set $\sigma(X):=C(\xi,PX)$, for all $X\in\Gamma(TM)$. To that end, we have the following result.

Proposition 4.2. *Let (M,g) be a screen integrable null hypersurface of an indefinite Kaehler space form $\bar{M}(c)$ of dimension >3.*

1. *If M is Hopf, i.e. $A^*_\xi U=\alpha U$, then $c=0$. Moreover, the function α satisfies the differential equations*

$$\xi\alpha+\alpha\tau(\xi)-\alpha^2=B(V,JA_N\xi), \tag{4.1}$$

$$\text{and}\quad PX\alpha+\alpha\tau(PX)=B(V,JA_N PX). \tag{4.2}$$

2. *If M is quasi Hopf, i.e. $A_N U=\beta U$, then β satisfies*

$$\beta^2=-U\sigma(U)+\sigma(U)\tau(U)-\sigma(U)^2, \tag{4.3}$$

where $\sigma(U) = C(\xi, U)$, *and*

$$\xi\beta - \beta\tau(\xi) - (3c/4) = C(V, A_N V) + 2C(V, PJA_N\xi). \tag{4.4}$$

Proof. Assume that M is Hopf, then by a direct calculation while considering (2.1), (2.8) and Definition 4.1, we derive

$$\begin{aligned}(X\alpha)v(Y) + \alpha Xv(Y) - \alpha v(\nabla_X Y) - B(Y, JA_N X)\\ - (Y\alpha)v(X) - \alpha Yv(X) + \alpha v(\nabla_Y X) + B(X, JA_N Y)\\ = (c/4)[u(Y)\eta(X) - u(X)\eta(Y) + 2g(X, \bar{J}Y)],\end{aligned} \tag{4.5}$$

for all $X, Y \in \Gamma(TM)$. On the other hand, using (2.4) and Definition 4.1, we have

$$Xv(Y) - v(\nabla_X Y) = v(X)\eta(Y) + g(Y, JA_N X) + \tau(X)v(Y), \tag{4.6}$$

for all $X, Y \in \Gamma(TM)$. Substituting (4.6) in (4.5) gives

$$\begin{aligned}(X\alpha)v(Y) + \alpha^2 v(X)\eta(Y) + \alpha g(Y, JA_N X) + \alpha\tau(X)v(Y)\\ - B(Y, JA_N X) - (Y\alpha)v(X) - \alpha^2 v(Y)\eta(X) - \alpha g(X, JA_N Y)\\ - \alpha\tau(Y)v(X) + B(X, JA_N Y) = (c/4)[u(Y)\eta(X)\\ - u(X)\eta(Y) + 2g(X, \bar{J}Y)], \quad \forall\, X, Y \in \Gamma(TM).\end{aligned} \tag{4.7}$$

Setting $X = \xi$ and $Y = U$ in (4.7) and then use the fact that $JU = 0$, we get $3c/4 = 0$, i.e. $c = 0$, which was also obtained by Jin [8, Theorem 5.4]. Then putting $X = \xi$ and $Y = V$, gives $\xi\alpha + \alpha\tau(\xi) - B(V, JA_N\xi) - \alpha^2 = 0$. This proves (4.1). On the other hand, putting $X = PX$ and $Y = PY$ in (4.7) leads to

$$\begin{aligned}g([PX\alpha + \alpha\tau(PX)]U + \alpha JA_N PX - A^*_\xi JA_N PX, PY)\\ = g([PY\alpha + \alpha\tau(PY)]U + \alpha JA_N PY - A^*_\xi JA_N PY, PX),\end{aligned} \tag{4.8}$$

for all $X, Y \in \Gamma(TM)$. Note that both sides of (4.8) vanish since $\dim S(TM) > 1$ from the fact that $\dim \bar{M} > 3$. Thus, the nondegeneracy of $S(TM)$ implies that

$$[PX\alpha + \alpha\tau(PX)]U + \alpha PJA_N PX - A^*_\xi JA_N PX = 0, \tag{4.9}$$

for all $X \in \Gamma(TM)$. Taking the g-product of (4.9) with respect to V gives (4.2), which proves (1) of our proposition. Turning to part (2), we have, from (2.10) and Definition 4.1, that

$$\begin{aligned}\bar{g}(\bar{R}(X, U)U, N) = -C(JA_N X, U) + 2C(X, U)\tau(U) - UC(X, U)\\ + C(\nabla_U X, U), \quad \forall\, X \in \Gamma(TM).\end{aligned} \tag{4.10}$$

On the other hand, from (2.1), with $X = \xi$, $Y = Z = U$, we have

$$\bar{R}(\xi, U)U = (3c/4)N. \tag{4.11}$$

Considering (4.10) and (4.11), we have

$$-C(JA_N\xi, U) + 2C(\xi, U)\tau(U) - UC(\xi, U) + C(\nabla_U\xi, U) = 0. \tag{4.12}$$

