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Mechanical Systems on Manifolds

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Mechanical Systems on Manifolds

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Dedicated to the memory of my father and mother

Abstract. As-well known, modern differential geometry is a suitable frame for studying Lagrangian and Hamiltonian formalisms of classical mechanics. More clearly, dynamics of Lagrangians and Hamiltonians is explicitly explained by differential geometry tools. Therefore, this study has intended to collect the analogues of Euler- Lagrange and Hamilton equations about mechanical systems on manifolds produced by Author. Also the geometrical and physical results on related mechanical systems are presented. Mechanical systems introduced here can be used to model problems in electrical, magnetical and gravitational fields of quantum and classical mechanics of physics.

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Brief presentation of the contents

This monograph collects analogues of Euler-Lagrange and Hamilton equations, mechanical systems, energy functions and fields obtained by means of the differentiable structures on manifolds, tangent and cotangent bundles.

In Chapter 1, preliminaries and notations are given. Clearly, Lagrangian and Hamiltonian formalisms, quaternion and Clifford manifolds are shortly introduced.

In Chapter 2, we introduce Euler-Lagrange and Hamilton equations on $(\mathbf{R}^2, g, J)$ and $(\mathbf{R}_n^{2n}, g, J)$ being models of para-Kähler space forms. Finally, some geometrical and physical results on the related mechanical systems have been derived.

In Chapter 3, we present standard Clifford Kähler analogues of Hamiltonian and Lagrangian mechanics. Also, the some geometric and physical results related to the standard Clifford Kähler dynamical systems are given.

In Chapter 4, Clifford Kähler analogues of Lagrangian and Hamiltonian dynamics are introduced. Also, the some geometrical and physical results over the obtained Clifford Kähler dynamical systems are discussed.

In Chapter 5, we give the further steps of the previously done studies taking into consideration analogues of Lagrangian and Hamiltonian mechanics. Presently, considering quaternion Kähler manifolds, we introduce quaternion

Kähler analogue of Lagrangian mechanics. Then a quaternion Kähler version of Hamilton equations is obtained. Finally, the some results related to quaternion Kähler Lagrangian and Hamiltonian dynamical systems are also given.

In Chapter 6, we present equations related to Lagrangian and Hamiltonian mechanical systems on para-quaternion Kähler manifold. Finally, the some results related to para- quaternion Kähler mechanical systems are also given.

In Chapter 7, we make a contribution to the modern development of Lagrangian formalisms of classical mechanics in terms of differential-geometric methods on differentiable manifolds. So, we obtain complex and paracomplex Euler-Lagrange equations with constraints on the (para) Kähler manifold.

In Chapter 8, by means of an almost product structure, we present Euler-Lagrange and Hamilton equations related to mechanical systems on the horizontal and vertical distributions of the bundles used in obtaining geometric quantization. In conclusion, we give some results related to mechanical systems.

In Chapter 9, equations related to bi-para-mechanical systems on the bi-Lagrangian manifold used in obtaining geometric quantization have been presented. Finally, some geometric and physical results related to dynamical systems are given.

This book addresses to mathematicians, engineers, physics researches and graduate students within the field, as primary comprehensive resource.

Prof.Dr. Mehmet Tekkoyun

Contents

CHAPTER 1

Preliminaries and Notations

Modern differential geometry is a suitable frame for studying Lagrangian formalisms of classical mechanics. More clearly, dynamics of Lagrangians is explicitly explained by differential geometry terms. It is well-known that the dynamics of Lagrangian systems is characterized by a convenient vector field ξ defined on the tangent bundles which are phase-spaces of velocities of a given configuration manifold Q. If Q is an m-dimensional configuration manifold and $L : TQ \to \mathbf{R}$ is a regular Lagrangian function, then there is a unique vector field ξ on TQ such that dynamics equations is determined by

$$i_\xi \Phi_L = dE_L \tag{0.1}$$

where Φ_L indicates the symplectic form and E_L is energy associated to L. The triple, either (TQ, ω_L, ξ) or (TQ, ω_L, L), is called *Lagrangian system* on the tangent bundle TQ.

The so-called Euler-Lagrange vector field ξ is a semispray (or *second order differential equation*) on Q since its integral curves are the solutions of the *Euler-Lagrange equations* as follows:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0. \tag{0.2}$$

Also, differential geometry provides a good framework in which develop the dynamics of Hamiltonians. One may say that Hamiltonian systems are characterized by a suitable vector field X defined on the cotangent bundles which are phase-spaces of momentum of a given configuration manifold Q. Therefore, if Q is an m-dimensional configuration manifold and $\mathbf{H} : T^*Q \to \mathbf{R}$ is a Hamiltonian energy function, then there is a unique vector field X on T^*Q such that dynamics equations are given by

$$i_X \Phi = d\mathbf{H} \tag{0.3}$$

where Φ indicates the symplectic form. The triple, either (T^*Q, ω, Z_H) or (T^*Q, ω, H), is called *Hamiltonian system* on the cotangent bundle T^*Q endowed with symplectic form ω.

The paths of the Hamiltonian vector field X are the solutions of the *Hamilton equations* shown by

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \ \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \tag{0.4}$$

where q_i and (q_i, p_i) are respectively coordinates of Q and T^*Q.

It is well-known that quaternions are useful for representing rotations in both quantum and classical mechanics [**1**]. Quaternions are introduced by Sir William Rowan Hamiltonian. Hamiltonian's expression is as follows:

$$i^2 = j^2 = k^2 = ijk = -1. \tag{0.5}$$

If it is compared to the calculus of vectors, quaternions have slipped into the realm of obscurity. They do however still find use in the computation of rotations. By means of quaternions, it is possible to state many physical laws in classical, relativistic, and quantum mechanics. Some researches hope to find deeper understanding of the universe using quaternion algebra.

It is well known that Clifford manifold is a quaternion manifold. So, all properties defined on quaternion manifold of dimension $8n$ also is valid for Clifford manifold.

As well-known, there are many studies about Lagrangian and Hamiltonian mechanics, formalisms, systems and equations such that time-dependent or not, constraint, real, complex, paracomplex and other analogues [**2**]-[**10**] and there in. So, we see that it is possible to produce different analogues in different spaces.

We may say that the goal of finding new dynamics equations is both a new expansion and contribution to science to explain physics and cosmos events.

Throughout this paper, all mathematical objects and mappings are assumed to be smooth, i.e. infinitely differentiable and Einstein convention of summarizing is adopted. $\mathcal{F}(M)$, $\chi(M)$ and $\Lambda^1(M)$ denote the set of functions on M, the set of vector fields on M and the set of 1-forms on M, respectively.

CHAPTER 2

Mechanical Systems on Para-Kähler Space Forms

In this chapter, we introduce Euler-Lagrange and Hamilton equations on $(\mathbf{R}^2, g, J)$ and $(\mathbf{R}_n^{2n}, g, J)$ being models of para-Kähler space forms given by [**11, 12**]. Finally, some geometrical and physical results on the related mechanical systems have been derived.

1. Mechanical Systems on (R^2, g, J)

The aim of this section is to introduce Euler-Lagrange and Hamilton equations on $\mathbf{R}^2$ which is a model of para-Kähler manifolds of a para-Kähler space form or constant J-sectional curvature. In conclusion, some geometrical and physical results on the related mechanic systems are given.

1.1. Para-Kähler Space Forms. Let M be a manifold endowed with an almost product structure $J \neq \mp Id$; which is a (1; 1)-tensor field such that $J^2 = Id$. We say that (M, J) (resp.(M, J, g)) is an almost product (resp. almost Hermitian) manifold, where g is a semi-Riemannian metric on M with respect to which J is skew-symmetric, that is

$$g(JX, Y) + g(X, JY) = 0, \ \forall X, Y \in \chi(M). \tag{1.1}$$

Then (M, J, g) is para-Kähler if J is parallel with respect to the Levi-Civita connection.

Let (M, J, g) be a para-Kähler manifold and let denote the curvature (0, 4)-tensor field by

$$\mathcal{R}(X, Y, Z, V) = g(\mathcal{R}(X, Y)Z, V), \ \forall X, Y, Z, V \in \chi(M),$$

where the Riemannian curvature (1, 3)-tensor field associated to the Levi-Civita connection ∇ of g is given by $\mathcal{R}$ =$[\nabla, \nabla]$ -$\nabla_{[\ ,\ ]}$. Then

$$\mathcal{R}(X, Y, Z, V) = -\mathcal{R}(Y, X, Z, V) = -\mathcal{R}(X, Y, V, Z) = \mathcal{R}(JX, JY, Z, V)$$
$$\text{and } \sum_{\sigma} \mathcal{R}(X, Y, Z, V) = 0,$$

where σ denotes the sum over all cyclic permutations. We know that the following (0,4)-tensor field is defined by

$$\mathcal{R}_0(X, Y, Z, V) = \tfrac{1}{4}\{g(X, Z)g(Y, V) - g(X, V)g(Y, Z)$$
$$-g(X, JZ)g(Y, JV) + g(X, JV)g(Y, JZ) - 2g(X, JY)g(Z, JV)\}$$

where $\forall X, Y, Z, V \in \chi(M)$. For any $p \in M$, a subspace $S \subset T_pM$ is called non-degenerate if g restricted to S is non-degenerate. If $\{u, v\}$ is a basis of a plane $\sigma \subset T_pM$, then σ is non-degenerate iff $g(u,u)g(v,v) - [g(u,v)]^2 \neq 0$. In this case the sectional curvature of $\sigma = span\,\{u,v\}$ is

$$k(\sigma) = \frac{\mathcal{R}(u,v,u,v)}{g(u,u)g(v,v) - [g(u,v)]^2}$$

From (1.1) it follows that X and JX are orthogonal for any $X \in \chi(M)$. By a J-plane we mean a plane which is invariant by J. For any $p \in M$, a vector $u \in T_pM$ is isotropic provided $g(u,u) = 0$. If $u \in T_pM$ is not isotropic, then the sectional curvature $H(u)$ of the J-plane $span\,\{u, Ju\}$ is called the J-sectional curvature defined by u. When $H(u)$ is constant, then (M,J,g) is called of constant J-sectional curvature, or a para-Kähler space form [**13, 14**].

Theorem 1. [**13, 15**] *Let (M,J,g) be a para-Kähler manifold such that for each $p \in M$, there exists $c_p \in \mathcal{R}$ satisfying $H(u) = c_p$ for $u \in T_pM$ such that $g(u,u)g(Ju,Ju) \neq 0$.Then the Riemann- Christoffel tensor $\mathcal{R}$ satisfies $\mathcal{R} = c\mathcal{R}_0$, where c is the function defined by $p \to c_p$. And conversely.*

Definition 1. *A para-Kähler manifold (M,J,g) is said to be of constant paraholomorphic sectional curvature c if it satisfies the conditions of **Theorem 1**.*

Theorem 2. [**13, 15**]. *Let (M,J,g) be a para-Kähler manifold with $dimM > 2$. Then the following properties are equivalent:*

1) M is a space of constant paraholomorphic sectional curvature c
2) The Riemann- Christoffel tensor curvature tensor R has the expression

$$\begin{aligned}\mathcal{R}(X,Y,Z,V) \;=\; & \frac{c}{4}\{g(X,Z)g(Y,V) - g(X,V)g(Y,Z) - g(X,JZ)g(Y,JV) \\ & + g(X,JV)g(Y,JZ) - 2g(X,JY)g(Z,JV),\end{aligned}$$

where $\forall X,Y,Z,V \in \chi(M)$.

Let $(x,\ y)$ be a real coordinate system on a neighborhood U of any point p of $\mathbf{R}^2$, and $\{(\frac{\partial}{\partial x})_p, (\frac{\partial}{\partial y})_p\}$ and $\{(dx)_p, (dy)_p\}$ natural bases over $\mathbf{R}$ of the tangent space $T_p(\mathbf{R}^2)$ and the cotangent space $T_p^*(\mathbf{R}^2)$ of $\mathbf{R}^2$, respectively.

The space $(\mathbf{R}^2, g, J)$, is the model of the para-Kähler space forms of dimension 2 and paraholomorphic sectional curvature $c \neq 0$, where g is the metric given by

$$g = \frac{4}{c}\left(\cosh^2 2y dx \otimes dx - dy \otimes dy\right), 0 \neq c \in \mathbf{R},$$

and J is the almost product structure determined by

$$J = -\frac{1}{\cosh 2y}\frac{\partial}{\partial x} \otimes dy - \cosh 2y \frac{\partial}{\partial y} \otimes dx.$$

Then we have

$$J(\frac{\partial}{\partial x}) = -\cosh 2y \frac{\partial}{\partial y},\ J(\frac{\partial}{\partial y}) = -\frac{1}{\cosh 2y}\frac{\partial}{\partial x}. \tag{1.2}$$

The dual endomorphism J^* of the cotangent space $T_p^*(\mathbf{R}^2)$ at any point p of manifold $\mathbf{R}^2$ satisfies $J^{*2} = Id$ and is defined by

$$J^*(dx) = -\cosh 2y dy,\ J^*(dy) = -\frac{1}{\cosh 2y} dx. \tag{1.3}$$

1.2. Lagrangian Mechanics. In this subsection, we find Euler-Lagrange equations for classical mechanics constructed on para-Kähler space form $(\mathbf{R}^2, g, J)$.

Denote by J the almost product structure and by (x, y) the coordinates of $\mathbf{R}^2$. Assume that semispray be a vector field as follows:

$$\xi = X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y},\ X = \dot{x} = y,\ Y = \dot{y}.$$

By *Liouville vector field* on para-Kähler space form $(\mathbf{R}^2, g, J)$, we call the vector field determined by $V = J\xi$ and calculated by

$$J\xi = -\frac{1}{\cosh 2y}.Y\frac{\partial}{\partial x} - \cosh 2y.X\frac{\partial}{\partial y},$$

Given T by *the kinetic energy* and P by *the potential energy of mechanics system* on para-Kähler space form. Then we write by $L = T - P$ *Lagrangian function* and by $E_L = V(L) - L$ *the energy function* associated L.

Operator i_J defined by

$$i_J : \wedge^2\mathbf{R}^2 \to \wedge^1\mathbf{R}^2,\ i_J(\omega)(X) = \omega(X, JX)$$

is called the *interior product* with J, or sometimes the *insertion operator*, or *contraction* by J, where $\omega \in \wedge^2\mathbf{R}^2$, $X \in \chi(\mathbf{R}^2)$. The exterior vertical derivation d_J is defined by

$$d_J = [i_J, d] = i_J d - d i_J,$$

where d is the usual exterior derivation. For almost product structure J determined by (1.2), the para-Kähler form is the closed 2-form given by $\Phi_L = -dd_J L$ such that

$$d_J = -\cosh 2y.\frac{\partial}{\partial y}dx - \frac{1}{\cosh 2y}.\frac{\partial}{\partial x}dy : \mathcal{F}(\mathbf{R}^2) \to \wedge^1\mathbf{R}^2.$$

Thus we get

$$\begin{aligned}\Phi_L \quad = \quad & \cosh 2y\frac{\partial^2 L}{\partial a\partial y}da \wedge dx + \cosh 2y\frac{\partial^2 L}{\partial b\partial y}db \wedge dx \\ & + \frac{1}{\cosh 2y}\frac{\partial^2 L}{\partial a\partial x}da \wedge dy + \frac{1}{\cosh 2y}\frac{\partial^2 L}{\partial b\partial x}db \wedge dy.\end{aligned}$$

where (a, b) is other coordinates of $\mathbf{R}^2$. Also, one may find

$$E_L = -\frac{1}{\cosh 2y}.Y\frac{\partial L}{\partial x} + \cosh 2y.X\frac{\partial L}{\partial y} - L.$$

Considering (0.1), we calculate

$$\cosh 2y.X\tfrac{\partial^2 L}{\partial a\partial y}dx+\cosh 2y.Y\tfrac{\partial^2 L}{\partial b\partial y}dx$$
$$+\tfrac{1}{\cosh 2y}.X\tfrac{\partial^2 L}{\partial a\partial x}dy+\tfrac{1}{\cosh 2y}.Y\tfrac{\partial^2 L}{\partial b\partial x}dy+\tfrac{\partial L}{\partial x}dx+\tfrac{\partial L}{\partial y}dy=0.$$

If the curve $\alpha:\mathbf{I}\subset\mathbf{R}\to\mathbf{R}^2$ be integral curve of ξ, we get equations

$$\cosh 2y\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial y}\right)+\frac{\partial L}{\partial x}=0,\frac{1}{\cosh 2y}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x}\right)+\frac{\partial L}{\partial y}=0. \tag{1.4}$$

Thus we may prove the following:

PROPOSITION 1. *Let ξ the semispray on $(\mathbf{R}^2,g,J)$. Then the paths of ξ are solutions of Euler-Lagrange equations given by*

$$\cosh 2y\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial y}\right)+\frac{\partial L}{\partial x}=0,\frac{1}{\cosh 2y}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x}\right)+\frac{\partial L}{\partial y}=0,$$

on para-Kähler space form $(\mathbf{R}^2,g,J)$.

PROPOSITION 2. *Let J almost product structure on para-Kähler space form $(\mathbf{R}^2,g,J)$.Also let (f_1,f_2) be natural bases of $\mathbf{R}^2$.Then it follows*

$$\cosh 2y.J(f_2)+f_1=0\Longleftrightarrow\quad \cosh 2y.\dot{f}_{2,L}+f_{1,L}=0,$$
$$\tfrac{1}{\cosh 2y}J(f_1)+f_2=0\Longleftrightarrow\quad \tfrac{1}{\cosh 2y}.\dot{f}_{1,L}+f_{2,L}=0,$$

where $f_{1,L}=\frac{\partial L}{\partial x},\ \ f_{2,L}=\frac{\partial L}{\partial y},\ \dot{f}_{1,L}=\frac{\partial}{\partial t}(\frac{\partial L}{\partial x}),\ \ \dot{f}_{2,L}=\frac{\partial}{\partial t}(\frac{\partial L}{\partial y})$.

Finally one may say that the triple $(\mathbf{R}^2,\Phi_L,\xi)$ is *mechanical system* on para-Kähler space form $(\mathbf{R}^2,g,J)$.

1.3. Hamiltonian Mechanics. In this subsection, we conclude Hamilton equations for classical mechanics structured on para-Kähler space form $(\mathbf{R}^2,g,J)$.

Let J^* be an almost product structure defined by (1.3) and λ Liouville form determined by $J^*(\omega)=-x\cosh 2ydy-y\frac{1}{\cosh 2y}dx$ such that $\omega=xdx+ydy$ 1-form on $\mathbf{R}^2$. If $\Phi=-d\lambda$ is closed para-Kähler form, then it is also a para-symplectic structure on $\mathbf{R}^2$.

Let $(\mathbf{R}^2,g,J)$ be para-Kähler space form fixed with closed para-Kähler form Φ. Suppose that Hamiltonian vector field Z_H associated to Hamiltonian energy H is given by

$$Z_H=X\frac{\partial}{\partial x}+Y\frac{\partial}{\partial y}.$$

For the closed para-Kähler form Φ on $\mathbf{R}^2$, we have

$$\Phi=\frac{\cosh^2 2y-1}{\cosh 2y}dx\wedge dy.$$

Then it follows

$$i_{Z_H}\Phi = -\frac{\cosh^2 2y - 1}{\cosh 2y} Y dx + \frac{\cosh^2 2y - 1}{\cosh 2y} X dy. \tag{1.5}$$

Otherwise, we find the differential of Hamiltonian energy as follows:

$$dH = \frac{\partial H}{\partial x} dx + \frac{\partial H}{\partial y} dy. \tag{1.6}$$

From (1.5) and (1.6) with respect to (0.3), we find para-Hamiltonian vector field on para-Kähler space form to be

$$Z_H = \frac{\cosh 2y}{\cosh^2 2y - 1} \frac{\partial H}{\partial y} \frac{\partial}{\partial x} - \frac{\cosh 2y}{\cosh^2 2y - 1} \frac{\partial H}{\partial x} \frac{\partial}{\partial y}. \tag{1.7}$$

Assume that the curve

$$\beta : I \subset \mathbf{R} \to \mathbf{R}^2$$

be an integral curve of Hamiltonian vector field Z_H, i.e.,

$$Z_H(\beta(t)) = \dot{\beta},\ t \in I. \tag{1.8}$$

In the local coordinates we get

$$\beta(t) = (x(t), y(t)),$$

and

$$\dot{\beta}(t) = \frac{dx}{dt} \frac{\partial}{\partial x} + \frac{dy}{dt} \frac{\partial}{\partial y}. \tag{1.9}$$

Now, by means of (1.8), from (1.7) and (1.9), we deduce the equations

$$\frac{dx}{dt} = \frac{\cosh 2y}{\cosh^2 2y - 1} \frac{\partial H}{\partial y},\ \frac{dy}{dt} = -\frac{\cosh 2y}{\cosh^2 2y - 1} \frac{\partial H}{\partial x}. \tag{1.10}$$

Thus we may prove the following:

PROPOSITION 3. *Let Z_H be the vector field on $(\mathbf{R}^2, g, J)$. Then the paths of Z_H are solutions of Hamilton equations determined by*

$$\frac{dx}{dt} = \frac{\cosh 2y}{\cosh^2 2y - 1} \frac{\partial H}{\partial y},\ \frac{dy}{dt} = -\frac{\cosh 2y}{\cosh^2 2y - 1} \frac{\partial H}{\partial x}$$

on para-Kähler space form (R^2, g, J).

In the end, we may say to be *para-mechanical system* $(\mathbf{R}^2, \Phi, Z_H)$ triple on para-Kähler space form $(\mathbf{R}^2, g, J)$.

CONCLUSION 1. *From above, we understand that Lagrangian and Hamiltonian formalisms in generalized classical mechanics and field theory can be intrinsically characterized on $(\mathbf{R}^2, g, J)$ being a model of para-Kähler space forms. So, the paths of semispray ξ on $\mathbf{R}^2$ are the solutions of the Euler-Lagrange equations given by (1.4) on the mechanical system $(\mathbf{R}^2, \Phi_L, \xi)$. Also, the solutions of the Hamilton equations determined by (1.10) on the mechanical system $(\mathbf{R}^2, \Phi, Z_H)$ are the paths of vector field Z_H on $\mathbf{R}^2$.*

2. Mechanical Systems on (R_n^{2n}, g, J)

The goal of this section is to present Euler-Lagrange and Hamilton equations on $\mathbf{R}_n^{2n}$ which is a model of para-Kähler manifolds of constant J-sectional curvature or a para-Kähler space form. In conclusion, some differential geometrical and physical results on the related mechanic systems have been given.

Let (x_i, y_i) be a real coordinate system on a neighborhood U of any point p of $\mathbf{R}_n^{2n}$, and $\{(\frac{\partial}{\partial x_i})_p, (\frac{\partial}{\partial y_i})_p\}$ and $\{(dx_i)_p, (dy_i)_p\}$ natural bases over $\mathbf{R}$ of the tangent space $T_p(\mathbf{R}_n^{2n})$ and the cotangent space $T_p^*(\mathbf{R}_n^{2n})$ of $\mathbf{R}_n^{2n}$, respectively.

The space $(\mathbf{R}_n^{2n}, g, J)$, is the model of the para-Kähler space forms of dimension $2n \geq 2$ and paraholomorphic sectional curvature $c = 0$, where g is the metric given by

$$g = dx_i \otimes dy_i + dy_i \otimes dx_i,$$

and J is the almost product structure defined by

$$J = \frac{\partial}{\partial x_i} \otimes dx_i - \frac{\partial}{\partial y_i} \otimes dx_i.$$

Then we have

$$J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_i}, \; J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial y_i}. \tag{2.1}$$

The dual endomorphism J^* of the cotangent space $T_p^*(\mathbf{R}_n^{2n})$ at any point p of manifold $\mathbf{R}_n^{2n}$ satisfies $J^{*2} = Id$ and is defined by

$$J^*(dx_i) = dx_i, \; J^*(dy_i) = -dy_i. \tag{2.2}$$

2.1. Lagrangian Mechanics Systems. In this subsection, we introduce Euler-Lagrange equations on para-Kähler manifolds of para-Kähler space form $(\mathbf{R}_n^{2n}, g, J)$.

Given by J almost product structure and by (x_i, y_i) the coordinates of $\mathbf{R}_n^{2n}$. Let semispray be a vector field as follows:

$$\xi = X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}, \; X_i = \dot{x_i} = y_i, \; Y_i = \dot{y_i}.$$

By *Liouville vector field* on para-Kähler space form $(\mathbf{R}_n^{2n}, g, J)$, we call the vector field determined by $V = J\xi$ and calculated by

$$J\xi = X_i \frac{\partial}{\partial x_i} - Y_i \frac{\partial}{\partial y_i},$$

Denote T by *the kinetic energy* and P by *the potential energy of mechanics system* on para-Kähler space of para-Kähler space form. Then we write by $L = T - P$ *Lagrangian function* and by $E_L = V(L) - L$ *the energy function* associated L.

Operator i_J defined by

$$i_J : \wedge^2 \mathbf{R}_n^{2n} \to \wedge^1 \mathbf{R}_n^{2n}$$

is called the interior product with J, or sometimes the insertion operator, or contraction by J. The exterior vertical derivation d_J is defined by

$$d_J = [i_J, d] = i_J d - d i_J,$$

where d is the usual exterior derivation. For almost product structure J determined by (2.1), the para-Kähler form is the closed 2-form given by $\Phi_L = -dd_J L$ such that

$$d_J = \frac{\partial}{\partial x_i} dx_i - \frac{\partial}{\partial y_i} dy_i : \mathcal{F}(\mathbf{R}_n^{2n}) \to \wedge^1 \mathbf{R}_n^{2n}.$$

Thus we get

$$\begin{aligned}\Phi_L \quad = \quad & -\frac{\partial^2 L}{\partial x_j \partial x_i} dx_j \wedge dx_i - \frac{\partial^2 L}{\partial y_j \partial x_i} dy_j \wedge dx_i \\ & + \frac{\partial^2 L}{\partial x_j \partial y_i} dx_j \wedge dy_i + \frac{\partial^2 L}{\partial y_j \partial y_i} dy_j \wedge dy_i.\end{aligned}$$

Also, one may obtain

$$E_L = X_i \frac{\partial L}{\partial x_i} - Y_i \frac{\partial L}{\partial y_i} - L,$$

Taking care of (0.1), we have

$$\begin{aligned}&-X_i \tfrac{\partial^2 L}{\partial x_j \partial x_i} dx_j - Y_i \tfrac{\partial^2 L}{\partial y_j \partial x_i} dx_j + \tfrac{\partial L}{\partial x_j} dx_j \\ &+X_i \tfrac{\partial^2 L}{\partial x_j \partial y_i} dy_j + Y_i \tfrac{\partial^2 L}{\partial y_j \partial y_i} dy_j + \tfrac{\partial L}{\partial y_j} dy_j = 0.\end{aligned}$$

If α on $\mathbf{R}_n^{2n}$ is an integral curve of ξ, it follows

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_j}\right) - \frac{\partial L}{\partial x_j} = 0, \ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial y_j}\right) + \frac{\partial L}{\partial y_j} = 0, \tag{2.3}$$

so-called *Euler-Lagrange equations* whose solutions are the paths of the semispray ξ on para-Kähler space form $(\mathbf{R}_n^{2n}, g, J)$. Finally one may say that the triple $(\mathbf{R}_n^{2n}, \Phi_L, \xi)$ is *mechanical system* on para-Kähler manifolds of para-Kähler space form $(\mathbf{R}_n^{2n}, g, J)$.Therefore we say

PROPOSITION 4. *Let J almost product structure on para-Kähler space of para-Kähler space form $(\mathbf{R}_n^{2n}, g, J)$. Also let (f_1, f_2) be natural bases of $\mathbf{R}_n^{2n}$. Then it follows*

$$\begin{aligned}J(f_1) - f_1 = 0 \Longleftrightarrow \quad & \dot{f}_{1,L} - f_{1,L} = 0, \\ J(f_2) + f_2 = 0 \Longleftrightarrow \quad & \dot{f}_{2,L} + f_{2,L} = 0,\end{aligned}$$

where $f_{1,L} = \frac{\partial L}{\partial x_i}$, $f_{2,L} = \frac{\partial L}{\partial y_i}$, $\dot{f}_{1,L} = \frac{\partial}{\partial t}(\frac{\partial L}{\partial x_i})$, $\dot{f}_{2,L} = \frac{\partial}{\partial t}(\frac{\partial L}{\partial y_i})$.

2.2. Hamiltonian Mechanics Systems. In this subsection, we present Hamilton equations on para-Kähler manifolds of para-Kähler space form $(\mathbf{R}_n^{2n}, g, J)$.

Let J^* be an almost product structure defined by (2.2) and λ Liouville form determined by $J^*(\omega) = \frac{1}{2}y_i dx_i - \frac{1}{2}x_i dy_i$ such that $\omega = \frac{1}{2}y_i dx_i + \frac{1}{2}x_i dy_i$ 1-form on $\mathbf{R}_n^{2n}$. If $\Phi = -d\lambda$ is closed para-Kähler form, then it is also a para-symplectic structure on $\mathbf{R}_n^{2n}$.

Let $(\mathbf{R}_n^{2n}, g, J)$ be para-Kähler manifolds of para-Kähler space form with closed para-Kähler form Φ. Suppose that Hamiltonian vector field Z_H associated to Hamiltonian energy H is given by

$$Z_H = X_i \frac{\partial}{\partial x_i} + Y_i \frac{\partial}{\partial y_i}.$$

For the closed para-Kähler form Φ on $\mathbf{R}_n^{2n}$, we have

$$\Phi = dx_i \wedge dy_i.$$

Then it follows

$$i_{Z_H}\Phi = -Y_i dx_i + X_i dy_i. \tag{2.4}$$

Otherwise, one may calculate the differential of Hamiltonian energy as follows:

$$dH = \frac{\partial H}{\partial x_i} dx_i + \frac{\partial H}{\partial y_i} dy_i. \tag{2.5}$$

From (2.4) and (2.5) with respect to $i_{Z_H}\Phi = dH$, we find para-Hamiltonian vector field on para-Kähler space of para-Kähler space form to be

$$Z_H = \frac{\partial H}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial y_i}. \tag{2.6}$$

Suppose that the curve γ on $\mathbf{R}_n^{2n}$ is an integral curve of Hamiltonian vector field Z_H, i.e.,

$$Z_H(\gamma(t)) = \dot{\gamma}, \ t \in I. \tag{2.7}$$

In the local coordinates we have

$$\gamma(t) = (x_i(t), y_i(t)),$$

and

$$\dot{\gamma}(t) = \frac{dx_i}{dt}\frac{\partial}{\partial x_i} + \frac{dy_i}{dt}\frac{\partial}{\partial y_i}. \tag{2.8}$$

Now, by means of (2.7), from (2.6) and (2.8), we deduce the equations so-called *para-Hamilton equations*

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial y_i}, \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i}. \tag{2.9}$$

In the end, we may say to be *para-mechanical system* $(\mathbf{R}_n^{2n}, \Phi, Z_H)$ triple on para-Kähler manifolds of para-Kähler space form $(\mathbf{R}_n^{2n}, g, J)$.

CONCLUSION 2. *From the above, we obtain that Lagrangian and Hamiltonian formalisms in generalized classical mechanics and field theory can be intrinsically characterized on* $(\mathbf{R}_n^{2n}, g, J)$ *being a model of para-Kähler space of para-Kähler space form. So, the paths of semispray* ξ *on* $\mathbf{R}_n^{2n}$ *are the solutions of the Euler-Lagrange equations given by (2.3) on the mechanical system* $(\mathbf{R}_n^{2n}, \Phi_L, \xi)$*. Also, the solutions of the Hamilton equations determined by (2.9) on the mechanical system* $(\mathbf{R}_n^{2n}, \Phi, Z_H)$ *are the paths of vector field* Z_H *on* $\mathbf{R}_n^{2n}$*.*

CHAPTER 3

Mechanical Systems on Standard Clifford Kähler Manifolds

This chapter deals with the notation of a Clifford structure on an 8n-dimensional Riemannian manifold (as introduced in a previous paper of Burdujan given in [**19**]) and the construction of some Lagrangian and Hamiltonian mechanical systems related to such structure in given [**16, 17**]. Also, a discussion on some geometrical and physical results about Euler-Lagrange and Hamilton equations and fields obtained on standard Clifford Kähler manifold is given.

0.3. Clifford Kähler Manifolds. Here, we will recall the main concepts and structures given in [**18, 19**]. Let M be a real manifold of dimension m. Suppose that there is a 6-dimensional vector bundle V consisting of $J_i(i=\overline{1,6})$ tensors of type (1,1) over M. Such a local basis $\{J_i\}(i=\overline{1,6})$ is called a canonical local basis of the bundle V in a neighborhood U of M. Then V is called an almost Clifford structure in M. The pair (M,V) is named an almost Clifford manifold with V. Hence, an almost Clifford manifold M is of dimension $m=8n$. If there exists on (M,V) a global basis $\{J_i\}(i=\overline{1,6})$, then (M,V) is said to be an almost Clifford manifold and the basis $\{J_i\}(i=\overline{1,6})$ is called a global basis for V.

An almost Clifford connection on the almost Clifford manifold (M,V) is a linear connection ∇ on M which preserves by parallel transport the vector bundle V. This means that if Φ is a cross-section (local-global) of the bundle V, then $\nabla_X\Phi$ is also a cross-section (local-global, respectively) of V, X being an arbitrary vector field of M.

If for any canonical basis $\{J_i\}(i=\overline{1,6})$ of V in a coordinate neighborhood U, the identities

$$g(J_iX,J_iY)=g(X,Y),\ \forall X,Y\in\chi(M),\ (i=\overline{1,6}),$$

hold, the triple (M,g,V) is named an almost Clifford Hermitian manifold or metric Clifford manifold denoting by V an almost Clifford structure V and by g a Riemannian metric and by (g,V) an almost Clifford metric structure.

Since each $J_i(i=\overline{1,6})$ is almost Hermitian structure with respect to g, setting

$$\Phi_i(X,Y)=g(J_iX,Y),(i=\overline{1,6}),$$

for any vector fields X and Y, we see that Φ_i are 6 local 2-forms.

If the Levi-Civita connection $\nabla = \nabla^g$ on (M, g, V) preserves the vector bundle V by parallel transport, then (M, g, V) is called a Clifford Kähler manifold, and an almost Clifford structure Φ_i of M is called a Clifford Kähler structure.

A Clifford Kähler manifold is Riemannian manifold (M^{8n}, g). For example, we say that $\mathbf{R}^{8n}$ is the simplest example of Clifford Kähler manifold. Suppose that let

$$\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}, x_{4n+i}, x_{5n+i}, x_{6n+i}, x_{7n+i}\},$$

$i = \overline{1,n}$ be a real coordinate system on $\mathbf{R}^{8n}$. Then we define by

$$\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{n+i}}, \frac{\partial}{\partial x_{2n+i}}, \frac{\partial}{\partial x_{3n+i}}, \frac{\partial}{\partial x_{4n+i}}, \frac{\partial}{\partial x_{5n+i}}, \frac{\partial}{\partial x_{6n+i}}, \frac{\partial}{\partial x_{7n+i}}\right\}$$

and

$$\{dx_i, dx_{n+i}, dx_{2n+i}, dx_{3n+i}, dx_{4n+i}, dx_{5n+i}, dx_{6n+i}, dx_{7n+i}\}$$

be natural bases over $\mathbf{R}$ of the tangent space $T(\mathbf{R}^{8n})$ and the cotangent space $T^*(\mathbf{R}^{8n})$ of $\mathbf{R}^{8n}$, respectively. By structures $J_i (i = \overline{1,3})$, the following expressions are obtained

(0.10)

$$\begin{array}{ccc}
J_1(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_{n+i}} & J_2(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_{2n+i}} & J_3(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_{3n+i}} \\
J_1(\frac{\partial}{\partial x_{n+i}}) = -\frac{\partial}{\partial x_i} & J_2(\frac{\partial}{\partial x_{n+i}}) = -\frac{\partial}{\partial x_{4n+i}} & J_3(\frac{\partial}{\partial x_{n+i}}) = -\frac{\partial}{\partial x_{5n+i}} \\
J_1(\frac{\partial}{\partial x_{2n+i}}) = \frac{\partial}{\partial x_{4n+i}} & J_2(\frac{\partial}{\partial x_{2n+i}}) = -\frac{\partial}{\partial x_i} & J_3(\frac{\partial}{\partial x_{2n+i}}) = -\frac{\partial}{\partial x_{6n+i}} \\
J_1(\frac{\partial}{\partial x_{3n+i}}) = \frac{\partial}{\partial x_{5n+i}} & J_2(\frac{\partial}{\partial x_{3n+i}}) = \frac{\partial}{\partial x_{6n+i}} & J_3(\frac{\partial}{\partial x_{3n+i}}) = -\frac{\partial}{\partial x_i} \\
J_1(\frac{\partial}{\partial x_{4n+i}}) = -\frac{\partial}{\partial x_{2n+i}} & J_2(\frac{\partial}{\partial x_{4n+i}}) = \frac{\partial}{\partial x_{n+i}} & J_3(\frac{\partial}{\partial x_{4n+i}}) = \frac{\partial}{\partial x_{7n+i}} \\
J_1(\frac{\partial}{\partial x_{5n+i}}) = -\frac{\partial}{\partial x_{3n+i}} & J_2(\frac{\partial}{\partial x_{5n+i}}) = -\frac{\partial}{\partial x_{7n+i}} & J_3(\frac{\partial}{\partial x_{5n+i}}) = \frac{\partial}{\partial x_{n+i}} \\
J_1(\frac{\partial}{\partial x_{6n+i}}) = \frac{\partial}{\partial x_{7n+i}} & J_2(\frac{\partial}{\partial x_{6n+i}}) = -\frac{\partial}{\partial x_{3n+i}} & J_3(\frac{\partial}{\partial x_{6n+i}}) = \frac{\partial}{\partial x_{2n+i}} \\
J_1(\frac{\partial}{\partial x_{7n+i}}) = -\frac{\partial}{\partial x_{6n+i}} & J_2(\frac{\partial}{\partial x_{7n+i}}) = \frac{\partial}{\partial x_{5n+i}} & J_3(\frac{\partial}{\partial x_{7n+i}}) = -\frac{\partial}{\partial x_{4n+i}}
\end{array}$$

A canonical local basis$\{J_i^*\}(i = \overline{1,3})$ of V^* of the cotangent space $T^*(\mathbf{R}^{8n})$ of manifold $\mathbf{R}^{8n}$ satisfies the condition:

$$J_1^{*2} = J_2^{*2} = J_3^{*2} = J_1^* J_2^* J_3^{*2} J_2^* J_1^* = -I,$$

defining

(0.11)

$$\begin{array}{ccc}
J_1^*(dx_i) = dx_{n+i} & J_2^*(dx_i) = dx_{2n+i} & J_3^*(dx_i) = dx_{3n+i} \\
J_1^*(dx_{n+i}) = -dx_i & J_2^*(dx_{n+i}) = -dx_{4n+i} & J_3^*(dx_{n+i}) = -dx_{5n+i} \\
J_1^*(dx_{2n+i}) = dx_{4n+i} & J_2^*(dx_{2n+i}) = -dx_i & J_3^*(dx_{2n+i}) = -dx_{6n+i} \\
J_1^*(dx_{3n+i}) = dx_{5n+i} & J_2^*(dx_{3n+i}) = dx_{6n+i} & J_3^*(dx_{3n+i}) = -dx_i \\
J_1^*(dx_{4n+i}) = -dx_{2n+i} & J_2^*(dx_{4n+i}) = dx_{n+i} & J_3^*(dx_{4n+i}) = dx_{7n+i} \\
J_1^*(dx_{5n+i}) = -dx_{3n+i} & J_2^*(dx_{5n+i}) = -dx_{7n+i} & J_3^*(dx_{5n+i}) = dx_{n+i} \\
J_1^*(dx_{6n+i}) = dx_{7n+i} & J_2^*(dx_{6n+i}) = -dx_{3n+i} & J_3^*(dx_{6n+i}) = dx_{2n+i} \\
J_1^*(dx_{7n+i}) = -dx_{6n+i} & J_2^*(dx_{7n+i}) = dx_{5n+i} & J_3^*(dx_{7n+i}) = -dx_{4n+i}
\end{array}$$

0.4. Standard Clifford Lagrangian Mechanics. Here, we obtain Euler-Lagrange equations for quantum and classical mechanics by means of a canonical local basis $\{J_i\}(i = \overline{1,3})$ of V on the standard Clifford Kähler manifold $(\mathbf{R}^{8n}, V)$.

Firstly, let J_1 take a local basis component on the standard Clifford Kähler manifold $(\mathbf{R}^{8n}, V)$. Let semispray be the vector field ξ determined by

$$\begin{array}{c} \xi = X^i \frac{\partial}{\partial x_i} + X^{n+i} \frac{\partial}{\partial x_{n+i}} + X^{2n+i} \frac{\partial}{\partial x_{2n+i}} + X^{3n+i} \frac{\partial}{\partial x_{3n+i}} \\ +X^{4n+i} \frac{\partial}{\partial x_{4n+i}} + X^{5n+i} \frac{\partial}{\partial x_{5n+i}} + X^{6n+i} \frac{\partial}{\partial x_{6n+i}} + X^{7n+i} \frac{\partial}{\partial x_{7n+i}}. \end{array} \tag{0.12}$$

Where

$$\begin{array}{rcl} X^i & = & \dot{x}_i, X^{n+i} = \dot{x}_{n+i}, X^{2n+i} = \dot{x}_{2n+i}, X^{3n+i} = \dot{x}_{3n+i}, \\ X^{4n+i} & = & \dot{x}_{4n+i}, X^{5n+i} = \dot{x}_{5n+i}, X^{6n+i} = \dot{x}_{6n+i}, X^{7n+i} = \dot{x}_{7n+i} \end{array}$$

and the dot indicates the derivative with respect to time t. The vector field defined by

$$\begin{array}{c} V_{J_1} = J_1(\xi) = X^i \frac{\partial}{\partial x_{n+i}} - X^{n+i} \frac{\partial}{\partial x_i} + X^{2n+i} \frac{\partial}{\partial x_{4n+i}} + X^{3n+i} \frac{\partial}{\partial x_{5n+i}} \\ -X^{4n+i} \frac{\partial}{\partial x_{2n+i}} - X^{5n+i} \frac{\partial}{\partial x_{3n+i}} + X^{6n+i} \frac{\partial}{\partial x_{7n+i}} - X^{7n+i} \frac{\partial}{\partial x_{6n+i}}, \end{array}$$

is called *Liouville vector field* on the standard Clifford Kähler manifold $(\mathbf{R}^{8n}, V)$. The maps given by $T, P : \mathbf{R}^{8n} \to \mathbf{R}$ such that

$$T = \frac{1}{2} m_i (\dot{x}_i{}^2 + \dot{x}^2_{n+i} + \dot{x}^2_{2n+i} + \dot{x}^2_{3n+i} + \dot{x}^2_{4n+i} + \dot{x}^2_{5n+i} + \dot{x}^2_{6n+i} + \dot{x}^2_{7n+i}), \ P = m_i g h$$

are called *the kinetic energy* and *the potential energy of the system,* respectively. Here m_i, g and h stand for mass of a mechanical system having m particles, the gravity acceleration and distance to the origin of a mechanical system on the standard Clifford Kähler manifold $(\mathbf{R}^{8n}, V)$, respectively. Then $L : \mathbf{R}^{8n} \to R$ is a map that satisfies the conditions; i) $L = T - P$ is a *Lagrangian function, ii)* the function given by $E_L^{J_1} = V_{J_1}(L) - L$, is *energy function.*

The operator i_{J_1} induced by J_1 and given by

$$i_{J_1} \omega(X_1, X_2, ..., X_r) = \sum_{i=1}^{r} \omega(X_1, ..., J_1 X_i, ..., X_r),$$

is said to be *vertical derivation,* where $\omega \in \wedge^r \mathbf{R}^{8n}$, $X_i \in \chi(\mathbf{R}^{8n})$. The *vertical differentiation* d_{J_1} is defined by

$$d_{J_1} = [i_{J_1}, d] = i_{J_1} d - d i_{J_1},$$

where d is the usual exterior derivation and $[\ ,]$ is Lie bracket. The Clifford Kähler form is the closed 2-form given by $\Phi_L^{J_1} = -dd_{J_1} L$ such that

$$\begin{aligned} d_{J_1} &= \frac{\partial}{\partial x_{n+i}} dx_i - \frac{\partial}{\partial x_i} dx_{n+i} + \frac{\partial}{\partial x_{4n+i}} dx_{2n+i} + \frac{\partial}{\partial x_{5n+i}} dx_{3n+i} \\ &\quad - \frac{\partial}{\partial x_{2n+i}} dx_{4n+i} - \frac{\partial}{\partial x_{3n+i}} dx_{5n+i} + \frac{\partial}{\partial x_{7n+i}} dx_{6n+i} - \frac{\partial}{\partial x_{6n+i}} dx_{7n+i} \end{aligned}$$

and

$$d_{J_1} : \mathcal{F}(\mathbf{R}^{8n}) \rightarrow \wedge^1 \mathbf{R}^{8n}.$$

Then

$$\begin{aligned} \Phi_L^{J_1} &= -\tfrac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j \wedge dx_i + \tfrac{\partial^2 L}{\partial x_j \partial x_i} dx_j \wedge dx_{n+i} - \tfrac{\partial^2 L}{\partial x_j \partial x_{4n+i}} dx_j \wedge dx_{2n+i} \\ &\quad - \tfrac{\partial^2 L}{\partial x_j \partial x_{5n+i}} dx_j \wedge dx_{3n+i} + \tfrac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j \wedge dx_{4n+i} + \tfrac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j \wedge dx_{5n+i} \\ &\quad - \tfrac{\partial^2 L}{\partial x_j \partial x_{7n+i}} dx_j \wedge dx_{6n+i} + \tfrac{\partial^2 L}{\partial x_j \partial x_{6n+i}} dx_j \wedge dx_{7n+i} - \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \wedge dx_i \\ &\quad + \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} \wedge dx_{n+i} - \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{4n+i}} dx_{n+j} \wedge dx_{2n+i} - \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{5n+i}} dx_{n+j} \wedge dx_{3n+i} \\ &\quad + \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \wedge dx_{4n+i} + \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} \wedge dx_{5n+i} - \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}} dx_{n+j} \wedge dx_{6n+i} \\ &\quad + \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{6n+i}} dx_{n+j} \wedge dx_{7n+i} - \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} \wedge dx_i + \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} \wedge dx_{n+i} \\ &\quad - \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{4n+i}} dx_{2n+j} \wedge dx_{2n+i} - \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{5n+i}} dx_{2n+j} \wedge dx_{3n+i} + \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} \wedge dx_{4n+i} \\ &\quad + \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} \wedge dx_{5n+i} - \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{7n+i}} dx_{2n+j} \wedge dx_{6n+i} + \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{6n+i}} dx_{2n+j} \wedge dx_{7n+i} \\ &\quad - \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \wedge dx_i + \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \wedge dx_{n+i} - \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{4n+i}} dx_{3n+j} \wedge dx_{2n+i} \\ &\quad - \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{5n+i}} dx_{3n+j} \wedge dx_{3n+i} + \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \wedge dx_{4n+i} + \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} \wedge dx_{5n+i} \\ &\quad - \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{7n+i}} dx_{3n+j} \wedge dx_{6n+i} + \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{6n+i}} dx_{3n+j} \wedge dx_{7n+i} - \tfrac{\partial L}{\partial x_{4n+j} \partial x_{n+i}} dx_{4n+j} \wedge dx_i \\ &\quad + \tfrac{\partial L}{\partial x_{4n+j} \partial x_i} dx_{4n+j} \wedge dx_{n+i} - \tfrac{\partial L}{\partial x_{4n+j} \partial x_{4n+i}} dx_{4n+j} \wedge dx_{2n+i} - \tfrac{\partial L}{\partial x_{4n+j} \partial x_{5n+i}} dx_{4n+j} \wedge dx_{3n+i} \\ &\quad + \tfrac{\partial^2 L}{\partial x_{4n+j} \partial x_{2n+i}} dx_{4n+j} \wedge dx_{4n+i} + \tfrac{\partial^2 L}{\partial x_{4n+j} \partial x_{3n+i}} dx_{4n+j} \wedge dx_{5n+i} - \tfrac{\partial^2 L}{\partial x_{4n+j} \partial x_{7n+i}} dx_{4n+j} \wedge dx_{6n+i} \\ &\quad + \tfrac{\partial^2 L}{\partial x_{4n+j} \partial x_{6n+i}} dx_{4n+j} \wedge dx_{7n+i} - \tfrac{\partial^2 L}{\partial x_{5n+j} \partial x_{n+i}} dx_{5n+j} \wedge dx_i + \tfrac{\partial^2 L}{\partial x_{5n+j} \partial x_i} dx_{5n+j} \wedge dx_{n+i} \\ &\quad - \tfrac{\partial^2 L}{\partial x_{5n+j} \partial x_{4n+i}} dx_{5n+j} \wedge dx_{2n+i} - \tfrac{\partial^2 L}{\partial x_{5n+j} \partial x_{5n+i}} dx_{5n+j} \wedge dx_{3n+i} + \tfrac{\partial^2 L}{\partial x_{5n+j} \partial x_{2n+i}} dx_{5n+j} \wedge dx_{4n+i} \\ &\quad + \tfrac{\partial^2 L}{\partial x_{5n+j} \partial x_{3n+i}} dx_{5n+j} \wedge dx_{5n+i} - \tfrac{\partial^2 L}{\partial x_{5n+j} \partial x_{7n+i}} dx_{5n+j} \wedge dx_{6n+i} + \tfrac{\partial^2 L}{\partial x_{5n+j} \partial x_{6n+i}} dx_{5n+j} \wedge dx_{7n+i} \end{aligned}$$