But, using the fact $X = PX + \eta(X)\xi$, the symmetry of A_N and $JU = 0$, we have

$$C(JA_N\xi, U) = \eta(JA_N\xi)C(\xi, U) = C(\xi, U)^2. \tag{4.13}$$

Furthermore, a direct calculation yields

$$C(\nabla_U \xi, U) = -\beta^2 - C(\xi, U)\tau(U), \tag{4.14}$$

in which we have used (2.3), 2.14 and Definition 4.1. Replacing relations (4.13) and (4.14) in (4.12), we get $-\sigma(U)^2 + \sigma(U)\tau(U) - U\sigma(U) - \beta^2 = 0$, which proves (4.3). Also, by (4.11), (2.8), (2.14) and Definition 4.1, we have

$$3c/4 = \bar{g}(\bar{R}(\xi, U)U, \xi) = \xi\beta - 2B(\nabla^*_\xi U, U) - B(A^*_\xi U, U) + \beta\tau(\xi),$$

from which we get (4.4), and hence all the claims in proposition are proved. □

The following is direct consequence of Theorem 4.2.

Corollary 4.3. *There exist no any real quasi Hopf null hypersurface of an indefinite Kaehler space forms $\bar{M}(c)$, with $U\sigma(U) - \sigma(U)\tau(U) + \sigma(U)^2 > 0$. Moreover, if $A_N\xi = 0$ then $\beta = 0$. Furthermore, $c < 0$, $c = 0$ and $c > 0$ if and only if A_NV is spacelike, null and timelike vector field of $S(TM)$, respectively.*

Theorem 4.4. *Let (M, g) be a screen conformal null hypersurface of an indefinite Kaehler manifold $\bar{M}(c)$. If A^*_ξ either commutes or anti-commutes with J then, the immersion of M as a null hypersurface is affinely equivalent to the graph immersion of a certain function $F : M \longrightarrow \mathbb{R}$.*

Proof. Assume that J commutes with A^*_ξ, then, by the fact $JU = 0$, we have $JA^*_\xi U = A^*_\xi JU = 0$. Applying J to this relation and (2.12), we get $A^*_\xi U = B(U, V)U$. It follows from this last relation that M is Hopf with $\alpha = B(U, V)$. As M is screen conformal, we see that $\alpha U = A^*_\xi U = \varphi^{-1}A_N U$, from which we see that M is also quasi Hopf with $\beta = \alpha\varphi$. As $C(\xi, U) = 0$ on a screen conformal null hypersurface, we note, from (4.3) that $\beta^2 = \alpha^2\varphi^2 = 0$. But $\varphi \neq 0$, and therefore, $\alpha = 0$. Therefore, from (4.9), we have $JA^*_\xi A^*_\xi PX = 0$, for all $X \in \Gamma(TM)$. Applying J to the last relation and using (2.12), we get $A^*_\xi A^*_\xi PX = u(A^*_\xi A^*_\xi PX)U = g(A^*_\xi PX, A^*_\xi V)U$. From the fact that $A_NV = A^*_\xi U$, we see that $\varphi A^*_\xi V = \alpha U = 0$, which implies that $A^*_\xi V = 0$. Hence, $A^*_\xi A^*_\xi PX = 0$, for all $X \in \Gamma(TM)$. Furthermore, we may assume that $A^*_\xi e_i = \mu_i e_i$, for $i \in \{1, \dots, 2m-4\}$, where $\{V, U, e_i\}$ is a quasi-orthonormal basis of $S(TM)$. It follows that $\mu_i^2 = 0$. Since $A^*_\xi V = 0$ and $A^*_\xi U = 0$, we see that $B = 0$ and, hence M is totally geodesic. The assumption of screen conformality then implies that $C = 0$, i.e. M is totally screen geodesic and $S(TM)$ parallel. We then note that $R = 0$ and thus, by Proposition 3.1, the immersion of M as a null hypersurface is affinely equivalent to the graph immersion of a certain function $F : M \longrightarrow \mathbb{R}$. Similar conclusions can be arrived at when J anti-commutes with A^*_ξ, which completes the proof. □

Remark 4.2. In view of Theorems 3.2, 3.4 and 4.2 we note that the well-known classes of null hypersurfaces, such as the totally screen umbilic, screen conformal and the recently introduced Hopf null hypersurfaces of an indefinite Kaehler space form $\bar{M}(c)$ do exist only when $c = 0$, that is $\bar{M}(c)$ is $\mathbb{R}^{2(m+1)}_{2s}$. Moreover, similar conclusions have been reached if the null hypersurface is totally umbilic (see Theorem 2.5, of [2]).

Remark 4.2 highlights the need to study and perhaps describe, where possible, the nature of null hypersurfaces in indefinite Kaehler space forms $\bar{M}(c \neq 0)$.

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