$-\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_{6n+j}\wedge dx_i+\frac{\partial^2 L}{\partial x_{6n+j}\partial x_i}dx_{6n+j}\wedge dx_{n+i}-\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_{6n+j}\wedge dx_{2n+i}$
$-\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}dx_{6n+j}\wedge dx_{3n+i}+\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{6n+j}\wedge dx_{4n+i}+\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}dx_{6n+j}\wedge dx_{5n+i}$
$-\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{6n+j}\wedge dx_{6n+i}+\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{6n+j}\wedge dx_{7n+i}-\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_{7n+j}\wedge dx_i$
$+\frac{\partial^2 L}{\partial x_{7n+j}\partial x_i}dx_{7n+j}\wedge dx_{n+i}-\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_{7n+j}\wedge dx_{2n+i}-\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_{7n+j}\wedge dx_{3n+i}$
$+\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{7n+j}\wedge dx_{4n+i}+\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{7n+j}\wedge dx_{5n+i}-\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{7n+j}\wedge dx_{6n+i}$
$+\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{7n+j}\wedge dx_{7n+i}.$

Also, we obtain

$E_L^{J_1}=X^i\frac{\partial L}{\partial x_{n+i}}-X^{n+i}\frac{\partial L}{\partial x_i}+X^{2n+i}\frac{\partial L}{\partial x_{4n+i}}+X^{3n+i}\frac{\partial L}{\partial x_{5n+i}}$
$-X^{4n+i}\frac{\partial L}{\partial x_{2n+i}}-X^{5n+i}\frac{\partial L}{\partial x_{3n+i}}+X^{6n+i}\frac{\partial L}{\partial x_{7n+i}}-X^{7n+i}\frac{\partial L}{\partial x_{6n+i}}-L.$

With the use of **Eq.** (0.1), the following expressions can be obtained:

$-X^i\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}dx_j+X^i\frac{\partial^2 L}{\partial x_j\partial x_i}dx_{n+j}-X^i\frac{\partial^2 L}{\partial x_j\partial x_{4n+i}}dx_{2n+j}-X^i\frac{\partial^2 L}{\partial x_j\partial x_{5n+i}}dx_{3n+j}$
$+X^i\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}dx_{4n+j}+X^i\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}dx_{5n+j}-X^i\frac{\partial^2 L}{\partial x_j\partial x_{7n+i}}dx_{6n+j}$
$+X^i\frac{\partial^2 L}{\partial x_j\partial x_{6n+i}}dx_{7n+j}-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}dx_j+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}dx_{n+j}$
$-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}}dx_{2n+j}-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}dx_{3n+j}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}dx_{4n+j}$
$+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{3n+i}}dx_{5n+j}-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}dx_{6n+j}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{6n+i}}dx_{7n+j}$
$-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_j+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}dx_{n+j}-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}}dx_{2n+j}$
$-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}dx_{3n+j}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{4n+j}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{3n+i}}dx_{5n+j}$
$-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}dx_{6n+j}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}dx_{7n+j}-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_j$
$+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}dx_{n+j}-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}dx_{2n+j}-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}dx_{3n+j}$
$+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{4n+j}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{3n+i}}dx_{5n+j}-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}dx_{6n+j}$
$+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}dx_{7n+j}-X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{n+i}}dx_j+X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_i}dx_{n+j}$
$-X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{4n+i}}dx_{2n+j}-X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{5n+i}}dx_{3n+j}+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{2n+i}}dx_{4n+j}$
$+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}dx_{5n+j}-X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{7n+i}}dx_{6n+j}+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}dx_{7n+j}$
$-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}dx_j+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_i}dx_{n+j}-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}dx_{2n+j}$
$-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}dx_{3n+j}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}dx_{4n+j}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}dx_{5n+j}$
$-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}dx_{6n+j}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}dx_{7n+j}-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_j$
$+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_i}dx_{n+j}-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_{2n+j}-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}dx_{3n+j}$
$+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{4n+j}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{3n+i}}dx_{5n+j}-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{6n+j}$
$+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{7n+j}-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_j+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_i}dx_{n+j}$

$$-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_{2n+j}-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_{3n+j}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{4n+j}$$
$$+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{5n+j}-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{6n+j}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{7n+j}$$
$$+\frac{\partial L}{\partial x_j}dx_j+\frac{\partial L}{\partial x_{n+j}}dx_{n+j}+\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}+\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}+\frac{\partial L}{\partial x_{4n+j}}dx_{4n+j}$$
$$+\frac{\partial L}{\partial x_{5n+j}}dx_{5n+j}+\frac{\partial L}{\partial x_{6n+j}}dx_{6n+j}+\frac{\partial L}{\partial x_{7n+j}}dx_{7n+j}=0.$$

If a curve α on $\mathbf{R}^8$ is considered to be an integral curve of ξ, then we calculate the following equations:

(0.13)
$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{n+i}}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right)-\frac{\partial L}{\partial x_i}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right)+\frac{\partial L}{\partial x_{4n+i}}=0,$$
$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right)+\frac{\partial L}{\partial x_{5n+i}}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{4n+i}}\right)-\frac{\partial L}{\partial x_{2n+i}}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{5n+i}}\right)-\frac{\partial L}{\partial x_{3n+i}}=0,$$
$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{6n+i}}\right)+\frac{\partial L}{\partial x_{7n+i}}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{7n+i}}\right)-\frac{\partial L}{\partial x_{6n+i}}=0,$$

such that the equations obtained in **Eq.** (0.13) are said to be *Euler-Lagrange equations* structured on the standard Clifford Kähler manifold $(\mathbf{R}^8, V)$ by means of $\Phi_L^{J_1}$ and in the case, the triple $(\mathbf{R}^8, \Phi_L^{J_1}, \xi)$ is called a *mechanical system* on the standard Clifford Kähler manifold $(\mathbf{R}^8, V)$.

Secondly, we find Euler-Lagrange equations for quantum and classical mechanics by means of Φ_L^G on the standard Clifford Kähler manifold $(\mathbf{R}^8, V)$.

Consider J_2 be another local basis component on the Clifford Kähler manifold $(\mathbf{R}^8, V)$. Let ξ take as in **Eq.** (0.12). In the case, the vector field given by

$$V_{J_2}=J_2(\xi)=X^i\frac{\partial}{\partial x_{2n+i}}-X^{n+i}\frac{\partial}{\partial x_{4n+i}}-X^{2n+i}\frac{\partial}{\partial x_i}+X^{3n+i}\frac{\partial}{\partial x_{6n+i}}$$
$$+X^{4n+i}\frac{\partial}{\partial x_{n+i}}-X^{5n+i}\frac{\partial}{\partial x_{7n+i}}-X^{6n+i}\frac{\partial}{\partial x_{3n+i}}+X^{7n+i}\frac{\partial}{\partial x_{5n+i}},$$

is *Liouville vector field* on the standard Clifford Kähler manifold $(\mathbf{R}^8, V)$. The function given by $E_L^{J_2}=V_{J_2}(L)-L$ is *energy function.* Then the operator i_{J_2} induced by J_2 and denoted by

$$i_{J_2}\omega(X_1,X_2,...,X_r)=\sum_{i=1}^{r}\omega(X_1,...,J_2X_i,...,X_r),$$

is *vertical derivation,* where $\omega\in\wedge^r\mathbf{R}^8$, $X_i\in\chi(\mathbf{R}^8)$. The *vertical differentiation* d_{J_2} is defined by

$$d_{J_2}=[i_{J_2},d]=i_{J_2}d-di_{J_2}.$$

Since taking into considering J_2, the standard Clifford Kähler form is the closed 2-form given by $\Phi_L^{J_2}=-dd_{J_2}L$ such that

$$\begin{aligned}d_{J_2}=&\frac{\partial}{\partial x_{2n+i}}dx_i-\frac{\partial}{\partial x_{4n+i}}dx_{n+i}-\frac{\partial}{\partial x_i}dx_{2n+i}+\frac{\partial}{\partial x_{6n+i}}dx_{3n+i}\\&+\frac{\partial}{\partial x_{n+i}}dx_{4n+i}-\frac{\partial}{\partial x_{7n+i}}dx_{5n+i}-\frac{\partial}{\partial x_{3n+i}}dx_{6n+i}+\frac{\partial}{\partial x_{5n+i}}dx_{7n+i}\end{aligned}\tag{0.14}$$

and

$$d_{J_2}:\mathcal{F}(\mathbf{R}^8)\rightarrow\wedge^1\mathbf{R}^8.$$

The closed standard Clifford Kähler form $\Phi_L^{J_2}$ on $\mathbf{R}^8$ is the symplectic structure. So it holds

$$\begin{array}{c} E_L^{J_2} = X^i \frac{\partial L}{\partial x_{2n+i}} - X^{n+i} \frac{\partial L}{\partial x_{4n+i}} - X^{2n+i} \frac{\partial L}{\partial x_i} + X^{3n+i} \frac{\partial L}{\partial x_{6n+i}} \\ + X^{4n\ +i} \frac{\partial L}{\partial x_{n+i}} - X^{5n+i} \frac{\partial L}{\partial x_{7n+i}} - X^{6n+i} \frac{\partial L}{\partial x_{3n+i}} + X^{7n+i} \frac{\partial L}{\partial x_{5n+i}} - L. \end{array} \tag{0.15}$$

By means of **Eq.** (0.1), using (0.12), (0.14) and (0.15), also taking into consideration the above first case we calculate the equations
(0.16)

$$\begin{array}{c} \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right) + \frac{\partial L}{\partial x_{2n+i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right) - \frac{\partial L}{\partial x_{4n+i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right) - \frac{\partial L}{\partial x_i} = 0, \\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right) + \frac{\partial L}{\partial x_{6n+i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{4n+i}}\right) + \frac{\partial L}{\partial x_{n+i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{5n+i}}\right) - \frac{\partial L}{\partial x_{7n+i}} = 0, \\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{6n+i}}\right) - \frac{\partial L}{\partial x_{3n+i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{7n+i}}\right) + \frac{\partial L}{\partial x_{5n+i}} = 0, \end{array}$$

Hence the equations obtained in **Eq.** (0.16) are called *Euler-Lagrange equations* structured by means of $\Phi_L^{J_2}$ on the standard Clifford Kähler manifold $(\mathbf{R}^8, V)$ and so, the triple $(\mathbf{R}^8, \Phi_L^{J_2}, \xi)$ is said to be a *mechanical system* on the standard Clifford Kähler manifold $(\mathbf{R}^8, V)$.

Thirdly, we introduce Euler-Lagrange equations for quantum and classical mechanics by means of $\Phi_L^{J_3}$ on the standard Clifford Kähler manifold $(\mathbf{R}^8, V)$.

Let J_3 be a local basis on the standard Clifford Kähler manifold $(\mathbf{R}^8, V)$. Let semispray ξ give as in **Eq.**(0.12). Therefore, *Liouville vector field* on the standard Clifford Kähler manifold $(\mathbf{R}^8, V)$ is the vector field given by

$$\begin{array}{c} V_{J_3} = J_3(\xi) = X^i \frac{\partial}{\partial x_{3n+i}} - X^{n+i} \frac{\partial}{\partial x_{5n+i}} - X^{2n+i} \frac{\partial}{\partial x_{6n+i}} - X^{3n+i} \frac{\partial}{\partial x_i} \\ + X^{4n+i} \frac{\partial}{\partial x_{7n+i}} + X^{5n+i} \frac{\partial}{\partial x_{n+i}} + X^{6n+i} \frac{\partial}{\partial x_{2n+i}} - X^{7n+i} \frac{\partial}{\partial x_{4n+i}}. \end{array}$$

The function given by $E_L^{J_3} = V_{J_3}(L) - L$ is *energy function* and calculated by

$$\begin{array}{c} E_L^{J_3} = X^i \frac{\partial L}{\partial x_{3n+i}} - X^{n+i} \frac{\partial L}{\partial x_{5n+i}} - X^{2n+i} \frac{\partial L}{\partial x_{6n+i}} - X^{3n+i} \frac{\partial L}{\partial x_i} \\ + X^{4n+i} \frac{\partial L}{\partial x_{7n+i}} + X^{5n+i} \frac{\partial L}{\partial x_{n+i}} + X^{6n+i} \frac{\partial L}{\partial x_{2n+i}} - X^{7n+i} \frac{\partial L}{\partial x_{4n+i}} - L. \end{array}$$

The function i_{J_3} induced by J_3 and shown by

$$i_{J_3}\omega(X_1, X_2, ..., X_r) = \sum_{i=1}^{r} \omega(X_1, ..., J_3 X_i, ..., X_r),$$

is said to be *vertical derivation,* where $\omega \in \wedge^r \mathbf{R}^8$, $X_i \in \chi(\mathbf{R}^8)$. The *vertical differentiation* d_{J_3} is denoted by

$$d_{J_3} = [i_{J_3}, d] = i_{J_3} d - d i_{J_3}.$$

Considering J_3, the Kähler form is the closed 2-form given by $\Phi_L^{J_3} = -dd_{J_3} L$ such that

$$\begin{array}{c} d_{J_3} = \frac{\partial}{\partial x_{3n+i}} dx_i - \frac{\partial}{\partial x_{5n+i}} dx_{n+i} - \frac{\partial}{\partial x_{6n+i}} dx_{2n+i} - \frac{\partial}{\partial x_i} dx_{3n+i} \\ + \frac{\partial}{\partial x_{7n+i}} dx_{4n+i} + \frac{\partial}{\partial x_{n+i}} dx_{5n+i} + \frac{\partial}{\partial x_{2n+i}} dx_{6n+i} - \frac{\partial}{\partial x_{4n+i}} dx_{7n+i} \end{array}$$

and

$$d_{J_3} : \mathcal{F}(\mathbf{R}^8) \to \wedge^1 \mathbf{R}^8.$$

Using **Eq.** (0.1), similar to the above first and second cases , we find the following equations

(0.17)

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right) + \frac{\partial L}{\partial x_{3n+i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right) - \frac{\partial L}{\partial x_{5n+i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right) - \frac{\partial L}{\partial x_{6n+i}} = 0,$$
$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right) - \frac{\partial L}{\partial x_i} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{4n+i}}\right) + \frac{\partial L}{\partial x_{7n+i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{5n+i}}\right) + \frac{\partial L}{\partial x_{n+i}} = 0,$$
$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{6n+i}}\right) + \frac{\partial L}{\partial x_{2n+i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{7n+i}}\right) - \frac{\partial L}{\partial x_{4n+i}} = 0.$$

Thus the equations given in **Eq.** (0.17) infer *Euler-Lagrange equations* structured by means of $\Phi_L^{J_3}$ on the standard Clifford Kähler manifold $(\mathbf{R}^8, V)$ and therefore the triple $(\mathbf{R}^8, \Phi_L^{J_3}, \xi)$ is named a *mechanical system* on the standard Clifford Kähler manifold $(\mathbf{R}^8, V)$.

0.5. Standard Clifford Hamiltonian Mechanics. Here, we obtain Hamilton equations and Hamiltonian mechanical system for quantum and classical mechanics structured on the standard Clifford Kähler manifold (R^{8n}, V^*).

Firstly, let $(\mathbf{R}^{8n}, V^*)$ be the standard Clifford Kähler manifold. Assume that a component of the almost Clifford structure V^*, a Liouville form and a 1-form on the standard Clifford Kähler manifold $(\mathbf{R}^{8n}, V^*)$ are given by J_1^*, $\lambda_{J_1^*} = J_1^*(\omega)$ and ω, respectively. Then

$$\begin{aligned} \omega = \tfrac{1}{2}(x_i dx_i + x_{n+i} dx_{n+i} + x_{2n+i} dx_{2n+i} + x_{3n+i} dx_{3n+i} \\ + x_{4n+i} dx_{4n+i} + x_{5n+i} dx_{5n+i} + x_{6n+i} dx_{6n+i} + x_{7n+i} dx_{7n+i}), \end{aligned} \tag{0.18}$$

and

$$\begin{aligned} \lambda_{J_1^*} &= \frac{1}{2}(x_i dx_{n+i} - x_{n+i} dx_i + x_{2n+i} dx_{4n+i} + x_{3n+i} dx_{5n+i} \\ &\quad - x_{4n+i} dx_{2n+i} - x_{5n+i} dx_{3n+i} + x_{6n+i} dx_{7n+i} - x_{7n+i} dx_{6n+i}). \end{aligned}$$

It is well-known that if $\Phi_{J_1^*} = -d\lambda_{J_1^*}$ is a closed Kähler form on the standard Clifford Kähler manifold $(\mathbf{R}^{8n}, V^*)$, then $\Phi_{J_1^*}$ is also a symplectic structure on Clifford Kähler manifold $(\mathbf{R}^{8n}, V^*)$.

Consider that Hamilton vector field X associated with Hamiltonian energy $\mathbf{H}$ given by

$$\begin{aligned} X = X^i \frac{\partial}{\partial x_i} + X^{n+i} \frac{\partial}{\partial x_{n+i}} + X^{2n+i} \frac{\partial}{\partial x_{2n+i}} + X^{3n+i} \frac{\partial}{\partial x_{3n+i}} \\ + X^{4n+i} \frac{\partial}{\partial x_{4n+i}} + X^{5n+i} \frac{\partial}{\partial x_{5n+i}} + X^{6n+i} \frac{\partial}{\partial x_{6n+i}} + X^{7n+i} \frac{\partial}{\partial x_{7n+i}}. \end{aligned}$$

Then

$$\Phi_{J_1^*} = dx_{n+i} \wedge dx_i + dx_{4n+i} \wedge dx_{2n+i} + dx_{5n+i} \wedge dx_{3n+i} + dx_{7n+i} \wedge dx_{6n+i},$$

and

$$\begin{aligned} i_X \Phi_{J_1^*} = X^{n+i} dx_i - X^i dx_{n+i} + X^{4n+i} dx_{2n+i} - X^{2n+i} dx_{4n+i} \\ + X^{5n+i} dx_{3n+i} - X^{3n+i} dx_{5n+i} + X^{7n+i} dx_{6n+i} - X^{6n+i} dx_{7n+i}. \end{aligned} \tag{0.19}$$

Moreover, the differential of Hamiltonian energy function is obtained as follows:

$$\begin{aligned} d\mathbf{H} &= \frac{\partial \mathbf{H}}{\partial x_i}dx_i + \frac{\partial \mathbf{H}}{\partial x_{n+i}}dx_{n+i} + \frac{\partial \mathbf{H}}{\partial x_{2n+i}}dx_{2n+i} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}}dx_{3n+i} \\ &+ \frac{\partial \mathbf{H}}{\partial x_{4n+i}}dx_{4n+i} + \frac{\partial \mathbf{H}}{\partial x_{5n+i}}dx_{5n+i} + \frac{\partial \mathbf{H}}{\partial x_{6n+i}}dx_{6n+i} + \frac{\partial \mathbf{H}}{\partial x_{7n+i}}dx_{7n+i}. \end{aligned} \tag{0.20}$$

According to **Eq.**(0.3), if **Eq.** (0.19) and **Eq.** (0.20) are equaled, the Hamiltonian vector field is found as follows:

$$\begin{aligned} X &= -\frac{\partial \mathbf{H}}{\partial x_{n+i}}\frac{\partial}{\partial x_i} + \frac{\partial \mathbf{H}}{\partial x_i}\frac{\partial}{\partial x_{n+i}} - \frac{\partial \mathbf{H}}{\partial x_{4n+i}}\frac{\partial}{\partial x_{2n+i}} - \frac{\partial \mathbf{H}}{\partial x_{5n+i}}\frac{\partial}{\partial x_{3n+i}} \\ &+ \frac{\partial \mathbf{H}}{\partial x_{2n+i}}\frac{\partial}{\partial x_{4n+i}} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}}\frac{\partial}{\partial x_{5n+i}} - \frac{\partial \mathbf{H}}{\partial x_{7n+i}}\frac{\partial}{\partial x_{6n+i}} + \frac{\partial \mathbf{H}}{\partial x_{6n+i}}\frac{\partial}{\partial x_{7n+i}}. \end{aligned} \tag{0.21}$$

Suppose that a curve

$$\theta : \mathbf{R} \to \mathbf{R}^{8n}$$

be an integral curve of the Hamiltonian vector field X, i.e.,

$$X(\theta(t)) = \dot{\theta}, \ t \in \mathbf{R}. \tag{0.22}$$

In the local coordinates, it holds

$$\theta(t) = (x_i, x_{n+i}, x_{2n+i}, x_{3n+i}, x_{4n+i}, x_{5n+i}, x_{6n+i}, x_{7n+i}),$$

and

$$\begin{aligned} \dot{\theta}(t) &= \frac{dx_i}{dt}\frac{\partial}{\partial x_i} + \frac{dx_{n+i}}{dt}\frac{\partial}{\partial x_{n+i}} + \frac{dx_{2n+i}}{dt}\frac{\partial}{\partial x_{2n+i}} + \frac{dx_{3n+i}}{dt}\frac{\partial}{\partial x_{3n+i}} \\ &+ \frac{dx_{4n+i}}{dt}\frac{\partial}{\partial x_{4n+i}} + \frac{dx_{5n+i}}{dt}\frac{\partial}{\partial x_{5n+i}} + \frac{dx_{6n+i}}{dt}\frac{\partial}{\partial x_{6n+i}} + \frac{dx_{7n+i}}{dt}\frac{\partial}{\partial x_{7n+i}}. \end{aligned} \tag{0.23}$$

Considering **Eq.** (0.22), if **Eq.** (0.21) and **Eq.** (0.23) are equaled, it follows

$$\begin{aligned} &\frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}}, \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i}, \frac{dx_{2n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{4n+i}}, \frac{dx_{3n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{5n+i}}, \\ &\frac{dx_{4n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{2n+i}}, \frac{dx_{5n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{3n+i}}, \frac{dx_{6n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{7n+i}}, \frac{dx_{7n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{6n+i}}. \end{aligned} \tag{0.24}$$

Thus, the equations obtained in **Eq.** (0.24) are seen to be *Hamilton equations* with respect to component J_1^* of almost Clifford structure V^* on Clifford Kähler manifold $(\mathbf{R}^{8n}, V^*)$, and then the triple $(\mathbf{R}^{8n}, \Phi_{J_1^*}, X)$ is seen to be a *Hamiltonian mechanical system* on Clifford Kähler manifold $(\mathbf{R}^{8n}, V^*)$.

Secondly, suppose that an element of the almost Clifford structure V^* and a Liouville form on the standard Clifford Kähler manifold $(\mathbf{R}^{8n}, V^*)$ are denoted by J_2^* and $\lambda_{J_2^*} = J_2^*(\omega)$, respectively.

we have

$$\begin{aligned} \lambda_{J_2^*} &= \frac{1}{2}(x_i dx_{2n+i} - x_{n+i}dx_{4n+i} - x_{2n+i}dx_i + x_{3n+i}dx_{6n+i} \\ &+ x_{4n+i}dx_{n+i} - x_{5n+i}dx_{7n+i} - x_{6n+i}dx_{3n+i} + x_{7n+i}dx_{5n+i}). \end{aligned}$$

Considering

$$\Phi_{J_2^*} = dx_{n+i} \wedge dx_{4n+i} + dx_{2n+i} \wedge dx_i + dx_{5n+i} \wedge dx_{7n+i} + dx_{6n+i} \wedge dx_{3n+i},$$

then we calculate

$$\begin{aligned} i_X\Phi_{J_2^*} &= X^{n+i}dx_{4n+i} - X^{4n+i}dx_{n+i} + X^{2n+i}dx_i - X^i dx_{2n+i} \\ &+ X^{5n+i}dx_{7n+i} - X^{7n+i}dx_{5n+i} + X^{6n+i}dx_{3n+i} - X^{3n+i}dx_{6n+i}. \end{aligned} \tag{0.25}$$

According to **Eq.**(0.3), if we equal **Eq.** (0.20) and **Eq.** (0.25), it follows

$$X = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}}\frac{\partial}{\partial x_i} + \frac{\partial \mathbf{H}}{\partial x_{4n+i}}\frac{\partial}{\partial x_{n+i}} + \frac{\partial \mathbf{H}}{\partial x_i}\frac{\partial}{\partial x_{2n+i}} - \frac{\partial \mathbf{H}}{\partial x_{6n+i}}\frac{\partial}{\partial x_{3n+i}} - \frac{\partial \mathbf{H}}{\partial x_{n+i}}\frac{\partial}{\partial x_{4n+i}} + \frac{\partial \mathbf{H}}{\partial x_{7n+i}}\frac{\partial}{\partial x_{5n+i}} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}}\frac{\partial}{\partial x_{6n+i}} - \frac{\partial \mathbf{H}}{\partial x_{5n+i}}\frac{\partial}{\partial x_{7n+i}}. \tag{0.26}$$

Considering **Eq.** (0.22), if **Eq.** (0.23) and **Eq.** (0.26) are equaled, we find equations

$$\frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}}, \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{4n+i}}, \frac{dx_{2n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i}, \frac{dx_{3n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{6n+i}}, \frac{dx_{4n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}}, \frac{dx_{5n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{7n+i}}, \frac{dx_{6n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{3n+i}}, \frac{dx_{7n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{5n+i}}. \tag{0.27}$$

In the end, the equations obtained in **Eq.** (0.27) are known to be *Hamilton equations* with respect to component J_2^* of the standard almost Clifford structure V^* on the standard Clifford Kähler manifold $(\mathbf{R}^{8n}, V^*)$, and then the triple $(\mathbf{R}^{8n}, \Phi_{J_2^*}, X)$ is a *Hamiltonian mechanical system* on the standard Clifford Kähler manifold $(\mathbf{R}^{8n}, V^*)$.

Thirdly, by J_3^* and $\lambda_{J_3^*} = J_3^*(\omega)$, we denote a component of almost Clifford structure $V^*$and a Liouville form on the standard Clifford Kähler manifold $(\mathbf{R}^{8n}, V^*)$, respectively.

Then it holds

$$\begin{aligned}\lambda_{J_3^*} &= \frac{1}{2}(x_i dx_{3n+i} - x_{n+i}dx_{5n+i} - x_{2n+i}dx_{6n+i} - x_{3n+i}dx_i \\ &\quad + x_{4n+i}dx_{7n+i} + x_{5n+i}dx_{n+i} + x_{6n+i}dx_{2n+i} - x_{7n+i}dx_{4n+i}).\end{aligned}$$

As known if $\Phi_{J_3^*} = -d\lambda_{J_3^*}$ is a closed Kähler form on the standard Clifford Kähler manifold $(\mathbf{R}^{8n}, V^*)$, then $\Phi_{J_3^*}$ is also a symplectic structure on Clifford Kähler manifold $(\mathbf{R}^{8n}, V^*)$.

Taking into

$$\Phi_{J_3^*} = dx_{3n+i} \wedge dx_i + dx_{n+i} \wedge dx_{5n+i} + dx_{2n+i} \wedge dx_{6n+i} + dx_{7n+i} \wedge dx_{4n+i},$$

we find

$$i_X \Phi_{J_3^*} = X^{3n+i}dx_i - X^i dx_{3n+i} + X^{n+i}dx_{5n+i} - X^{5n+i}dx_{n+i} + X^{2n+i}dx_{6n+i} - X^{6n+i}dx_{2n+i} + X^{7n+i}dx_{4n+i} - X^{4n+i}dx_{7n+i}. \tag{0.28}$$

According to **Eq.**(0.3), if **Eq.** (0.20) and **Eq.** (0.28) are equaled, we obtain a Hamiltonian vector field given by

$$X = -\frac{\partial \mathbf{H}}{\partial x_{3n+i}}\frac{\partial}{\partial x_i} + \frac{\partial \mathbf{H}}{\partial x_{5n+i}}\frac{\partial}{\partial x_{n+i}} + \frac{\partial \mathbf{H}}{\partial x_{6n+i}}\frac{\partial}{\partial x_{2n+i}} + \frac{\partial \mathbf{H}}{\partial x_i}\frac{\partial}{\partial x_{3n+i}} - \frac{\partial \mathbf{H}}{\partial x_{7n+i}}\frac{\partial}{\partial x_{4n+i}} - \frac{\partial \mathbf{H}}{\partial x_{n+i}}\frac{\partial}{\partial x_{5n+i}} - \frac{\partial \mathbf{H}}{\partial x_{2n+i}}\frac{\partial}{\partial x_{6n+i}} + \frac{\partial \mathbf{H}}{\partial x_{4n+i}}\frac{\partial}{\partial x_{7n+i}}. \tag{0.29}$$

Taking into **Eq.** (0.22), if we equal **Eq.** (0.23) and **Eq.** (0.29), it yields

$$\frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{3n+i}}, \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{5n+i}}, \frac{dx_{2n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{6n+i}}, \frac{dx_{3n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i}, \frac{dx_{4n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{7n+i}}, \frac{dx_{5n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}}, \frac{dx_{6n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}}, \frac{dx_{7n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{4n+i}}. \tag{0.30}$$

Finally, the equations obtained in **Eq.** (0.30) are obtained to be *Hamilton equations* with respect to component J_3^* of the almost Clifford structure

V^* on the standard Clifford Kähler manifold $(\mathbf{R}^{8n}, V^*)$, and then the triple $(\mathbf{R}^{8n}, \Phi_{J_3^*}, X)$ is a *Hamiltonian mechanical system* on the standard Clifford Kähler manifold $(\mathbf{R}^{8n}, V^*)$.

CONCLUSION 3. *From above, Lagrangian mechanics has intrinsically been described taking into account a canonical local basis* $\{J_1, J_2, J_3\}$ *of* V *on the standard Clifford Kähler manifold* $(\mathbf{R}^8, V)$*. The paths of semispray* ξ *on the standard Clifford Kähler manifold are the solutions Euler–Lagrange equations raised in (0.13), (0.16) and (0.17), and also obtained by a canonical local basis* $\{J_1, J_2, J_3\}$ *of vector bundle* V *on the standard Clifford Kähler manifold* $(\mathbf{R}^8, V)$*. One can be proved that these equations are very important to explain the rotational spatial mechanics problems. Formalism of Hamiltonian mechanics has intrinsically been described with taking into account the basis* $\{J_1^*, J_2^*, J_3^*\}$ *of almost Clifford structure* V^* *on the standard Clifford Kähler manifold* $(\mathbf{R}^{8n}, V^*)$*. Hamiltonian models arise to be a very important tool since they present a simple method to describe the model for mechanical systems. In solving problems in classical mechanics, the rotational mechanical system will then be easily usable model. Since physical phenomena, as well-known, do not take place all over the space, a new model for dynamic systems on subspaces is needed. Therefore, equations ((0.24), (0.27) and (0.30) are only considered to be a first step to realize how Clifford geometry has been used in solving problems in different physical area. For further research, the Hamiltonian vector fields derived here are suggested to deal with problems in electrical, magnetical and gravitational fields of quantum and classical mechanics of physics.*

CHAPTER 4

Mechanical Systems on Clifford Kähler Manifolds

In this chapter, Clifford Kähler analogues of Lagrangian and Hamiltonian dynamics in given [**20, 21**] are introduced. Also, the some geometrical and physical results over the obtained Clifford Kähler dynamical systems are discussed.

0.6. Clifford Kähler Manifolds. Now, here we extend and rewrite the main concepts and structures given in [**18, 19**] . Let M be a real smooth manifold of dimension m. Assume that there is a 6-dimensional vector bundle V consisting of $J_i(i = \overline{1,6})$ tensors of type (1,1) over M. Such a local basis $\{J_i\}(i = \overline{1,6})$ is named a canonical local basis of the bundle V in a neighborhood U of M. Then V is said to be an almost Clifford structure in M. The pair (M, V) is called an almost Clifford manifold with V. Thus, an almost Clifford manifold M is of dimension $m = 8n$. If there exists on (M, V) a global basis $\{J_i\}(i = \overline{1,6})$, then (M, V) is said to be an almost Clifford manifold; the basis $\{J_i\}(i = \overline{1,6})$ is called a global basis for V.

An almost Clifford connection on the almost Clifford manifold (M, V) is a linear connection ∇ on M which preserves by parallel transport the vector bundle V. This means that if Φ is a cross-section (local-global) of the bundle V, then $\nabla_X \Phi$ is also a cross-section (local-global, respectively) of V, X being an arbitrary vector field of M.

If for any canonical basis $\{J_i\}(i = \overline{1,6})$ of V in a coordinate neighborhood U, the identities

$$g(J_iX, J_iY) = g(X, Y),\ \forall X, Y \in \chi(M),\ (i = \overline{1,6})$$

hold, the triple (M, g, V) is said to be an almost Clifford Hermitian manifold or metric Clifford manifold denoting by V an almost Clifford structure V and by g a Riemannian metric and by (g, V) an almost Clifford metric structure.

Since each $J_i(i = \overline{1,6})$ is almost Hermitian structure with respect to g, setting

$$\Phi_i(X, Y) = g(J_iX, Y),\ (i = \overline{1,6}),$$

for any vector fields X and Y, we see that Φ_i are 6 local 2-forms.

If the Levi-Civita connection $\nabla = \nabla^g$ on (M, g, V) preserves the vector bundle V by parallel transport, then (M, g, V) is named a Clifford Kähler manifold, and an almost Clifford structure Φ_i of M is said to be a Clifford

Kähler structure. Assume that let

$$\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}, x_{4n+i}, x_{5n+i}, x_{6n+i}, x_{7n+i}\}, i=\overline{1,n}$$

be a real coordinate system on (M,V). Then we determine by

$$\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{n+i}}, \frac{\partial}{\partial x_{2n+i}}, \frac{\partial}{\partial x_{3n+i}}, \frac{\partial}{\partial x_{4n+i}}, \frac{\partial}{\partial x_{5n+i}}, \frac{\partial}{\partial x_{6n+i}}, \frac{\partial}{\partial x_{7n+i}}\right\}$$

and

$$\{dx_i, dx_{n+i}, dx_{2n+i}, dx_{3n+i}, dx_{4n+i}, dx_{5n+i}, dx_{6n+i}, dx_{7n+i}\}$$

the natural bases over $\mathbf{R}$ of the tangent space $T(M)$ and the cotangent space $T^*(M)$ of M, respectively. The definition of structures $\{J_i\}$ and $\{J_i^*\}(i=\overline{1,3})$ is given in **Chapter 3.** The expressions of $\{J_i\}$ $(i=\overline{4,6})$ are as follows:
(0.31)

$$\begin{array}{lll}
J_4(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_{4n+i}} & J_5(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_{5n+i}} & J_6(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_{6n+i}} \\
J_4(\frac{\partial}{\partial x_{n+i}}) = -\frac{\partial}{\partial x_{2n+i}} & J_5(\frac{\partial}{\partial x_{n+i}}) = -\frac{\partial}{\partial x_{3n+i}} & J_6(\frac{\partial}{\partial x_{n+i}}) = -\frac{\partial}{\partial x_{7n+i}} \\
J_4(\frac{\partial}{\partial x_{2n+i}}) = \frac{\partial}{\partial x_{n+i}} & J_5(\frac{\partial}{\partial x_{2n+i}}) = -\frac{\partial}{\partial x_{7n+i}} & J_6(\frac{\partial}{\partial x_{2n+i}}) = -\frac{\partial}{\partial x_{3n+i}} \\
J_4(\frac{\partial}{\partial x_{3n+i}}) = -\frac{\partial}{\partial x_{7n+i}} & J_5(\frac{\partial}{\partial x_{3n+i}}) = \frac{\partial}{\partial x_{n+i}} & J_6(\frac{\partial}{\partial x_{3n+i}}) = \frac{\partial}{\partial x_{2n+i}} \\
J_4(\frac{\partial}{\partial x_{4n+i}}) = -\frac{\partial}{\partial x_i} & J_5(\frac{\partial}{\partial x_{4n+i}}) = \frac{\partial}{\partial x_{6n+i}} & J_6(\frac{\partial}{\partial x_{4n+i}}) = \frac{\partial}{\partial x_{5n+i}} \\
J_4(\frac{\partial}{\partial x_{5n+i}}) = \frac{\partial}{\partial x_{6n+i}} & J_5(\frac{\partial}{\partial x_{5n+i}}) = -\frac{\partial}{\partial x_i} & J_6(\frac{\partial}{\partial x_{5n+i}}) = -\frac{\partial}{\partial x_{4n+i}} \\
J_4(\frac{\partial}{\partial x_{6n+i}}) = -\frac{\partial}{\partial x_{5n+i}} & J_5(\frac{\partial}{\partial x_{6n+i}}) = -\frac{\partial}{\partial x_{4n+i}} & J_6(\frac{\partial}{\partial x_{6n+i}}) = -\frac{\partial}{\partial x_i} \\
J_4(\frac{\partial}{\partial x_{7n+i}}) = \frac{\partial}{\partial x_{3n+i}} & J_5(\frac{\partial}{\partial x_{7n+i}}) = \frac{\partial}{\partial x_{2n+i}} & J_6(\frac{\partial}{\partial x_{7n+i}}) = \frac{\partial}{\partial x_{n+i}}.
\end{array}$$

A canonical local basis$\{J_i^*\}(i=\overline{4,6})$ of V^* of the cotangent space $T^*(M)$ of manifold M satisfies the following condition:
(0.32)

$$\begin{array}{lll}
J_4^*(dx_i) = dx_{4n+i} & J_5^*(dx_i) = dx_{5n+i} & J_6^*(dx_i) = dx_{6n+i} \\
J_4^*(dx_{n+i}) = -dx_{2n+i} & J_5^*(dx_{n+i}) = -dx_{3n+i} & J_6^*(dx_{n+i}) = -dx_{7n+i} \\
J_4^*(dx_{2n+i}) = dx_{n+i} & J_5^*(dx_{2n+i}) = -dx_{7n+i} & J_6^*(dx_{2n+i}) = -dx_{3n+i} \\
J_4^*(dx_{3n+i}) = -dx_{7n+i} & J_5^*(dx_{3n+i}) = dx_{n+i} & J_6^*(dx_{3n+i}) = dx_{2n+i} \\
J_4^*(dx_{4n+i}) = -dx_i & J_5^*(dx_{4n+i}) = dx_{6n+i} & J_6^*(dx_{4n+i}) = dx_{5n+i} \\
J_4^*(dx_{5n+i}) = dx_{6n+i} & J_5^*(dx_{5n+i}) = -dx_i & J_6^*(dx_{5n+i}) = -dx_{4n+i} \\
J_4^*(dx_{6n+i}) = -dx_{5n+i} & J_5^*(dx_{6n+i}) = -dx_{4n+i} & J_6^*(dx_{6n+i}) = -dx_i \\
J_4^*(dx_{7n+i}) = dx_{3n+i} & J_5^*(dx_{7n+i}) = dx_{2n+i} & J_6^*(dx_{7n+i}) = dx_{n+i}.
\end{array}$$

and

$$J_4^{*2} = J_5^{*2} = J_6^{*2} = -I.$$

0.7. Clifford Lagrangian Mechanics. In this section, we introduce Euler-Lagrange equations for quantum and classical mechanics by means of a canonical local basis $\{J_i\}$, $i=\overline{1,6}$ of V on Clifford Kähler manifold (M,V). The Euler-Lagrange equations using basis $\{J_1,J_2,J_3\}$ of V on $(\mathbf{R}^{8n},V)$ are introduced in **Chapter 3**. We see that they are the same as the equations obtained by operators J_1,J_2,J_3 of V on Clifford Kähler manifold (M,V).

Therefore, here, only we derive Euler-Lagrange equations using operators J_4, J_5, J_6 of V on Clifford Kähler manifold (M, V).

Fourth, let J_4 take a local basis component on Clifford Kähler manifold (M, V).Let semispray be the vector field ξ given by (0.12) in **Chapter 3**. The vector field determined by

$$\begin{aligned} V_{J_4} = J_4(\xi) = X^i \tfrac{\partial}{\partial x_{4n+i}} - X^{n+i} \tfrac{\partial}{\partial x_{2n+i}} + X^{2n+i} \tfrac{\partial}{\partial x_{n+i}} - X^{3n+i} \tfrac{\partial}{\partial x_{7n+i}} \\ -X^{4n+i} \tfrac{\partial}{\partial x_i} + X^{5n+i} \tfrac{\partial}{\partial x_{6n+i}} - X^{6n+i} \tfrac{\partial}{\partial x_{5n+i}} + X^{7n+i} \tfrac{\partial}{\partial x_{3n+i}}, \end{aligned}$$

is named *Liouville vector field* on Clifford Kähler manifold (M, V). The maps explained by $T, P : M \to \mathbf{R}$ such that

$$T = \frac{1}{2} m_i (\dot{x}_i{}^2 + \dot{x}^2_{n+i} + \dot{x}^2_{2n+i} + \dot{x}^2_{3n+i} + \dot{x}^2_{4n+i} + \dot{x}^2_{5n+i} + \dot{x}^2_{6n+i} + \dot{x}^2_{7n+i}),\ P = m_i g h$$

are said to be *the kinetic energy* and *the potential energy of the system,* respectively. Here m_i, g and h stand for mass of a mechanical system having m particles, the gravity acceleration and the distance to the origin of a mechanical system on Clifford Kähler manifold (M, V), respectively. Then $L : M \to R$ is a map that satisfies the conditions; i) $L = T - P$ is a *Lagrangian function, ii)* the function given by $E_L^{J_4} = V_{J_4}(L) - L$, is *energy function.*

The operator i_{J_4} induced by J_4 and defined by

$$i_{J_4}\omega(X_1, X_2, ..., X_r) = \sum_{i=1}^{r} \omega(X_1, ..., J_4 X_i, ..., X_r),$$

is called *vertical derivation,* where $\omega \in \wedge^r M$, $X_i \in \chi(M)$. The *vertical differentiation* d_{J_4} is determined by

$$d_{J_4} = [i_{J_4}, d] = i_{J_4} d - d i_{J_4}.$$

We saw that the Clifford Kähler form is the closed 2-form given by $\Phi_L^{J_4} = -dd_{J_4} L$ such that

$$\begin{aligned} d_{J_4} \quad = \quad & \frac{\partial}{\partial x_{4n+i}} dx_i - \frac{\partial}{\partial x_{2n+i}} dx_{n+i} + \frac{\partial}{\partial x_{n+i}} dx_{2n+i} - \frac{\partial}{\partial x_{7n+i}} dx_{3n+i} \\ & - \frac{\partial}{\partial x_i} dx_{4n+i} + \frac{\partial}{\partial x_{6n+i}} dx_{5n+i} - \frac{\partial}{\partial x_{5n+i}} dx_{6n+i} + \frac{\partial}{\partial x_{3n+i}} dx_{7n+i} \end{aligned}$$

determined by operator

$$d_{J_4} : \mathcal{F}(M) \to \wedge^1 M.$$

Then

$$\begin{aligned} \Phi_L^{J_4} = & -\tfrac{\partial^2 L}{\partial x_j \partial x_{4n+i}} dx_j \wedge dx_i + \tfrac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j \wedge dx_{n+i} - \tfrac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j \wedge dx_{2n+i} \\ & + \tfrac{\partial^2 L}{\partial x_j \partial x_{7n+i}} dx_j \wedge dx_{3n+i} + \tfrac{\partial^2 L}{\partial x_j \partial x_i} dx_j \wedge dx_{4n+i} - \tfrac{\partial^2 L}{\partial x_j \partial x_{6n+i}} dx_j \wedge dx_{5n+i} \\ & + \tfrac{\partial^2 L}{\partial x_j \partial x_{5n+i}} dx_j \wedge dx_{6n+i} - \tfrac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j \wedge dx_{7n+i} - \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{4n+i}} dx_{n+j} \wedge dx_i \\ & + \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \wedge dx_{n+i} - \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \wedge dx_{2n+i} \\ & + \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{7n+i}} dx_{n+j} \wedge dx_{3n+i} + \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} \wedge dx_{4n+i} \end{aligned}$$

$-\frac{\partial^2 L}{\partial x_{n+j}\partial x_{6n+i}}dx_{n+j}\wedge dx_{5n+i}+\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}dx_{n+j}\wedge dx_{6n+i}$
$-\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}dx_{n+j}\wedge dx_{7n+i}-\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}}dx_{2n+j}\wedge dx_{i}$
$+\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}dx_{2n+j}\wedge dx_{n+i}-\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}dx_{2n+j}\wedge dx_{2n+i}$
$+\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}dx_{2n+j}\wedge dx_{3n+i}+\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{i}}dx_{2n+j}\wedge dx_{4n+i}$
$-\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}dx_{2n+j}\wedge dx_{5n+i}+\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}dx_{2n+j}\wedge dx_{6n+i}$
$-\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}dx_{2n+j}\wedge dx_{7n+i}-\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}dx_{3n+j}\wedge dx_{i}$
$+\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}dx_{3n+j}\wedge dx_{n+i}-\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}dx_{3n+j}\wedge dx_{2n+i}$
$+\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}dx_{3n+j}\wedge dx_{3n+i}+\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{i}}dx_{3n+j}\wedge dx_{4n+i}$
$-\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}dx_{3n+j}\wedge dx_{5n+i}+\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}dx_{3n+j}\wedge dx_{6n+i}$
$-\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}dx_{3n+j}\wedge dx_{7n+i}-\frac{\partial L}{\partial x_{4n+j}\partial x_{4n+i}}dx_{4n+j}\wedge dx_{i}$
$+\frac{\partial L}{\partial x_{4n+j}\partial x_{2n+i}}dx_{4n+j}\wedge dx_{n+i}-\frac{\partial L}{\partial x_{4n+j}\partial x_{n+i}}dx_{4n+j}\wedge dx_{2n+i}$
$+\frac{\partial L}{\partial x_{4n+j}\partial x_{7n+i}}dx_{4n+j}\wedge dx_{3n+i}+\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{i}}dx_{4n+j}\wedge dx_{4n+i}$
$-\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}dx_{4n+j}\wedge dx_{5n+i}+\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}}dx_{4n+j}\wedge dx_{6n+i}$
$-\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}dx_{4n+j}\wedge dx_{7n+i}-\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}dx_{5n+j}\wedge dx_{i}$
$+\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}dx_{5n+j}\wedge dx_{n+i}-\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}dx_{5n+j}\wedge dx_{2n+i}$
$+\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}dx_{5n+j}\wedge dx_{3n+i}+\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{i}}dx_{5n+j}\wedge dx_{4n+i}$
$-\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}dx_{5n+j}\wedge dx_{5n+i}+\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}dx_{5n+j}\wedge dx_{6n+i}$
$-\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}dx_{5n+j}\wedge dx_{7n+i}-\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}dx_{6n+j}\wedge dx_{i}$
$+\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}dx_{6n+j}\wedge dx_{n+i}-\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}dx_{6n+j}\wedge dx_{2n+i}$
$+\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{6n+j}\wedge dx_{3n+i}+\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{i}}dx_{6n+j}\wedge dx_{4n+i}$
$-\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}dx_{6n+j}\wedge dx_{5n+i}+\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}dx_{6n+j}\wedge dx_{6n+i}$
$-\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}dx_{6n+j}\wedge dx_{7n+i}-\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}dx_{7n+j}\wedge dx_{i}$
$+\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}dx_{7n+j}\wedge dx_{n+i}-\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}dx_{7n+j}\wedge dx_{2n+i}$
$+\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}dx_{7n+j}\wedge dx_{3n+i}+\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{i}}dx_{7n+j}\wedge dx_{4n+i}$
$-\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}dx_{7n+j}\wedge dx_{5n+i}+\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}dx_{7n+j}\wedge dx_{6n+i}$
$-\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}dx_{7n+j}\wedge dx_{7n+i}.$

Also, we have energy function as follows:
$E_L^{J_4}=X^i\frac{\partial L}{\partial x_{4n+i}}-X^{n+i}\frac{\partial L}{\partial x_{2n+i}}+X^{2n+i}\frac{\partial L}{\partial x_{n+i}}-X^{3n+i}\frac{\partial L}{\partial x_{7n+i}}$
$-X^{4n+i}\frac{\partial L}{\partial x_{i}}+X^{5n+i}\frac{\partial L}{\partial x_{6n+i}}-X^{6n+i}\frac{\partial L}{\partial x_{5n+i}}+X^{7n+i}\frac{\partial L}{\partial x_{3n+i}}-L$

By means of **Eq.** (0.1), we calculate the following expressions
$-X^i\frac{\partial^2 L}{\partial x_j\partial x_{4n+i}}\delta_i^j dx_i+X^i\frac{\partial^2 L}{\partial x_j\partial x_{2n+i}}\delta_i^j dx_{n+i}-X^i\frac{\partial^2 L}{\partial x_j\partial x_{n+i}}\delta_i^j dx_{2n+i}$
$+X^i\frac{\partial^2 L}{\partial x_j\partial x_{7n+i}}\delta_i^j dx_{3n+i}+X^i\frac{\partial^2 L}{\partial x_j\partial x_{i}}\delta_i^j dx_{4n+i}-X^i\frac{\partial^2 L}{\partial x_j\partial x_{6n+i}}\delta_i^j dx_{5n+i}$

$$
\begin{aligned}
&+X^i\frac{\partial^2 L}{\partial x_j\partial x_{5n+i}}\delta_i^j dx_{6n+i}-X^i\frac{\partial^2 L}{\partial x_j\partial x_{3n+i}}\delta_i^j dx_{7n+i}-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{4n+i}}\delta_{n+i}^{n+j}dx_i\\
&+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{2n+i}}\delta_{n+i}^{n+j}dx_{n+i}-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{n+i}}\delta_{n+i}^{n+j}dx_{2n+i}\\
&+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}\delta_{n+i}^{n+j}dx_{3n+i}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_i}\delta_{n+i}^{n+j}dx_{4n+i}\\
&-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{6n+i}}\delta_{n+i}^{n+j}dx_{5n+i}+X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{5n+i}}\delta_{n+i}^{n+j}dx_{6n+i}\\
&-X^{n+i}\frac{\partial^2 L}{\partial x_{n+j}\partial x_{7n+i}}\delta_{n+i}^{n+j}dx_{7n+i}-\ X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{4n+i}}\delta_{2n+i}^{2n+j}dx_i\\
&+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{2n+i}}\delta_{2n+i}^{2n+j}dx_{n+i}-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{n+i}}\delta_{2n+i}^{2n+j}dx_{2n+i}\\
&+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}\delta_{2n+i}^{2n+j}dx_{3n+i}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_i}\delta_{2n+i}^{2n+j}dx_{4n+i}\\
&-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{6n+i}}\delta_{2n+i}^{2n+j}dx_{5n+i}+X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{5n+i}}\delta_{2n+i}^{2n+j}dx_{6n+i}\\
&-X^{2n+i}\frac{\partial^2 L}{\partial x_{2n+j}\partial x_{7n+i}}\delta_{2n+i}^{2n+j}dx_{7n+i}-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{4n+i}}\delta_{3n+i}^{3n+j}dx_i\\
&+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{2n+i}}\delta_{3n+i}^{3n+j}dx_{n+i}-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{n+i}}\delta_{3n+i}^{3n+j}dx_{2n+i}\\
&+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}\delta_{3n+i}^{3n+j}dx_{3n+i}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_i}\delta_{3n+i}^{3n+j}dx_{4n+i}\\
&-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{6n+i}}\delta_{3n+i}^{3n+j}dx_{5n+i}+X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{5n+i}}\delta_{3n+i}^{3n+j}dx_{6n+i}\\
&-X^{3n+i}\frac{\partial^2 L}{\partial x_{3n+j}\partial x_{7n+i}}\delta_{3n+i}^{3n+j}dx_{7n+i}-X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{4n+i}}\delta_{4n+i}^{4n+j}dx_i\\
&+X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{2n+i}}\delta_{4n+i}^{4n+j}dx_{n+i}-X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{n+i}}\delta_{4n+i}^{4n+j}dx_{2n+i}\\
&+X^{4n+i}\frac{\partial L}{\partial x_{4n+j}\partial x_{7n+i}}\delta_{4n+i}^{4n+j}dx_{3n+i}+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_i}\delta_{4n+i}^{4n+j}dx_{4n+i}\\
&-X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{6n+i}}\delta_{4n+i}^{4n+j}dx_{5n+i}+X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{5n+i}}\delta_{4n+i}^{4n+j}dx_{6n+i}\\
&-X^{4n+i}\frac{\partial^2 L}{\partial x_{4n+j}\partial x_{3n+i}}\delta_{4n+i}^{4n+j}dx_{7n+i}-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{4n+i}}\delta_{5n+i}^{5n+j}dx_i\\
&+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{2n+i}}\delta_{5n+i}^{5n+j}dx_{n+i}-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{n+i}}\delta_{5n+i}^{5n+j}dx_{2n+i}\\
&+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{7n+i}}\delta_{5n+i}^{5n+j}dx_{3n+i}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_i}\delta_{5n+i}^{5n+j}dx_{4n+i}\\
&-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{6n+i}}\delta_{5n+i}^{5n+j}dx_{5n+i}+X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{5n+i}}\delta_{5n+i}^{5n+j}dx_{6n+i}\\
&-X^{5n+i}\frac{\partial^2 L}{\partial x_{5n+j}\partial x_{3n+i}}\delta_{5n+i}^{5n+j}dx_{7n+i}-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{4n+i}}\delta_{6n+i}^{6n+j}dx_i\\
&+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{2n+i}}\delta_{6n+i}^{6n+j}dx_{n+i}-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{n+i}}\delta_{6n+i}^{6n+j}dx_{2n+i}\\
&+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}\delta_{6n+i}^{6n+j}dx_{3n+i}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_i}\delta_{6n+i}^{6n+j}dx_{4n+i}\\
&-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{6n+i}}\delta_{6n+i}^{6n+j}dx_{5n+i}+X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{5n+i}}\delta_{6n+i}^{6n+j}dx_{6n+i}\\
&-X^{6n+i}\frac{\partial^2 L}{\partial x_{6n+j}\partial x_{7n+i}}\delta_{6n+i}^{6n+j}dx_{7n+i}-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{4n+i}}\delta_{7n+i}^{7n+j}dx_i\\
&+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{2n+i}}\delta_{7n+i}^{7n+j}dx_{n+i}-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{n+i}}\delta_{7n+i}^{7n+j}dx_{2n+i}\\
&+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{7n+i}}\delta_{7n+i}^{7n+j}dx_{3n+i}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_i}\delta_{7n+i}^{7n+j}dx_{4n+i}\\
&-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{6n+i}}\delta_{7n+i}^{7n+j}dx_{5n+i}+X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{5n+i}}\delta_{7n+i}^{7n+j}dx_{6n+i}\\
&-X^{7n+i}\frac{\partial^2 L}{\partial x_{7n+j}\partial x_{3n+i}}\delta_{7n+i}^{7n+j}dx_{7n+i}+\frac{\partial L}{\partial x_j}dx_j+\frac{\partial L}{\partial x_{n+j}}dx_{n+j}\\
&+\frac{\partial L}{\partial x_{2n+j}}dx_{2n+j}+\frac{\partial L}{\partial x_{3n+j}}dx_{3n+j}+\frac{\partial L}{\partial x_{4n+j}}dx_{4n+j}+\frac{\partial L}{\partial x_{5n+j}}dx_{5n+j}\\
&+\frac{\partial L}{\partial x_{6n+j}}dx_{6n+j}+\frac{\partial L}{\partial x_{7n+j}}dx_{7n+j}=0.
\end{aligned}
$$

By a curve α on M being an integral curve of ξ, we found equations as follows:

(0.33)
$$\begin{array}{c}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{4n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right)-\frac{\partial L}{\partial x_{2n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right)+\frac{\partial L}{\partial x_{n+i}}=0,\\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right)-\frac{\partial L}{\partial x_{7n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{4n+i}}\right)-\frac{\partial L}{\partial x_i}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{5n+i}}\right)+\frac{\partial L}{\partial x_{6n+i}}=0,\\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{6n+i}}\right)-\frac{\partial L}{\partial x_{5n+i}}=0,\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{7n+i}}\right)+\frac{\partial L}{\partial x_{3n+i}}=0,\end{array}$$

such that the equations expressed in **Eq.** (0.33) are named *Euler-Lagrange equations* structured on Clifford Kähler manifold (M,V) by means of $\Phi_L^{J_4}$ and in the case, the triple $(M,\Phi_L^{J_4},\xi)$ is said to be a *mechanical system* on Clifford Kähler manifold (M,V).

Fifth, we obtain Euler-Lagrange equations for quantum and classical mechanics by means of $\Phi_L^{J_5}$ on Clifford Kähler manifold (M,V).

Let J_5 be another local basis component on the Clifford Kähler manifold (M,V). Let ξ take as in **Eq.** (0.12) given in **Chapter 3**. In the case, the vector field defined by

$$\begin{array}{c}V_{J_5}=J_5(\xi)=X^i\frac{\partial}{\partial x_{5n+i}}-X^{n+i}\frac{\partial}{\partial x_{3n+i}}-X^{2n+i}\frac{\partial}{\partial x_{7n+i}}+X^{3n+i}\frac{\partial}{\partial x_{n+i}}\\ +X^{4n+i}\frac{\partial}{\partial x_{6n+i}}-X^{5n+i}\frac{\partial}{\partial x_i}-X^{6n+i}\frac{\partial}{\partial x_{4n+i}}+X^{7n+i}\frac{\partial}{\partial x_{2n+i}},\end{array}$$

is *Liouville vector field* on Clifford Kähler manifold (M,V). The function given by $E_L^{J_5}=V_{J_5}(L)-L$ is *energy function.* Then the operator i_{J_5} induced by J_5 and defined by

$$i_{J_5}\omega(X_1,X_2,...,X_r)=\sum_{i=1}^{r}\omega(X_1,...,J_5X_i,...,X_r)$$

is *vertical derivation,* where $\omega\in\wedge^rM$, $X_i\in\chi(M)$. The *vertical differentiation* d_{J_5} is determined by

$$d_{J_5}=[i_{J_5},d]=i_{J_5}d-di_{J_5}.$$

Taking into consideration J_5, the Clifford Kähler form is the closed 2-form given by $\Phi_L^{J_5}=-dd_{J_5}L$ such that

$$\begin{array}{c}d_{J_5}=\frac{\partial}{\partial x_{5n+i}}dx_i-\frac{\partial}{\partial x_{3n+i}}dx_{n+i}-\frac{\partial}{\partial x_{7n+i}}dx_{2n+i}+\frac{\partial}{\partial x_{n+i}}dx_{3n+i}\\ +\frac{\partial}{\partial x_{6n+i}}dx_{4n+i}-\frac{\partial}{\partial x_i}dx_{5n+i}-\frac{\partial}{\partial x_{4n+i}}dx_{6n+i}+\frac{\partial}{\partial x_{2n+i}}dx_{7n+i},\end{array}\tag{0.34}$$

by means of the operator

$$d_{J_5}:\mathcal{F}(M)\to\wedge^1M.$$

The closed Clifford Kähler form $\Phi_L^{J_5}$ on M is the symplectic structure. So it yields

$$\begin{array}{c}E_L^{J_5}=X^i\frac{\partial L}{\partial x_{5n+i}}-X^{n+i}\frac{\partial L}{\partial x_{3n+i}}-X^{2n+i}\frac{\partial L}{\partial x_{7n+i}}+X^{3n+i}\frac{\partial L}{\partial x_{n+i}}\\ +X^{4n+i}\frac{\partial L}{\partial x_{6n+i}}-X^{5n+i}\frac{\partial L}{\partial x_i}-X^{6n+i}\frac{\partial L}{\partial x_{4n+i}}+X^{7n+i}\frac{\partial L}{\partial x_{2n+i}}-L.\end{array}\tag{0.35}$$

Using **Eq.** (0.1), using (0.12), (0.34) and (0.35), also similar to the above fourth case we obtain the equations
(0.36)

$$\begin{array}{c}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{5n+i}}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right)-\frac{\partial L}{\partial x_{3n+i}}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right)-\frac{\partial L}{\partial x_{7n+i}}=0,\\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right)+\frac{\partial L}{\partial x_{n+i}}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{4n+i}}\right)+\frac{\partial L}{\partial x_{6n+i}}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{5n+i}}\right)-\frac{\partial L}{\partial x_i}=0,\\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{6n+i}}\right)-\frac{\partial L}{\partial x_{4n+i}}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{7n+i}}\right)+\frac{\partial L}{\partial x_{2n+i}}=0.\end{array}$$

Thus the equations found in **Eq.** (0.36) are named *Euler-Lagrange equations* structured by means of $\Phi_L^{J_5}$ on Clifford Kähler manifold (M,V) and so, the triple $(M,\Phi_L^{J_5},\xi)$ is called a *mechanical system* on Clifford Kähler manifold (M,V).

Sixth, we present Euler-Lagrange equations for quantum and classical mechanics by means of $\Phi_L^{J_6}$ on Clifford Kähler manifold (M,V).

Let J_6 be a local basis on Clifford Kähler manifold (M,V). Let semispray ξ give as in **Eq.**(0.12). So, *Liouville vector field* on Clifford Kähler manifold (M,V) is the vector field defined by

$$\begin{array}{c}V_{J_6}=J_6(\xi)=X^i\frac{\partial}{\partial x_{6n+i}}-X^{n+i}\frac{\partial}{\partial x_{7n+i}}-X^{2n+i}\frac{\partial}{\partial x_{3n+i}}+X^{3n+i}\frac{\partial}{\partial x_{2n+i}}\\ +X^{4n+i}\frac{\partial}{\partial x_{5n+i}}-X^{5n+i}\frac{\partial}{\partial x_{4n+i}}-X^{6n+i}\frac{\partial}{\partial x_i}+X^{7n+i}\frac{\partial}{\partial x_{n+i}}.\end{array}$$

The function given by $E_L^{J_6}=V_{J_6}(L)-L$ is *energy function* and found by

$$\begin{array}{c}E_L^{J_6}=X^i\frac{\partial L}{\partial x_{6n+i}}-X^{n+i}\frac{\partial L}{\partial x_{7n+i}}-X^{2n+i}\frac{\partial L}{\partial x_{3n+i}}+X^{3n+i}\frac{\partial L}{\partial x_{2n+i}}\\ +X^{4n+i}\frac{\partial L}{\partial x_{5n+i}}-X^{5n+i}\frac{\partial L}{\partial x_{4n+i}}-X^{6n+i}\frac{\partial L}{\partial x_i}+X^{7n+i}\frac{\partial L}{\partial x_{n+i}}-L.\end{array}$$

The function i_{J_6} induced by J_6 and given by

$$i_{J_6}\omega(X_1,X_2,...,X_r)=\sum_{i=1}^{r}\omega(X_1,...,J_6X_i,...,X_r),$$

is said to be *vertical derivation,* where $\omega\in\wedge^rM,\ X_i\in\chi(M)$. The *vertical differentiation* d_{J_6} is determined by

$$d_{J_6}=[i_{J_6},d]=i_{J_6}d-di_{J_6},$$

We say the Kähler form is the closed 2-form given by $\Phi_L^{J_6}=-dd_{J_6}L$ such that

$$d_{J_6}:\mathcal{F}(M)\to\wedge^1M,$$

$$\begin{aligned}d_{J_6}\ =\ &\frac{\partial}{\partial x_{6n+i}}dx_i-\frac{\partial}{\partial x_{7n+i}}dx_{n+i}-\frac{\partial}{\partial x_{3n+i}}dx_{2n+i}+\frac{\partial}{\partial x_{2n+i}}dx_{3n+i}\\ &+\frac{\partial}{\partial x_{5n+i}}dx_{4n+i}-\frac{\partial}{\partial x_{4n+i}}dx_{5n+i}-\frac{\partial}{\partial x_i}dx_{6n+i}+\frac{\partial}{\partial x_{n+i}}dx_{7n+i}.\end{aligned}$$

Considering **Eq.** (0.1), similar to the above cases, we calculate the following equations

(0.37)

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{6n+i}}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right)-\frac{\partial L}{\partial x_{7n+i}}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right)-\frac{\partial L}{\partial x_{3n+i}}=0,$$
$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right)+\frac{\partial L}{\partial x_{2n+i}}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{4n+i}}\right)+\frac{\partial L}{\partial x_{5n+i}}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{5n+i}}\right)-\frac{\partial L}{\partial x_{4n+i}}=0,$$
$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{6n+i}}\right)-\frac{\partial L}{\partial x_i}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{7n+i}}\right)+\frac{\partial L}{\partial x_{n+i}}=0.$$

Thus the equations obtained in **Eq.** (0.37) infer *Euler-Lagrange equations* structured by means of $\Phi_L^{J_6}$ on Clifford Kähler manifold (M,V) and so, the triple $(M,\Phi_L^{J_6},\xi)$ is called a *mechanical system* on Clifford Kähler manifold (M,V).

0.8. Clifford Hamilton Mechanics. Here, we obtain Hamilton equations and Hamiltonian mechanical system for quantum and classical mechanics by means of a canonical local basis $\{J_i^*\}(i=\overline{1,6})$ of V^* on Clifford Kähler manifold (M,V^*). The Hamilton equations using basis $\{J_1^*,J_2^*,J_3^*\}$ of V on $(\mathbf{R}^{8n},V^*)$ are introduced in **Chapter 3**. We see that they are the same as the equations obtained by operators J_1^*,J_2^*,J_3^* of V^* on Clifford Kähler manifold (M,V^*).

Therefore, here, only we derive Hamilton equations using operators J_4^*,J_5^*,J_6^* of V^* on Clifford Kähler manifold (M,V^*).

Fourth, let (M,V^*) be a Clifford Kähler manifold. Suppose that a component of the almost Clifford structure $V^*$and a Liouville form and a 1-form on Clifford Kähler manifold (M,V^*) are given by J_4^* and $\lambda_{J_4^*}$, respectively.

Let ω be as given by **Eq.** (0.18) in **Chapter 3**.

$$\begin{aligned}\omega &= \frac{1}{2}(x_idx_i+x_{n+i}dx_{n+i}+x_{2n+i}dx_{2n+i}+x_{3n+i}dx_{3n+i}\\ &\quad +x_{4n+i}dx_{4n+i}+x_{5n+i}dx_{5n+i}+x_{6n+i}dx_{6n+i}+x_{7n+i}dx_{7n+i}),\end{aligned}$$

we have

$$\begin{aligned}\lambda_{J_4^*} &= J_4^*(\omega)=\frac{1}{2}(x_idx_{4n+i}-x_{n+i}dx_{2n+i}+x_{2n+i}dx_{n+i}-x_{3n+i}dx_{7n+i}\\ &\quad -x_{4n+i}dx_i+x_{5n+i}dx_{6n+i}-x_{6n+i}dx_{5n+i}+x_{7n+i}dx_{3n+i}).\end{aligned}$$

It is known that if $\Phi_{J_4^*}$ is a closed Kähler form on Clifford Kähler manifold (M,V^*), then $\Phi_{J_4^*}$ is also a symplectic structure on Clifford Kähler manifold (M,V^*).

Take into consideration that Hamilton vector field X associated with Hamilton energy $\mathbf{H}$ is given by

$$X=X^i\frac{\partial}{\partial x_i}+X^{n+i}\frac{\partial}{\partial x_{n+i}}+X^{2n+i}\frac{\partial}{\partial x_{2n+i}}+X^{3n+i}\frac{\partial}{\partial x_{3n+i}}$$
$$+X^{4n+i}\frac{\partial}{\partial x_{4n+i}}+X^{5n+i}\frac{\partial}{\partial x_{5n+i}}+X^{6n+i}\frac{\partial}{\partial x_{6n+i}}+X^{7n+i}\frac{\partial}{\partial x_{7n+i}}.$$

Then

$\Phi_{J_4^*} = -d\lambda_{J_4^*} = dx_{n+i} \wedge dx_{2n+i} + dx_{3n+i} \wedge dx_{7n+i} + dx_{4n+i} \wedge dx_i + dx_{6n+i} \wedge dx_{5n+i}$

and

(0.38)

$$\begin{aligned} i_X \Phi_{J_4^*} = \Phi_{J_4^*}(X) = X^{n+i} dx_{2n+i} - X^{2n+i} dx_{n+i} + X^{3n+i} dx_{7n+i} - X^{7n+i} dx_{3n+i} \\ + X^{4n+i} dx_i - X^i dx_{4n+i} + X^{6n+i} dx_{5n+i} - X^{5n+i} dx_{6n+i}. \end{aligned}$$

Furthermore, the differential of Hamilton energy is obtained as follows:

$$\begin{aligned} d\mathbf{H} = \frac{\partial \mathbf{H}}{\partial x_i} dx_i + \frac{\partial \mathbf{H}}{\partial x_{n+i}} dx_{n+i} + \frac{\partial \mathbf{H}}{\partial x_{2n+i}} dx_{2n+i} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}} dx_{3n+i} \\ + \frac{\partial \mathbf{H}}{\partial x_{4n+i}} dx_{4n+i} + \frac{\partial \mathbf{H}}{\partial x_{5n+i}} dx_{5n+i} + \frac{\partial \mathbf{H}}{\partial x_{6n+i}} dx_{6n+i} + \frac{\partial \mathbf{H}}{\partial x_{7n+i}} dx_{7n+i}. \end{aligned} \tag{0.39}$$

According to **Eq.**(0.3), if equaled **Eq.** (0.38) and **Eq.** (0.39), the Hamilton vector field is calculated as follows:

$$\begin{aligned} X = -\frac{\partial \mathbf{H}}{\partial x_{4n+i}} \frac{\partial}{\partial x_i} + \frac{\partial \mathbf{H}}{\partial x_{2n+i}} \frac{\partial}{\partial x_{n+i}} - \frac{\partial \mathbf{H}}{\partial x_{n+i}} \frac{\partial}{\partial x_{2n+i}} + \frac{\partial \mathbf{H}}{\partial x_{7n+i}} \frac{\partial}{\partial x_{3n+i}} \\ + \frac{\partial \mathbf{H}}{\partial x_i} \frac{\partial}{\partial x_{4n+i}} - \frac{\partial \mathbf{H}}{\partial x_{6n+i}} \frac{\partial}{\partial x_{5n+i}} + \frac{\partial \mathbf{H}}{\partial x_{5n+i}} \frac{\partial}{\partial x_{6n+i}} - \frac{\partial \mathbf{H}}{\partial x_{3n+i}} \frac{\partial}{\partial x_{7n+i}}. \end{aligned} \tag{0.40}$$

Assume that a curve

$$\alpha : \mathbf{R} \to M$$

be an integral curve of the Hamilton vector field X, i.e.,

$$X(\alpha(t)) = \dot{\alpha}, \ t \in \mathbf{R}. \tag{0.41}$$

In the local coordinates, it is found that

$$\alpha(t) = (x_i, x_{n+i}, x_{2n+i}, x_{3n+i}, x_{4n+i}, x_{5n+i}, x_{6n+i}, x_{7n+i})$$

and

$$\begin{aligned} \dot{\alpha}(t) = \frac{dx_i}{dt} \frac{\partial}{\partial x_i} + \frac{dx_{n+i}}{dt} \frac{\partial}{\partial x_{n+i}} + \frac{dx_{2n+i}}{dt} \frac{\partial}{\partial x_{2n+i}} + \frac{dx_{3n+i}}{dt} \frac{\partial}{\partial x_{3n+i}} \\ + \frac{dx_{4n+i}}{dt} \frac{\partial}{\partial x_{4n+i}} + \frac{dx_{5n+i}}{dt} \frac{\partial}{\partial x_{5n+i}} + \frac{dx_{6n+i}}{dt} \frac{\partial}{\partial x_{6n+i}} + \frac{dx_{7n+i}}{dt} \frac{\partial}{\partial x_{7n+i}}. \end{aligned} \tag{0.42}$$

Thinking out **Eq.** (0.41), if equaled **Eq.** (0.40) and **Eq.** (0.42), it follows

(0.43)

$$\begin{aligned} \frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{4n+i}}, \ \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{2n+i}}, \ \frac{dx_{2n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}}, \ \frac{dx_{3n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{7n+i}}, \\ \frac{dx_{4n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i}, \ \frac{dx_{5n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{6n+i}}, \ \frac{dx_{6n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{5n+i}}, \ \frac{dx_{7n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{3n+i}}. \end{aligned}$$

Hence, the equations obtained in **Eq.** (0.43) are shown to be *Hamilton equations* with respect to component J_4^* of almost Clifford structure V^* on Clifford Kähler manifold (M, V^*), and then the triple $(M, , \Phi_{J_4^*}, X)$ is said to be a *Hamiltonian mechanical system* on Clifford Kähler manifold (M, V^*).

Fifth, let (M, V^*) be a Clifford Kähler manifold. Assume that an element of the almost Clifford structure $V^*$and a Liouville form on Clifford Kähler manifold (M, V^*) are determined by J_5^* and $\lambda_{J_5^*}(= J_5^*(\omega))$, respectively.

we have

$$\begin{aligned} \lambda_{J_5^*} &= \frac{1}{2}(x_i dx_{5n+i} - x_{n+i} dx_{3n+i} - x_{2n+i} dx_{7n+i} + x_{3n+i} dx_{n+i} \\ &\quad + x_{4n+i} dx_{6n+i} - x_{5n+i} dx_i - x_{6n+i} dx_{4n+i} + x_{7n+i} dx_{2n+i}). \end{aligned}$$

Take into consideration

$$\Phi_{J_5^*} = -d\lambda_{J_5^*} = dx_{n+i} \wedge dx_{3n+i} + dx_{2n+i} \wedge dx_{7n+i} + dx_{5n+i} \wedge dx_i + dx_{6n+i} \wedge dx_{4n+i},$$

then we find

(0.44)
$$\begin{aligned} i_X \Phi_{J_5^*} = \Phi_{J_5^*}(X) = & X^{n+i} dx_{3n+i} - X^{3n+i} dx_{n+i} + X^{2n+i} dx_{7n+i} - X^{7n+i} dx_{2n+i} \\ & + X^{5n+i} dx_i - X^i dx_{5n+i} + X^{6n+i} dx_{4n+i} - X^{4n+i} dx_{6n+i}. \end{aligned}$$

According to **Eq.**(0.3), if we equal **Eq.** (0.39) and **Eq.** (0.44), it follows

$$\begin{aligned} X = & -\frac{\partial \mathbf{H}}{\partial x_{5n+i}} \frac{\partial}{\partial x_i} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}} \frac{\partial}{\partial x_{n+i}} + \frac{\partial \mathbf{H}}{\partial x_{7n+i}} \frac{\partial}{\partial x_{2n+i}} - \frac{\partial \mathbf{H}}{\partial x_{n+i}} \frac{\partial}{\partial x_{3n+i}} \\ & - \frac{\partial \mathbf{H}}{\partial x_{6n+i}} \frac{\partial}{\partial x_{4n+i}} + \frac{\partial \mathbf{H}}{\partial x_i} \frac{\partial}{\partial x_{5n+i}} + \frac{\partial \mathbf{H}}{\partial x_{4n+i}} \frac{\partial}{\partial x_{6n+i}} - \frac{\partial \mathbf{H}}{\partial x_{2n+i}} \frac{\partial}{\partial x_{7n+i}}. \end{aligned} \tag{0.45}$$

Taking **Eq.** (0.41), if **Eqs.** (0.42) and (0.45) are equaled, we obtain equations

(0.46)
$$\begin{aligned} &\frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{5n+i}}, \ \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{3n+i}}, \ \frac{dx_{2n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{7n+i}}, \ \frac{dx_{3n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}}, \\ &\frac{dx_{4n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{6n+i}}, \ \frac{dx_{5n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i}, \ \frac{dx_{6n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{4n+i}}, \ \frac{dx_{7n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}}. \end{aligned}$$

In the end, the equations found in **Eq.** (0.46) are seen to be *Hamilton equations* with respect to component J_5^* of the almost Clifford structure V^* on Clifford Kähler manifold (M, V^*), and then the triple $(M, \Phi_{J_5^*}, X)$ is named a *Hamiltonian mechanical system* on Clifford Kähler manifold (M, V^*).

Sixth, let (M, V^*) be a Clifford Kähler manifold. By J_6^*, $\lambda_{J_6^*}$, we denote a component of almost Clifford structure V^*, a Liouville form on Clifford Kähler manifold (M, V^*), respectively.

Then it yields

$$\begin{aligned} \lambda_{J_6^*} \quad = \quad & J_6^*(\omega) = \frac{1}{2}(x_i dx_{6n+i} - x_{n+i} dx_{7n+i} - x_{2n+i} dx_{3n+i} + x_{3n+i} dx_{2n+i} \\ & + x_{4n+i} dx_{5n+i} - x_{5n+i} dx_{4n+i} - x_{6n+i} dx_i + x_{7n+i} dx_{n+i}). \end{aligned}$$

It is known that if $\Phi_{J_6^*}$ is a closed Kähler form on Clifford Kähler manifold (M, V^*), then $\Phi_{J_6^*}$ is also a symplectic structure on Clifford Kähler manifold (M, V^*).

Considering

$$\Phi_{J_6^*} = -d\lambda_{J_6^*} = dx_{n+i} \wedge dx_{7n+i} + dx_{2n+i} \wedge dx_{3n+i} + dx_{5n+i} \wedge dx_{4n+i} + dx_{6n+i} \wedge dx_i,$$

we calculate

(0.47)
$$\begin{aligned} i_X \Phi_{J_6^*} = \Phi_{J_6^*}(X) = & X^{n+i} dx_{7n+i} - X^{7n+i} dx_{n+i} + X^{2n+i} dx_{3n+i} - X^{3n+i} dx_{2n+i} \\ & + X^{5n+i} dx_{4n+i} - X^{4n+i} dx_{5n+i} + X^{6n+i} dx_i - X^i dx_{6n+i}. \end{aligned}$$

According to **Eq.**(0.3), if **Eqs.** (0.39) and (0.47) are equaled, Hamilton vector field is found as follows:

$$\begin{aligned} X = & -\frac{\partial \mathbf{H}}{\partial x_{6n+i}} \frac{\partial}{\partial x_i} + \frac{\partial \mathbf{H}}{\partial x_{7n+i}} \frac{\partial}{\partial x_{n+i}} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}} \frac{\partial}{\partial x_{2n+i}} - \frac{\partial \mathbf{H}}{\partial x_{2n+i}} \frac{\partial}{\partial x_{3n+i}} \\ & - \frac{\partial \mathbf{H}}{\partial x_{5n+i}} \frac{\partial}{\partial x_{4n+i}} + \frac{\partial \mathbf{H}}{\partial x_{4n+i}} \frac{\partial}{\partial x_{5n+i}} + \frac{\partial \mathbf{H}}{\partial x_i} \frac{\partial}{\partial x_{6n+i}} - \frac{\partial \mathbf{H}}{\partial x_{n+i}} \frac{\partial}{\partial x_{7n+i}}. \end{aligned} \tag{0.48}$$

Considering **Eq.** (0.41), we equal **Eq.** (0.42) and **Eq.** (0.48), it holds

$$
(0.49)\quad \begin{array}{l}
\frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{6n+i}},\ \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{7n+i}},\ \frac{dx_{2n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{3n+i}},\ \frac{dx_{3n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}},\\
\frac{dx_{4n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{5n+i}},\ \frac{dx_{5n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{4n+i}},\ \frac{dx_{6n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i},\ \frac{dx_{7n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}}.
\end{array}
$$

Finally, the equations calculated in **Eq.** (0.49) are called to be *Hamilton equations* with respect to component J_6^* of the almost Clifford structure V^* on Clifford Kähler manifold (M, V^*), and then the triple $(M, \Phi_{J_6^*}, X)$ is said to be a *Hamiltonian mechanical system* on Clifford Kähler manifold (M, V^*).

CONCLUSION 4. *From above, Lagrangian formalisms has intrinsically been described taking into account a canonical local basis* $\{J_i\}$, $i = \overline{1,6}$ *of* V *on Clifford Kähler manifold* (M, V). *The paths of semispray* ξ *on Clifford Kähler manifold are the solutions Euler–Lagrange equations raised in (0.33), (0.36) and (0.37), and also obtained by a canonical local basis* $\{J_i\}$, $i = \overline{1,6}$ *of vector bundle* V *on Clifford Kähler manifold* (M, V). *One may be shown that these equations are very important to explain the rotational spatial mechanics problems. Hamilton Formalisms has intrinsically been described with taking into account the basis* $\{J_i^*\}$, $i = \overline{1,6}$ *of almost Clifford structure* V^* *on Clifford Kähler manifold* (M, V^*). *Hamilton models arise to be a very important tool since they present a simple method to describe the model for dynamical systems. In solving problems in classical mechanics, the rotational mechanical system will then be easily usable model. Since a new model for dynamic systems on subspaces and spaces is needed, equations (0.43), (0.46) and (0.49) are only considered to be a first step to realize how Clifford geometry has been used in understanding, modeling and solving problems in different physical fields.*

CHAPTER 5

Mechanical Systems on Quaternion Kähler Manifolds

This chapter presents the further steps of the previously done studies taking into consideration analogues of Lagrangian and Hamiltonian mechanics in given [**22, 23**]. Presently, considering quaternion Kähler manifolds, we introduce quaternion Kähler analogue of Lagrangian mechanics. And then a quaternion Kähler version of Hamilton equations is obtained. Finally, the some results related to quaternion Kähler Lagrangian and Hamiltonian dynamical systems are also given.

0.9. Quaternion Kähler Manifolds. Here, we recall some definitions given in [**18**]. Let M be an n-dimensional manifold. It has a 3-dimensional vector bundle V consisting of tensors of type (1,1). The manifold M satisfies the condition given by:

(a) In any coordinate neighborhood U of M, there exists a local basis $\{F, G, H\}$ of V such that

$$F^2 = G^2 = H^2 = FGH = -I.$$

I denotes the identity tensor of type (1,1) in M. Such a local basis $\{F, G, H\}$ is called a *canonical local basis* of the bundle V in U. Then V is said to be an *almost quaternion structure* in M, and M with V is an *almost quaternion manifold* denoted by (M, V). An almost quaternion manifold M is of dimension $n = 4m$ $(m \geq 1)$. In any almost quaternion manifold (M, V), there is a Riemannian metric tensor field g such that

$$g(\phi X, Y) + g(X, \phi Y) = 0$$

for any cross-section ϕ on M and any vector fields X, Y of M. An almost quaternion structure V fixed with a Riemannian metric g is called an *almost quaternion metric structure*. A manifold M endowed with an almost quaternion metric structure $\{g, V\}$ is said to be an *almost quaternion metric manifold* denoted by (M, g, V). Let $\{F, G, H\}$ be a canonical local basis of V an almost quaternion manifold (M, g, V). Since each of F, G and H is almost Hermitian with respect to g, setting

$$\Phi(X, Y) = g(FX, Y), \quad \Psi(X, Y) = g(GX, Y), \ \Theta(X, Y) = g(HX, Y)$$

for any vector fields X and Y, we see that Φ, Ψ and Θ are local 2-forms.

Assume that the Riemannian connection ∇ of (M, g, V) satisfies the conditions as follows:

(b) If ϕ is a cross-section (local or global) of the bundle V, then $V_X\phi$ is also a cross-section of V, where X is an arbitrary vector field in M. From (0.5) we see that the condition (b) is equivalent to the following condition:

(b') If F, G, H is a canonical local basis of V, then

$$\begin{aligned} \nabla_X F &= r(X)G - q(X)H, \quad \nabla_X G = -r(X)F + p(X)H, \\ \nabla_X H &= q(X)F - p(X)G \end{aligned}$$

for any vector field X, where p, q and r are certain local 1-forms. If an almost quaternion metric manifold M satisfies the condition (b) or (b'), then M is said to be a *quaternion Kähler manifold* and an almost quaternion structure of M is called a *quaternion Kähler structure*.

Let $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}$, $i = \overline{1,n}$ be a real coordinate system on a neighborhood U of M. Note that $\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{n+i}}, \frac{\partial}{\partial x_{2n+i}}, \frac{\partial}{\partial x_{3n+i}}\right\}$ and $\{dx_i, dx_{n+i}, dx_{2n+i}, dx_{3n+i}\}$ are natural bases over $\mathbf{R}$ of the tangent space $T(M)$ and the cotangent space $T^*(M)$ of M, respectively. The standard almost quaternion structure on $\mathbf{R}^n$ is given in [**19**]. Inspiring of [**19**], we can determine the existence of a local coordinate system connected with integrability of the almost quaternion structure as follows.

$$\begin{gathered} F(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_{n+i}}, \ F(\frac{\partial}{\partial x_{n+i}}) = -\frac{\partial}{\partial x_i}, \\ F(\frac{\partial}{\partial x_{2n+i}}) = \frac{\partial}{\partial x_{3n+i}}, \ F(\frac{\partial}{\partial x_{3n+i}}) = -\frac{\partial}{\partial x_{2n+i}}, \\ G(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_{2n+i}}, \ G(\frac{\partial}{\partial x_{n+i}}) = -\frac{\partial}{\partial x_{3n+i}}, \\ G(\frac{\partial}{\partial x_{2n+i}}) = -\frac{\partial}{\partial x_i}, \ G(\frac{\partial}{\partial x_{3n+i}}) = \frac{\partial}{\partial x_{n+i}}, \\ H(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_{3n+i}}, \ H(\frac{\partial}{\partial x_{n+i}}) = \frac{\partial}{\partial x_{2n+i}}, \\ H(\frac{\partial}{\partial x_{2n+i}}) = -\frac{\partial}{\partial x_{n+i}}, \ H(\frac{\partial}{\partial x_{3n+i}}) = -\frac{\partial}{\partial x_i}. \end{gathered} \tag{0.50}$$

A canonical local basis$\{F^*, G^*, H^*\}$ of V^* of the cotangent space $T^*(M)$ of manifold M satisfies the condition as follows:

$$F^{*2} = G^{*2} = H^{*2} = F^*G^*H^* = -I,$$

defining by

$$\begin{gathered} F^*(dx_i) = dx_{n+i}, \ F^*(dx_{n+i}) = -dx_i, \\ F^*(dx_{2n+i}) = dx_{3n+i}, \ F^*(dx_{3n+i}) = -dx_{2n+i}, \\ G^*(dx_i) = dx_{2n+i}, \ G^*(dx_{n+i}) = -dx_{3n+i}, \\ G^*(dx_{2n+i}) = -dx_i, \ G^*(dx_{3n+i}) = dx_{n+i}, \\ H^*(dx_i) = dx_{3n+i}, \ H^*(dx_{n+i}) = dx_{2n+i}, \\ H^*(dx_{2n+i}) = -dx_{n+i}, \ H^*(dx_{3n+i}) = -dx_i. \end{gathered} \tag{0.51}$$

So, we say to be a *quaternion manifold* M denoted by (M, V^*).

0.10. Quaternion Lagrangian Mechanics. In this subsection, we obtain Euler-Lagrange equations for quantum and classical mechanics by means of a canonical local basis $\{F, G, H\}$ of V on quaternion Kähler manifold (M, V).

Firstly, let F take a local basis component on the quaternion Kähler manifold (M, V), and $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}$ be its coordinate functions. Let semispray be the vector field ξ determined by

$$\xi = X^i \frac{\partial}{\partial x_i} + X^{n+i} \frac{\partial}{\partial x_{n+i}} + X^{2n+i} \frac{\partial}{\partial x_{2n+i}} + X^{3n+i} \frac{\partial}{\partial x_{3n+i}}, \tag{0.52}$$

where $X^i = \dot{x}_i, X^{n+i} = \dot{x}_{n+i}, X^{2n+i} = \dot{x}_{2n+i}, X^{3n+i} = \dot{x}_{3n+i}$and the dot indicates the derivative with respect to time t. The vector field defined by

$$V_F = F(\xi) = X^i \frac{\partial}{\partial x_{n+i}} - X^{n+i} \frac{\partial}{\partial x_i} + X^{2n+i} \frac{\partial}{\partial x_{3n+i}} - X^{3n+i} \frac{\partial}{\partial x_{2n+i}}$$

is called *Liouville vector field* on the quaternion Kähler manifold (M, V). The maps given by $T, P : M \to R$ such that $T = \frac{1}{2} m_i (\dot{x}_i{}^2 + \dot{x}_{n+i}^2 + x_{2n+i}^2 + \dot{x}_{3n+i}^2), P = m_i g h$ are called *the kinetic energy* and *the potential energy of the system,* respectively. Here m_i, g and h stand for mass of a mechanical system having m particles, the gravity acceleration and distance to the origin of a mechanical system on the quaternion Kähler manifold (M, V), respectively. Then $L : M \to R$ is a map that satisfies the conditions; i) $L = T - P$ is a *Lagrangian function, ii)* the function given by $E_L^F = V_F(L) - L$, is *energy function.*

The operator i_F induced by F and given by

$$i_F \omega(X_1, X_2, ..., X_r) = \sum_{i=1}^{r} \omega(X_1, ..., F X_i, ..., X_r),$$

is said to be *vertical derivation,* where $\omega \in \wedge^r M,\ X_i \in \chi(M)$. The *vertical differentiation* d_F is defined by

$$d_F = [i_F, d] = i_F d - d i_F,$$

where d is the usual exterior derivation. For F , the closed Kähler form is the closed 2-form given by $\Phi_L^F = -d d_{_F} L$ such that

$$d_{_F} = \frac{\partial}{\partial x_{n+i}} dx_i - \frac{\partial}{\partial x_i} dx_{n+i} + \frac{\partial}{\partial x_{3n+i}} dx_{2n+i} - \frac{\partial}{\partial x_{2n+i}} d_{3n+i} : \mathcal{F}(M) \to \wedge^1 M.$$

Then

$$\begin{aligned}\Phi_L^F = &-\tfrac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j \wedge dx_i + \tfrac{\partial^2 L}{\partial x_j \partial x_i} dx_j \wedge dx_{n+i} \\ &-\tfrac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j \wedge dx_{2n+i} + \tfrac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j \wedge dx_{3n+i} \\ &-\tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \wedge dx_i + \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} \wedge dx_{n+i} \\ &-\tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} \wedge dx_{2n+i} + \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \wedge dx_{3n+i} \\ &-\tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} \wedge dx_i + \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} \wedge dx_{n+i} \\ &-\tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} \wedge dx_{2n+i} + \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} \wedge dx_{3n+i} \\ &-\tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \wedge dx_i + \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \wedge dx_{n+i} \\ &-\tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} \wedge dx_{2n+i} + \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \wedge dx_{3n+i}.\end{aligned}$$

Also, we have

$$E_L^F = X^i \tfrac{\partial L}{\partial x_{n+i}} - X^{n+i} \tfrac{\partial L}{\partial x_i} + X^{2n+i} \tfrac{\partial L}{\partial x_{3n+i}} - X^{3n+i} \tfrac{\partial L}{\partial x_{2n+i}} - L.$$

With the use of **Eq.** (0.1), the following expressions can be obtained:

$$\begin{aligned}&-X^i \tfrac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j + X^i \tfrac{\partial^2 L}{\partial x_j \partial x_i} dx_{n+j} - X^i \tfrac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_{2n+j} + X^i \tfrac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_{3n+j} \\ &-X^{n+i} \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_j + X^{n+i} \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} - X^{n+i} \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{2n+j} \\ &+X^{n+i} \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{3n+j} - X^{2n+i} \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_j + X^{2n+i} \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{n+j} \\ &-X^{2n+i} \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} + X^{2n+i} \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{3n+j} - X^{3n+i} \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_j \\ &+X^{3n+i} \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{n+j} - X^{3n+i} \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{2n+j} + X^{3n+i} \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \\ &+\tfrac{\partial L}{\partial x_j} dx_j + \tfrac{\partial L}{\partial x_{n+j}} dx_{n+j} + \tfrac{\partial L}{\partial x_{2n+j}} dx_{2n+j} + \tfrac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0\end{aligned}$$

If a curve denoted by α on M being an integral curve of ξ, then we calculate the following equations:

$$\begin{aligned}&\tfrac{\partial}{\partial t}\left(\tfrac{\partial L}{\partial x_i}\right) + \tfrac{\partial L}{\partial x_{n+i}} = 0, \ \tfrac{\partial}{\partial t}\left(\tfrac{\partial L}{\partial x_{n+i}}\right) - \tfrac{\partial L}{\partial x_i} = 0, \\ &\tfrac{\partial}{\partial t}\left(\tfrac{\partial L}{\partial x_{2n+i}}\right) + \tfrac{\partial L}{\partial x_{3n+i}} = 0, \ \tfrac{\partial}{\partial t}\left(\tfrac{\partial L}{\partial x_{3n+i}}\right) - \tfrac{\partial L}{\partial x_{2n+i}} = 0,\end{aligned} \tag{0.53}$$

such that the equations obtained in **Eq.** (0.53) are said to be *Euler-Lagrange equations* structured on quaternion Kähler manifold (M, V) by means of Φ_L^F and thus the triple (M, Φ_L^F, ξ) is said to be a *mechanical system* on quaternion Kähler manifold (M, V).

Secondly, we find Euler-Lagrange equations for quantum and classical mechanics by means of Φ_L^G on quaternion Kähler manifold (M, V).

Consider G be another local basis component on the quaternion Kähler manifold (M, V). Let ξ take as in **Eq.** (0.52). In the case, the vector field given by

$$V_G = G(\xi) = X^i \frac{\partial}{\partial x_{2n+i}} - X^{n+i} \frac{\partial}{\partial x_{3n+i}} - X^{2n+i} \frac{\partial}{\partial x_i} + X^{3n+i} \frac{\partial}{\partial x_{n+i}}$$

is *Liouville vector field* on the quaternion Kähler manifold (M, V). The function given by $E_L^G = V_G(L) - L$ is *energy function*. Then the operator i_G

induced by G and denoted by

$$i_G\omega(X_1, X_2, ..., X_r) = \sum_{i=1}^{r} \omega(X_1, ..., GX_i, ..., X_r)$$

is *vertical derivation,* where $\omega \in \wedge^r M$, $X_i \in \chi(M)$. The *vertical differentiation* d_G is defined by

$$d_G = [i_G, d] = i_G d - d i_G.$$

Since taking into considering G, the closed Kähler form is the closed 2-form given by $\Phi_L^G = -dd_G L$ such that

$$d_G = \frac{\partial}{\partial x_{2n+i}} dx_i - \frac{\partial}{\partial x_{3n+i}} dx_{n+i} - \frac{\partial}{\partial x_i} dx_{2n+i} + \frac{\partial}{\partial x_{n+i}} d_{3n+i} : \mathcal{F}(M) \to \wedge^1 M.$$

Then we have

$$\begin{aligned}
\Phi_L^G = &-\tfrac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j \wedge dx_i + \tfrac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j \wedge dx_{n+i} + \tfrac{\partial^2 L}{\partial x_j \partial x_i} dx_j \wedge dx_{2n+i} \\
&-\tfrac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j \wedge dx_{3n+i} - \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \wedge dx_i + \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} \wedge dx_{n+i} \\
&+\tfrac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} \wedge dx_{2n+i} - \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \wedge dx_{3n+i} - \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} \wedge dx_i \\
&+\tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} \wedge dx_{n+i} + \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} \wedge dx_{2n+i} - \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} \wedge dx_{3n+i} \\
&-\tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \wedge dx_i + \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} \wedge dx_{n+i} + \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \wedge dx_{2n+i} \\
&-\tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \wedge dx_{3n+i}.
\end{aligned}$$

Also, we obtain

$$E_L^G = X^i \frac{\partial L}{\partial x_{2n+i}} - X^{n+i} \frac{\partial L}{\partial x_{3n+i}} - X^{2n+i} \frac{\partial L}{\partial x_i} + X^{3n+i} \frac{\partial L}{\partial x_{n+i}} - L.$$

By means of **Eq.** (0.1), we calculate

$$\begin{aligned}
&-X^i \tfrac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j + X^i \tfrac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_{n+j} + X^i \tfrac{\partial^2 L}{\partial x_j \partial x_i} dx_{2n+j} - X^i \tfrac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{3n+j} \\
&-X^{n+i} \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_j + X^{n+i} \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} + X^{n+i} \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{2n+j} \\
&-X^{n+i} \tfrac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{3n+j} - X^{2n+i} \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_j + X^{2n+i} \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{n+j} \\
&+X^{2n+i} \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} - X^{2n+i} \tfrac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{3n+j} - X^{3n+i} \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_j \\
&+X^{3n+i} \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{n+j} + X^{3n+i} \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{2n+j} - X^{3n+i} \tfrac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \\
&+\tfrac{\partial L}{\partial x_j} dx_j + \tfrac{\partial L}{\partial x_{n+j}} dx_{n+j} + \tfrac{\partial L}{\partial x_{2n+j}} dx_{2n+j} + \tfrac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0.
\end{aligned}$$

By α being an integral curve of ξ, then we obtain the equations:

$$\begin{aligned}
&\tfrac{\partial}{\partial t}\left(\tfrac{\partial L}{\partial x_i}\right) + \tfrac{\partial L}{\partial x_{2n+i}} = 0, \tfrac{\partial}{\partial t}\left(\tfrac{\partial L}{\partial x_{n+i}}\right) - \tfrac{\partial L}{\partial x_{3n+i}} = 0, \\
&\tfrac{\partial}{\partial t}\left(\tfrac{\partial L}{\partial x_{2n+i}}\right) - \tfrac{\partial L}{\partial x_i} = 0, \ \ \tfrac{\partial}{\partial t}\left(\tfrac{\partial L}{\partial x_{3n+i}}\right) + \tfrac{\partial L}{\partial x_{n+i}} = 0.
\end{aligned} \tag{0.54}$$

Thus the equations obtained in **Eq.** (0.54) are called *Euler-Lagrange equations* structured by means of Φ_L^G on quaternion Kähler manifold (M, V) and thus the triple (M, Φ_L^G, ξ) can be called to be a *mechanical system* on quaternion Kähler manifold (M, V).

Thirdly, we introduce Euler-Lagrange equations for quantum and classical mechanics by means of Φ_L^H on quaternion Kähler manifold (M, V).

Let H be a local basis on the quaternion Kähler manifold (M, V).Consider ξ. It is the semispray given in **Eq.**(0.52). Therefore, *Liouville vector field* on the quaternion Kähler manifold (M, V) is the vector field given by

$$V_H = H(\xi) = X^i \frac{\partial}{\partial x_{3n+i}} + X^{n+i} \frac{\partial}{\partial x_{2n+i}} - X^{2n+i} \frac{\partial}{\partial x_{n+i}} - X^{3n+i} \frac{\partial}{\partial x_i}.$$

The function given by $E_L^H = V_H(L) - L$ is *energy function.* The function i_H induced by H and shown by

$$i_H \omega(X_1, X_2, ..., X_r) = \sum_{i=1}^{r} \omega(X_1, ..., HX_i, ..., X_r),$$

is said to be *vertical derivation,* where $\omega \in \wedge^r M$, $X_i \in \chi(M)$. The *vertical differentiation* d_H is denoted by

$$d_H = [i_H, d] = i_H d - d i_H.$$

Then the closed Kähler form is the closed 2-form given by $\Phi_L^H = -dd_{_H} L$ such that

$$d_{_H} = \frac{\partial}{\partial x_{3n+i}} dx_i + \frac{\partial}{\partial x_{2n+i}} dx_{n+i} - \frac{\partial}{\partial x_{n+i}} dx_{2n+i} - \frac{\partial}{\partial x_i} d_{3n+i} : \mathcal{F}(M) \to \wedge^1 M$$

Then we get

$$\begin{array}{c} \Phi_L^H = -\frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j \wedge dx_i - \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j \wedge dx_{2n+i} \\ + \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} \wedge dx_i - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \wedge dx_{n+i} \\ + \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} \wedge dx_{3n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} \wedge dx_i \\ - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} \wedge dx_{3n+i} \\ - \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} \wedge dx_i - \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \wedge dx_{n+i} + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \wedge dx_{2n+i} \\ + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \wedge dx_{3n+i}. \end{array}$$

Also, we find

$$E_L^H = X^i \frac{\partial L}{\partial x_{3n+i}} + X^{n+i} \frac{\partial L}{\partial x_{2n+i}} - X^{2n+i} \frac{\partial L}{\partial x_{n+i}} - X^{3n+i} \frac{\partial L}{\partial x_i} - L.$$

Using **Eq.** (0.1), we calculate the following expression:

$$\begin{array}{c} -X^i \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j - X^i \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_{n+j} + X^i \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_{2n+j} + X^i \frac{\partial^2 L}{\partial x_j \partial x_i} dx_{3n+j} \\ -X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_j - X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} + X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{2n+j} \\ +X^{n+i} \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{3n+j} - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_j - X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{n+j} \\ +X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} + X^{2n+i} \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{3n+j} - X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_j \\ -X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{2n+j} + X^{3n+i} \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \\ + \frac{\partial L}{\partial x_j} dx_j + \frac{\partial L}{\partial x_{n+j}} dx_{n+j} + \frac{\partial L}{\partial x_{2n+j}} dx_{2n+j} + \frac{\partial L}{\partial x_{3n+j}} dx_{3n+j} = 0. \end{array}$$

By means of an integral curve α of ξ, then we find the equations:

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right)+\frac{\partial L}{\partial x_{3n+i}}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right)+\frac{\partial L}{\partial x_{2n+i}}=0,\\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right)-\frac{\partial L}{\partial x_{n+i}}=0,\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right)-\frac{\partial L}{\partial x_i}=0.\end{aligned} \tag{0.55}$$

Thus the equations given in **Eq.** (0.55) infer *Euler-Lagrange equations* structured by means of Φ_L^H on quaternion Kähler manifold (M, V) and thus the triple (M, Φ_L^H, ξ) is said to be a *mechanical system* on quaternion Kähler manifold (M, V).

0.11. Quaternion Hamiltonian Mechanics. Here, we introduce quaternion Kähler analogue of Hamilton equations given in (0.4).

Firstly, let (M, V^*) be a quaternion Kähler manifold. Assume that a component of almost quaternion structure V^*, a Liouville form and a 1-form on (M, V^*) are shown by F^*, $\lambda_{F^*} = F^*(\omega)$ and ω, respectively.

One puts

$$\omega = \frac{1}{2}(x_i dx_i + x_{n+i} dx_{n+i} + x_{2n+i} dx_{2n+i} + x_{3n+i} dx_{3n+i}). \tag{0.56}$$

Then we have

$$\lambda_{F^*} = \frac{1}{2}(x_i dx_{n+i} - x_{n+i} dx_i + x_{2n+i} dx_{3n+i} - x_{3n+i} dx_{2n+i}).$$

It is concluded that if $\Phi_{F^*} = -d\lambda_{F^*}$ is a closed Kähler form on the quaternion Kähler manifold (M, V^*), then Φ_{F^*} is also a symplectic structure on (M, V^*).

Consider that Hamiltonian vector fields $X_{F^*}, X_{G^*}, X_{H^*}$ associated with Hamiltonian energy $\mathbf{H}$ are given by

$$\begin{aligned}X_{F^*} &= X^i\frac{\partial}{\partial x_i}+X^{n+i}\frac{\partial}{\partial x_{n+i}}+X^{2n+i}\frac{\partial}{\partial x_{2n+i}}+X^{3n+i}\frac{\partial}{\partial x_{3n+i}},\\ X_{G^*} &= Y^i\frac{\partial}{\partial x_i}+Y^{n+i}\frac{\partial}{\partial x_{n+i}}+Y^{2n+i}\frac{\partial}{\partial x_{2n+i}}+Y^{3n+i}\frac{\partial}{\partial x_{3n+i}},\\ X_{H^*} &= Z^i\frac{\partial}{\partial x_i}+Z^{n+i}\frac{\partial}{\partial x_{n+i}}+X^{Z2n+i}\frac{\partial}{\partial x_{2n+i}}+Z^{3n+i}\frac{\partial}{\partial x_{3n+i}}.\end{aligned} \tag{0.57}$$

Then we have

$$\Phi_{F^*} = dx_{n+i}\wedge dx_i + dx_{3n+i}\wedge dx_{2n+i}$$

and

$$i_{X_{F^*}}\Phi_{F^*} = X^{n+i}dx_i - X^i dx_{n+i} + X^{3n+i}dx_{2n+i} - X^{2n+i}dx_{3n+i}. \tag{0.58}$$

Moreover, the differential of Hamiltonian energy is obtained as follows:

$$d\mathbf{H} = \frac{\partial \mathbf{H}}{\partial x_i}dx_i + \frac{\partial \mathbf{H}}{\partial x_{n+i}}dx_{n+i} + \frac{\partial \mathbf{H}}{\partial x_{2n+i}}dx_{2n+i} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}}dx_{3n+i}. \tag{0.59}$$

According to **Eq.** (0.3), by **Eq.** (0.58) and **Eq.** (0.59) the Hamiltonian vector field is found as follows:

$$X_{F^*} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}}\frac{\partial}{\partial x_i} + \frac{\partial \mathbf{H}}{\partial x_i}\frac{\partial}{\partial x_{n+i}} - \frac{\partial \mathbf{H}}{\partial x_{3n+i}}\frac{\partial}{\partial x_{2n+i}} + \frac{\partial \mathbf{H}}{\partial x_{2n+i}}\frac{\partial}{\partial x_{3n+i}}. \tag{0.60}$$

Suppose that a curve

$$\alpha : I \subset \mathbf{R} \to M$$

be an integral curve of the Hamiltonian vector fields $X_{F^*}, X_{G^*}, X_{H^*}$, i.e.,

$$X_{F^*}(\alpha(t)) = \dot{\alpha},\ X_{G^*}(\alpha(t)) = \dot{\alpha},\ X_{H^*}(\alpha(t)) = \dot{\alpha}, t \in I. \tag{0.61}$$

In the local coordinates, it is obtained that

$$\alpha(t) = (x_i, x_{n+i}, x_{2n+i}, x_{3n+i})$$

and

$$\dot{\alpha}(t) = \frac{dx_i}{dt}\frac{\partial}{\partial x_i} + \frac{dx_{n+i}}{dt}\frac{\partial}{\partial x_{n+i}} + \frac{dx_{2n+i}}{dt}\frac{\partial}{\partial x_{2n+i}} + \frac{dx_{3n+i}}{dt}\frac{\partial}{\partial x_{3n+i}}. \tag{0.62}$$

By **Eq.** (0.60), **Eq.** (0.61), **Eq.** (0.62) we have

$$\frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}},\ \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i},\ \frac{dx_{2n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{3n+i}},\ \frac{dx_{3n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{2n+i}}. \tag{0.63}$$

Thus, the equations obtained in **Eq.** (0.63) are seen to be *Hamilton equations* with respect to component F^* of the almost quaternion structure V^* on the quaternion Kähler manifold (M, V^*), and then the triple (M, Φ_{F^*}, X) is seen to be a *Hamiltonian mechanical system* on (M, V^*).

Secondly, suppose that an element of almost quaternion structure $V^*$and a Liouville form on (M, V^*) are denoted by $G^*$and $\lambda_{G^*} = G^*(\omega)$ respectively.

By (0.51) and (0.56) we calculate

$$\lambda_{G^*} = \frac{1}{2}(x_i dx_{2n+i} - x_{n+i}dx_{3n+i} - x_{2n+i}dx_i + x_{3n+i}dx_{n+i}).$$

It is known if $\Phi_{G^*} = -d\lambda_{G^*}$ is a closed Kähler form on the quaternion Kähler manifold (M, V^*), then Φ_{G^*} is also a symplectic structure on (M, V^*).

Considering

$$\Phi_{G^*} = dx_{2n+i} \wedge dx_i + dx_{n+i} \wedge dx_{3n+i},$$

then we calculate

$$i_{X_{G^*}}\Phi_{G^*} = Y^{2n+i}dx_i - Y^i dx_{2n+i} + Y^{n+i}dx_{3n+i} - Y^{3n+i}dx_{n+i}. \tag{0.64}$$

Taking account of **Eq.**(0.3), if we equal **Eq.** (0.59) and **Eq.** (0.64), it follows

$$X_{G^*} = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}}\frac{\partial}{\partial x_i} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}}\frac{\partial}{\partial x_{n+i}} + \frac{\partial \mathbf{H}}{\partial x_i}\frac{\partial}{\partial x_{2n+i}} - \frac{\partial \mathbf{H}}{\partial x_{n+i}}\frac{\partial}{\partial x_{3n+i}}. \tag{0.65}$$

Considering **Eq.** (0.61), if **Eq.** (0.62) and **Eq.** (0.65) are equaled, we find equations

$$\frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}},\ \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{3n+i}},\ \frac{dx_{2n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i},\ \frac{dx_{3n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}}. \tag{0.66}$$

In the end, the equations obtained in **Eq.** (0.66) are known to be *Hamilton equations* with respect to component G^* of the almost quaternion structure V^* on the quaternion Kähler manifold (M, V^*), and then the triple (M, Φ_{G^*}, X) is a *Hamiltonian mechanical system* on (M, V^*).

Thirdly, by $H^*$and $\lambda_{H^*} = H^*(\omega)$ we denote a component of almost quaternion structure V^* and a Liouville form on (M, V^*), respectively.

By means of (0.51) and (0.56) we find

$$\lambda_{H^*} = \frac{1}{2}(x_i dx_{3n+i} + x_{n+i} dx_{2n+i} - x_{2n+i} dx_{n+i} - x_{3n+i} dx_i).$$

It is well-known that if $\Phi_{H^*} = -d\lambda_{H^*}$ is a closed Kähler form on the quaternion Kähler manifold (M, V^*), then Φ_{H^*} is also a symplectic structure on (M, V^*).

Taking into

$$\Phi_{H^*} = dx_{3n+i} \wedge dx_i + dx_{2n+i} \wedge dx_{n+i},$$

we find

$$i_{X_{H^*}} \Phi_{H^*} = Z^{3n+i} dx_i - Z^i dx_{3n+i} + Z^{2n+i} dx_{n+i} - Z^{n+i} dx_{2n+i}. \tag{0.67}$$

By **Eq.**(0.3), **Eq.** (0.59) and **Eq.**(0.67), one obtains a Hamiltonian vector field given by

$$X_{H^*} = -\frac{\partial \mathbf{H}}{\partial x_{3n+i}} \frac{\partial}{\partial x_i} - \frac{\partial \mathbf{H}}{\partial x_{2n+i}} \frac{\partial}{\partial x_{n+i}} + \frac{\partial \mathbf{H}}{\partial x_{n+i}} \frac{\partial}{\partial x_{2n+i}} + \frac{\partial \mathbf{H}}{\partial x_i} \frac{\partial}{\partial x_{3n+i}}. \tag{0.68}$$

Taking into **Eq.** (0.61), if we equal **Eq.** (0.62) and **Eq.** (0.68), it yields

$$\frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{3n+i}}, \quad \frac{dx_{n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}}, \quad \frac{dx_{2n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{n+i}}, \quad \frac{dx_{3n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i}. \tag{0.69}$$

Finally, the equations obtained in **Eq.** (0.69) are obtained to be *Hamilton equations* with respect to component H^* of the almost quaternion structure V^* on the quaternion Kähler manifold (M, V^*), and then the triple (M, Φ_{H^*}, X) is a *Hamiltonian mechanical system* on (M, V^*).

CONCLUSION 5. *From above, Lagrangian mechanics has intrinsically been described taking into account a canonical local basis $\{F, G, H\}$ of V on the quaternion Kähler manifold (M, V). The paths of semispray ξ on the quaternion Kähler manifold are the solutions Euler–Lagrange equations raised in **Eq.** (0.53), **Eq.** (0.54) and **Eq.** (0.55), and obtained by a canonical local basis $\{F, G, H\}$ of the vector bundle V on the quaternion Kähler manifold (M, V). Formalism of Hamiltonian mechanics has intrinsically been described with taking into account the basis $\{F^*, G^*, H^*\}$ of the almost quaternion structure V^* on the quaternion Kähler manifold (M, V^*). The paths of Hamiltonian vector field on the quaternion Kähler manifold are the solutions Hamilton equations raised in (0.63), (0.66) and (0.69), and obtained by a canonical local basis $\{F^*, G^*, H^*\}$ of the vector bundle V^* on the quaternion Kähler manifold (M, V^*).*

CHAPTER 6

Mechanical Systems on Para-Quaternion Kähler Manifolds

In this chapter, we present equations related to Lagrangian and Hamiltonian mechanical systems on para-quaternion Kähler manifold given in [**24**].

The algebra B of split quaternions is a four-dimensional real vector space with basis $\{1, i, s, t\}$ given by

$$i^2 = -1, \; s^2 = 1 = t^2, \; is = t = -si.$$

This carries a natural indefinite inner product given by $< p, q >= Re\overline{p}q$, where $p = x+iy+su+tv$ has $\overline{p} = x-iy-su-tv$. We have $\|p\|^2 = x^2+y^2-s^2-t^2$, so a metric of signature $(2, 2)$. This norm is multiplicative, $\|pq\|^2 = \|p\|^2 \; \|q\|^2$, but the presence of elements of length zero means that B contains zero divisors. The fundamental structures $1, i, s, t$ are not the only split quaternions with square ± 1. Using the multiplication rules for B, one can calculate

$$p^2 = -1 \text{ if and only if } p = iy + su + tv, y^2 - s^2 - t^2 = 1,$$

$$p^2 = +1 \text{ if and only if } p = iy + su + tv, y^2 - s^2 - t^2 = -1 \text{ or } p = \pm 1.$$

The right B-module $B^n \cong R^{4n}$ inherits the inner product $< \xi, \eta >= Re\overline{\xi}^{T}\eta$ of signature $(2n, 2n)$. The automorphism group of $(B^n, \langle \cdot, \cdot \rangle)$ is $Sp(n, B) = \{A \in M_n(B) : \overline{A}^T A = 1\}$ which is a Lie group isomorphic to $Sp(2n, R)$, the symmetries of a symplectic vector space (R^{2n}, ω). Especially, $Sp(1, B) \cong SL(2, R)$ is the pseudo-sphere of $B = R^{2,2}$. The Lie algebra of $Sp(n, B)$ is $sp(n, B) = \{A \in M_n(B) : A + \overline{A}^T = 0\}$, so $sp(1, B) = ImB$. The group $Sp(n, B) \times Sp(1, B)$ acts on B^n via

$$(A, p).\xi = A\xi\overline{p}. \tag{0.70}$$

For detail see [**25**] .

0.12. Para-Quaternion Kähler Manifolds. Here, we recall hypersymplectic manifolds and para-quaternion Kähler manifolds given in [**25**]. Let $m = 4n$, identify R^{4n} with B^n and consider $\dot{G} = Sp(n, B) \subset GL(4n, R)$. An $Sp(n, B)$-structure $Sp_B(M)$ on M defines a metric g of signature $(2n, 2n)$ by

$g(u(v),u(w)) = < v,w >$. The right action of i,s and t on B^n define endomorphisms F,G and H of T_xM satisfying

$$F^2 = -I,\ G^2 = \ H^2 = I,\ FG = H = -GF, \tag{0.71}$$

and the compatibility equations, for $X,Y \in T_xM$

$$g(FX,FY) = g(X,Y),\ g(GX,GY) = -g(X,Y) = g(HX,HY), \tag{0.72}$$

where I denotes the identity tensor of type (1,1) in M, and g is Riemannian metric. Using (0.71), we obtain three 2-forms ω_F, ω_G and ω_H given by

$$\omega_F(X,Y) = g(FX,Y),\quad \omega_G(X,Y) = g(GX,Y),\ \omega_H(X,Y) = g(HX,Y).$$

The manifold M is said to be hypersymplectic if the 2-forms ω_F, ω_G and ω_H are all closed:

$$d\omega_F = 0,\ d\omega_G = 0 \text{ and } d\omega_H = 0.$$

Now we think of the larger structure group $Sp(n,B)Sp(1,B)$ acting on $B^n = R^{4n}$ via (0.70). Again we have metric of neutral signature $(2n,2n)$, but now we can not distinguish the endomorphisms F,G and H . Instead we have a bundle Ģ of endomorphisms of TM that locally admits a basis $\{F,G,H\}$ satisfying (0.71) and (0.72). $\{F,G,H\}$ is called a canonical local basis of the bundle V in any coordinate neighborhood U of M. Then V is called a para-quaternion structure in M. The pair (M,V) denotes a para-quaternion manifold with V. A para-quaternion manifold M is of dimension $m = 4n$ $(n \geq 1)$.A para-quaternion structure V with such a Riemannian metric g is called a para-quaternion metric structure. A manifold M with a para-quaternion metric structure $\{g,V\}$ is called a para-quaternion metric manifold. The triple (M,g,V) denotes a para-quaternion metric manifold. If $n > 1$, we say that M is para-quaternion Kähler if its holonomy lies in $Sp(n,B)Sp(1,B)$.

Let $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}$, $i = \overline{1,n}$ be a real coordinate system on a neighborhood U of M, and let $\left\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_{n+i}}, \frac{\partial}{\partial x_{2n+i}}, \frac{\partial}{\partial x_{3n+i}}\right\}$ and $\{dx_i, dx_{n+i}, dx_{2n+i}, dx_{3n+i}\}$ be natural bases over R of the tangent space $T(M)$ and the cotangent space $T^*(M)$ of M, respectively. Taking into consideration (0.71), then we can obtain the expressions as follows:

$$\begin{aligned}
F(\frac{\partial}{\partial x_i}) &= \frac{\partial}{\partial x_{n+i}},\ F(\frac{\partial}{\partial x_{n+i}}) = -\frac{\partial}{\partial x_i},\ F(\frac{\partial}{\partial x_{2n+i}}) = \frac{\partial}{\partial x_{3n+i}},\\
F(\frac{\partial}{\partial x_{3n+i}}) &= -\frac{\partial}{\partial x_{2n+i}} G(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_{2n+i}},\ G(\frac{\partial}{\partial x_{n+i}}) = -\frac{\partial}{\partial x_{3n+i}},\\
G(\frac{\partial}{\partial x_{2n+i}}) &= \frac{\partial}{\partial x_i},\ G(\frac{\partial}{\partial x_{3n+i}}) = -\frac{\partial}{\partial x_{n+i}}, H(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial x_{3n+i}},\\
H(\frac{\partial}{\partial x_{n+i}}) &= \frac{\partial}{\partial x_{2n+i}},\ H(\frac{\partial}{\partial x_{2n+i}}) = \frac{\partial}{\partial x_{n+i}},\ H(\frac{\partial}{\partial x_{3n+i}}) = \frac{\partial}{\partial x_i}.
\end{aligned}$$

A canonical local basis$\{F^*, G^*, H^*\}$ of V^* of the cotangent space $T^*(M)$ of manifold M satisfies the condition as follows:

$$F^{*2} = -I,\ G^{*2} = \ H^{*2} = I,\ F^*G^* = H^* = -G^*F^*,$$

defining by

$$\begin{aligned} F^*(dx_i) &= dx_{n+i},\ F^*(dx_{n+i}) = -dx_i,\ F^*(dx_{2n+i}) = dx_{3n+i}, \\ F^*(dx_{3n+i}) &= -dx_{2n+i}, G^*(dx_i) = dx_{2n+i},\ G^*(dx_{n+i}) = -dx_{3n+i}, \\ G^*(dx_{2n+i}) &= dx_i,\ G^*(dx_{3n+i}) = -dx_{n+i}, H^*(dx_i) = dx_{3n+i}, \\ H^*(dx_{n+i}) &= dx_{2n+i},\ H^*(dx_{2n+i}) = dx_{n+i},\ H^*(dx_{3n+i}) = dx_i. \end{aligned}$$

0.13. Para-Quaternion Lagrangians. Here, we obtain Euler-Lagrange equations for quantum and classical mechanics by means of a canonical local basis $\{F, G, H\}$ of V on para-quaternion Kähler manifold (M, g, V).

Firstly, let F take a local basis element on the para-quaternion Kähler manifold (M, g, V), and $\{x_i, x_{n+i}, x_{2n+i}, x_{3n+i}\}$ be its coordinate functions. Let semispray be the vector field X determined by

$$X = X^i \frac{\partial}{\partial x_i} + X^{n+i} \frac{\partial}{\partial x_{n+i}} + X^{2n+i} \frac{\partial}{\partial x_{2n+i}} + X^{3n+i} \frac{\partial}{\partial x_{3n+i}}, \tag{0.73}$$

where $X^i = \dot{x}_i, X^{n+i} = \dot{x}_{n+i}, X^{2n+i} = \dot{x}_{2n+i}, X^{3n+i} = \dot{x}_{3n+i}$ and the dot indicates the derivative with respect to time t. The vector field defined by

$$V_F = F(X) = X^i \frac{\partial}{\partial x_{n+i}} - X^{n+i} \frac{\partial}{\partial x_i} + X^{2n+i} \frac{\partial}{\partial x_{3n+i}} - X^{3n+i} \frac{\partial}{\partial x_{2n+i}}$$

is named *Liouville vector field* on the para-quaternion Kähler manifold (M, g, V). The maps given by $T, P : M \to R$ such that $T = \frac{1}{2} m_i(\dot{x}_i{}^2 + \dot{x}_{n+i}^2 + \dot{x}_{2n+i}^2 + \dot{x}_{3n+i}^2), P = m_i gh$ are said to be *the kinetic energy* and *the potential energy of the system,* respectively. Here m_i, g and h stand for mass of a mechanical system having m particles, the gravity acceleration and distance to the origin of a mechanical system on the para-quaternion Kähler manifold (M, g, V), respectively. Then $L : M \to R$ is a map that satisfies the conditions; i) $L = T - P$ is a *Lagrangian function, ii)* the function determined by $E_L^F = V_F(L) - L$, is *energy function.*

The function i_F induced by F and denoted by

$$i_F\omega(X_1, X_2, ..., X_r) = \sum_{i=1}^{r} \omega(X_1, ..., FX_i, ..., X_r),$$

is called *vertical derivation,* where $\omega \in \wedge^r M,\ X_i \in \chi(M)$. The *vertical differentiation* d_F is given by

$$d_F = [i_F, d] = i_F d - d i_F$$

where d is the usual exterior derivation. In the case the closed para-quaternion Kähler form is the closed 2-form given by $\Phi_L^F = -dd_F L$ such that

$$d_F = \frac{\partial}{\partial x_{n+i}} dx_i - \frac{\partial}{\partial x_i} dx_{n+i} + \frac{\partial}{\partial x_{3n+i}} dx_{2n+i} - \frac{\partial}{\partial x_{2n+i}} d_{3n+i} : \mathcal{F}(M) \to \wedge^1 M.$$

Then we have

$$\begin{array}{c} \Phi_L^F = -\frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j \wedge dx_i + \frac{\partial^2 L}{\partial x_j \partial x_i} dx_j \wedge dx_{n+i} \\ -\frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j \wedge dx_{3n+i} \\ -\frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} \wedge dx_{n+i} \\ -\frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \wedge dx_{3n+i} \\ -\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} \wedge dx_{n+i} \\ -\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} \wedge dx_{3n+i} \\ -\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \wedge dx_{n+i} \\ -\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \wedge dx_{3n+i}. \end{array}$$

Also we find energy function as follows:

$$E_L^F = V_F(L) - L = X^i \frac{\partial L}{\partial x_{n+i}} - X^{n+i} \frac{\partial L}{\partial x_i} + X^{2n+i} \frac{\partial L}{\partial x_{3n+i}} - X^{3n+i} \frac{\partial L}{\partial x_{2n+i}} - L$$

By means of α being an integral curve of X, then we obtain the equations given by

$$\begin{array}{c} \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right) - \frac{\partial L}{\partial x_{n+i}} = 0, \ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right) + \frac{\partial L}{\partial x_i} = 0, \\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right) - \frac{\partial L}{\partial x_{3n+i}} = 0, \ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right) + \frac{\partial L}{\partial x_{2n+i}} = 0, \end{array} \tag{0.74}$$

such that the equations calculated in (0.74) are named *Euler-Lagrange equations* constructed on the para-quaternion Kähler manifold (M, g, V) by means of Φ_L^F and thus the triple (M, Φ_L^F, X) is called a *mechanical system* on the para-quaternion Kähler manifold (M, g, V).

Secondly, we introduce Euler-Lagrange equations for quantum and classical mechanics by means of Φ_L^G on the para-quaternion Kähler manifold (M, g, V).

Take G. It is another local basis element on the para-quaternion Kähler manifold (M, g, V). Let us X which is the semispray in (0.73). In the case, the vector field determined by

$$V_G = G(X) = X^i \frac{\partial}{\partial x_{2n+i}} - X^{n+i} \frac{\partial}{\partial x_{3n+i}} + X^{2n+i} \frac{\partial}{\partial x_i} - X^{3n+i} \frac{\partial}{\partial x_{n+i}}$$

is *Liouville vector field* on the para-quaternion Kähler manifold (M, g, V). The operator given by $E_L^G = V_G(L) - L$ is *energy function*. Then the function i_G induced by G and given by

$$i_G \omega(X_1, X_2, ..., X_r) = \sum_{i=1}^{r} \omega(X_1, ..., GX_i, ..., X_r)$$

is *vertical derivation,* where $\omega \in \wedge^r M$, $X_i \in \chi(M)$. The *vertical differentiation* d_G is given by

$$d_G = [i_G, d] = i_G d - d i_G$$

where d is the usual exterior derivation. Since taking into consideration G, the closed para-quaternion Kähler form is the closed 2-form given by $\Phi_L^G = -dd_G L$ such that

$$d_G = \frac{\partial}{\partial x_{2n+i}} dx_i - \frac{\partial}{\partial x_{3n+i}} dx_{n+i} + \frac{\partial}{\partial x_i} dx_{2n+i} - \frac{\partial}{\partial x_{n+i}} d_{3n+i} : \mathcal{F}(M) \to \wedge^1 M.$$

Then we get

$$\begin{array}{c} \Phi_L^G = -\frac{\partial^2 L}{\partial x_j \partial x_{2n+i}} dx_j \wedge dx_i + \frac{\partial^2 L}{\partial x_j \partial x_{3n+i}} dx_j \wedge dx_{n+i} \\ -\frac{\partial^2 L}{\partial x_j \partial x_i} dx_j \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_j \partial x_{n+i}} dx_j \wedge dx_{3n+i} \\ -\frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}} dx_{n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}} dx_{n+j} \wedge dx_{n+i} \\ -\frac{\partial^2 L}{\partial x_{n+j} \partial x_i} dx_{n+j} \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}} dx_{n+j} \wedge dx_{3n+i} \\ -\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}} dx_{2n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}} dx_{2n+j} \wedge dx_{n+i} \\ -\frac{\partial^2 L}{\partial x_{2n+j} \partial x_i} dx_{2n+j} \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}} dx_{2n+j} \wedge dx_{3n+i} \\ -\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}} dx_{3n+j} \wedge dx_i + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}} dx_{3n+j} \wedge dx_{n+i} \\ -\frac{\partial^2 L}{\partial x_{3n+j} \partial x_i} dx_{3n+j} \wedge dx_{2n+i} + \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}} dx_{3n+j} \wedge dx_{3n+i}. \end{array}$$

Also, we calculate function

$$E_L^G = X^i \frac{\partial L}{\partial x_{2n+i}} - X^{n+i} \frac{\partial L}{\partial x_{3n+i}} + X^{2n+i} \frac{\partial L}{\partial x_i} - X^{3n+i} \frac{\partial L}{\partial x_{n+i}} - L.$$

By α an integral curve of X, then we obtain the equations:

$$\begin{array}{c} \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right) - \frac{\partial L}{\partial x_{2n+i}} = 0, \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right) + \frac{\partial L}{\partial x_{3n+i}} = 0, \\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right) - \frac{\partial L}{\partial x_i} = 0, \quad \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right) + \frac{\partial L}{\partial x_{n+i}} = 0. \end{array} \tag{0.75}$$

Hence the equations introduced in (0.75) are named *Euler-Lagrange equations* constructed by means of Φ_L^G on the para-quaternion Kähler manifold (M, g, V) and hence the triple (M, Φ_L^G, X) is said to be a *mechanical system* on the para-quaternion Kähler manifold (M, g, V).

Thirdly, we present Euler-Lagrange equations for quantum and classical mechanics by means of Φ_L^H on para-quaternion Kähler manifold (M, g, V).

Let H be a local basis element on the para-quaternion Kähler manifold (M, g, V). Consider X given by (0.73). So, *Liouville vector field* on the para-quaternion Kähler manifold (M, g, V) is the vector field determined by

$$V_H = H(X) = X^i \frac{\partial}{\partial x_{3n+i}} + X^{n+i} \frac{\partial}{\partial x_{2n+i}} + X^{2n+i} \frac{\partial}{\partial x_{n+i}} + X^{3n+i} \frac{\partial}{\partial x_i}.$$

The function given by $E_L^H = V_H(L) - L$ is *energy function.* The operator i_H induced by H and given by

$$i_H\omega(X_1, X_2, ..., X_r) = \sum_{i=1}^{r} \omega(X_1, ..., HX_i, ..., X_r),$$

is named *vertical derivation,* where $\omega \in \wedge^r M$, $X_i \in \chi(M)$. The *vertical differentiation* d_H is given by

$$d_H = [i_H, d] = i_H d - d i_H,$$

Thus, the closed para-quaternion Kähler form is the closed 2-form given by $\Phi_L^H = -dd_{_H}L$ such that

$$d_{_H} = \frac{\partial}{\partial x_{3n+i}}dx_i + \frac{\partial}{\partial x_{2n+i}}dx_{n+i} + \frac{\partial}{\partial x_{n+i}}dx_{2n+i} + \frac{\partial}{\partial x_i}d_{3n+i} : \mathcal{F}(M) \to \wedge^1 M.$$

Then we find

$$\begin{array}{c}
\Phi_L^H = -\frac{\partial^2 L}{\partial x_j \partial x_{3n+i}}dx_j \wedge dx_i - \frac{\partial^2 L}{\partial x_j \partial x_{2n+i}}dx_j \wedge dx_{n+i} \\
-\frac{\partial^2 L}{\partial x_j \partial x_{n+i}}dx_j \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_j \partial x_i}dx_j \wedge dx_{3n+i} \\
-\frac{\partial^2 L}{\partial x_{n+j} \partial x_{3n+i}}dx_{n+j} \wedge dx_i - \frac{\partial^2 L}{\partial x_{n+j} \partial x_{2n+i}}dx_{n+j} \wedge dx_{n+i} \\
-\frac{\partial^2 L}{\partial x_{n+j} \partial x_{n+i}}dx_{n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{n+j} \partial x_i}dx_{n+j} \wedge dx_{3n+i} \\
-\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{3n+i}}dx_{2n+j} \wedge dx_i - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_{2n+i}}dx_{2n+j} \wedge dx_{n+i} \\
-\frac{\partial^2 L}{\partial x_{2n+j} \partial x_{n+i}}dx_{2n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{2n+j} \partial x_i}dx_{2n+j} \wedge dx_{3n+i} \\
-\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{3n+i}}dx_{3n+j} \wedge dx_i - \frac{\partial^2 L}{\partial x_{3n+j} \partial x_{2n+i}}dx_{3n+j} \wedge dx_{n+i} \\
-\frac{\partial^2 L}{\partial x_{3n+j} \partial x_{n+i}}dx_{3n+j} \wedge dx_{2n+i} - \frac{\partial^2 L}{\partial x_{3n+j} \partial x_i}dx_{3n+j} \wedge dx_{3n+i}.
\end{array}$$

Also we have

$$E_L^H = X^i \frac{\partial L}{\partial x_{3n+i}} + X^{n+i} \frac{\partial L}{\partial x_{2n+i}} + X^{2n+i} \frac{\partial L}{\partial x_{n+i}} + X^{3n+i} \frac{\partial L}{\partial x_i} - L.$$

Taking α being an integral curve of X, then it follows:

$$\begin{array}{c}
\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_i}\right) - \frac{\partial L}{\partial x_{3n+i}} = 0, \; \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{n+i}}\right) - \frac{\partial L}{\partial x_{2n+i}} = 0, \\
\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{2n+i}}\right) - \frac{\partial L}{\partial x_{n+i}} = 0, \; \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial x_{3n+i}}\right) - \frac{\partial L}{\partial x_i} = 0.
\end{array} \tag{0.76}$$

Thus the equations introduced by (0.76) infer *Euler-Lagrange equations* constructed by means of Φ_L^H on the para-quaternion Kähler manifold (M, g, V) and then the triple (M, Φ_L^H, X) is named a *mechanical system* on the para-quaternion Kähler manifold (M, g, V).

0.14. Para-Quaternion Hamiltonians. Here, we present Hamilton equations and Hamiltonian mechanical systems for quantum and classical mechanics constructed on the para-quaternion Kähler manifold (M, g, V^*).

Firstly, let (M, g, V^*) be a para-quaternion Kähler manifold. Suppose that an element of para-quaternion structure V^*, a Liouville form and a 1-form on para-quaternion Kähler manifold (M, g, V^*) are shown by F^*, λ_{F^*} and ω_{F^*}, respectively.

Consider

$$\omega_{F^*} = \frac{1}{2}(x_i dx_i + x_{n+i} dx_{n+i} + x_{2n+i} dx_{2n+i} + x_{3n+i} dx_{3n+i}).$$

Then we have

$$\lambda_{F^*} = F^*(\omega_{F^*}) = \frac{1}{2}(x_i dx_{n+i} - x_{n+i} dx_i + x_{2n+i} dx_{3n+i} - x_{3n+i} dx_{2n+i}).$$

It is concluded that if Φ_{F^*} is a closed para-quaternion Kähler form on the para-quaternion Kähler manifold (M, g, V^*), then Φ_{F^*} is also a symplectic structure on the para-quaternion Kähler manifold (M, g, V^*).

Take X. It is Hamiltonian vector field associated with Hamiltonian energy $\mathbf{H}$ and determined by (0.73).

Then

$$\Phi_{F^*} = -d\lambda_{F^*} = dx_{n+i} \wedge dx_i + dx_{3n+i} \wedge dx_{2n+i},$$

and

$$(0.77)\quad i_X \Phi_{F^*} = \Phi_{F^*}(X) = X^{n+i} dx_i - X^i dx_{n+i} + X^{3n+i} dx_{2n+i} - X^{2n+i} dx_{3n+i}.$$

Furthermore, the differential of Hamiltonian energy is obtained by

$$(0.78)\qquad d\mathbf{H} = \frac{\partial \mathbf{H}}{\partial x_i} dx_i + \frac{\partial \mathbf{H}}{\partial x_{n+i}} dx_{n+i} + \frac{\partial \mathbf{H}}{\partial x_{2n+i}} dx_{2n+i} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}} dx_{3n+i}.$$

With respect to (0.3), if equaled (0.77) and (0.78), the Hamiltonian vector field is found as follows:

$$(0.79)\qquad X = -\frac{\partial \mathbf{H}}{\partial x_{n+i}} \frac{\partial}{\partial x_i} + \frac{\partial \mathbf{H}}{\partial x_i} \frac{\partial}{\partial x_{n+i}} - \frac{\partial \mathbf{H}}{\partial x_{3n+i}} \frac{\partial}{\partial x_{2n+i}} + \frac{\partial \mathbf{H}}{\partial x_{2n+i}} \frac{\partial}{\partial x_{3n+i}}.$$

Assume that a curve

$$\alpha : I \subset \mathbf{R} \to M$$

be an integral curve of the Hamiltonian vector field X, i.e.,

$$(0.80)\qquad X(\alpha(t)) = \dot{\alpha},\ t \in I.$$

In the local coordinates, it is obtained that

$$\alpha(t) = (x_i, x_{n+i}, x_{2n+i}, x_{3n+i})$$

and

$$(0.81)\qquad \dot{\alpha}(t) = \frac{dx_i}{dt} \frac{\partial}{\partial x_i} + \frac{dx_{n+i}}{dt} \frac{\partial}{\partial x_{n+i}} + \frac{dx_{2n+i}}{dt} \frac{\partial}{\partial x_{2n+i}} + \frac{dx_{3n+i}}{dt} \frac{\partial}{\partial x_{3n+i}}.$$

Taking (0.80), if we equal (0.79) and (0.81), it holds

(0.82)

$$\frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}},\ \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i},\ \frac{dx_{2n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{3n+i}},\ \frac{dx_{3n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{2n+i}}$$

Hence, the equations introduced in (0.82) are named *Hamilton equations* with respect to component F^* of the para-quaternion structure V^* on the para-quaternion Kähler manifold (M, g, V^*), and then the triple (M, Φ_{F^*}, X) is said to be a *Hamiltonian mechanical system* on para-quaternion Kähler manifold (M, g, V^*).

Secondly, assume that a component of para-quaternion structure V^*, a Liouville form and a 1-form on the para-quaternion Kähler manifold (M, g, V^*) are denoted by G^*, λ_{G^*} and ω_{G^*}, respectively.

Take

$$\omega_{G^*} = \frac{1}{2}(x_i dx_i + x_{n+i} dx_{n+i} - x_{2n+i} dx_{2n+i} - x_{3n+i} dx_{3n+i}).$$

Then we calculate

$$\lambda_{G^*} = G^*(\omega_{G^*}) = \frac{1}{2}(x_i dx_{2n+i} - x_{n+i} dx_{3n+i} - x_{2n+i} dx_i + x_{3n+i} dx_{n+i}).$$

It is well-known if Φ_{G^*} is a closed para-quaternion Kähler form on the para-quaternion Kähler manifold (M, g, V^*), then Φ_{G^*} is also a symplectic structure on para-quaternion Kähler manifold (M, g, V^*).

Let X a Hamiltonian vector field related to Hamiltonian energy $\mathbf{H}$ and given by (0.73).

Taking into consideration

$$\Phi_{G^*} = -d\lambda_{G^*} = dx_{2n+i} \wedge dx_i + dx_{n+i} \wedge dx_{3n+i},$$

then we calculate

$$(0.83)\quad i_X \Phi_{G^*} = \Phi_{G^*}(X) = X^{2n+i} dx_i - X^i dx_{2n+i} + X^{n+i} dx_{3n+i} - X^{3n+i} dx_{n+i}.$$

According to (0.3), if we equal (0.78) and (0.83), it yields

$$(0.84)\qquad X = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}} \frac{\partial}{\partial x_i} + \frac{\partial \mathbf{H}}{\partial x_{3n+i}} \frac{\partial}{\partial x_{n+i}} + \frac{\partial \mathbf{H}}{\partial x_i} \frac{\partial}{\partial x_{2n+i}} - \frac{\partial \mathbf{H}}{\partial x_{n+i}} \frac{\partial}{\partial x_{3n+i}}.$$

Taking (0.80), if (0.81) and (0.84) are equaled, we find equations

(0.85)

$$\frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}}, \ \frac{dx_{n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{3n+i}}, \ \frac{dx_{2n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i}, \ \frac{dx_{3n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{n+i}}$$

Finally, the equations found in (0.85) are called *Hamilton equations* with respect to component G^* of the para-quaternion structure V^* on the para-quaternion Kähler manifold (M, g, V^*), and then the triple (M, Φ_{G^*}, X) is named a *Hamiltonian mechanical system* on the para-quaternion Kähler manifold (M, g, V^*).

Thirdly, by H^*, λ_{H^*} and ω_{H^*}, we give a element of para-quaternion structure V^*, a Liouville form and a 1-form on para-quaternion Kähler manifold (M, g, V^*), respectively.

Let

$$\omega_{H^*} = \frac{1}{2}(x_i dx_i + x_{n+i} dx_{n+i} - x_{2n+i} dx_{2n+i} - x_{3n+i} dx_{3n+i}).$$

Then we find

$$\lambda_{H^*} = H^*(\omega_{H^*}) = \frac{1}{2}(x_i dx_{3n+i} + x_{n+i} dx_{2n+i} - x_{2n+i} dx_{n+i} - x_{3n+i} dx_i).$$

We know that if Φ_{H^*} is a closed para-quaternion Kähler form on the para-quaternion Kähler manifold (M, g, V^*), then Φ_{H^*} is also a symplectic structure on the para-quaternion Kähler manifold (M, g, V^*).

Let X a Hamiltonian vector field connected with Hamiltonian energy $\mathbf{H}$ and given by (0.73).

Calculating

$$\Phi_{H^*} = -d\lambda_{H^*} = dx_{3n+i} \wedge dx_i + dx_{2n+i} \wedge dx_{n+i}, \tag{0.86}$$

we have

$$i_X \Phi_{H^*} = \Phi_{H^*}(X) = X^{3n+i} dx_i - X^i dx_{3n+i} + X^{2n+i} dx_{n+i} - X^{n+i} dx_{2n+i}. \tag{0.87}$$

With respect to (0.3), if we equal (0.78) and (0.87), we find the Hamiltonian vector field given by

$$X = -\frac{\partial \mathbf{H}}{\partial x_{3n+i}} \frac{\partial}{\partial x_i} - \frac{\partial \mathbf{H}}{\partial x_{2n+i}} \frac{\partial}{\partial x_{n+i}} + \frac{\partial \mathbf{H}}{\partial x_{n+i}} \frac{\partial}{\partial x_{2n+i}} + \frac{\partial \mathbf{H}}{\partial x_i} \frac{\partial}{\partial x_{3n+i}}. \tag{0.88}$$

Considering (0.80), if (0.81) and (0.88) are equaled, it yields

$$\frac{dx_i}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{3n+i}}, \ \frac{dx_{n+i}}{dt} = -\frac{\partial \mathbf{H}}{\partial x_{2n+i}}, \ \frac{dx_{2n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_{n+i}}, \ \frac{dx_{3n+i}}{dt} = \frac{\partial \mathbf{H}}{\partial x_i} \tag{0.89}$$

In the end, the equations introduced in (0.89) are named *Hamilton equations* with respect to element H^* of the para-quaternion structure V^* on the para-quaternion Kähler manifold (M, g, V^*), and then the triple (M, Φ_{H^*}, X) is called a *Hamiltonian mechanical system* on the para-quaternion Kähler manifold (M, g, V^*).

CONCLUSION 6. *From above, Lagrangian mechanical systems have intrinsically been described taking into account a canonical local basis* $\{F, G, H\}$ *of* V *on the para-quaternion Kähler manifold* (M, g, V). *The paths of semispray* X *on the para-quaternion Kähler manifold are the solutions Euler-Lagrange equations raised in (0.74), (0.75) and (0.76), and introduced by a canonical local basis* $\{F, G, H\}$ *of vector bundle* V *on the para-quaternion Kähler manifold* (M, g, V). *Also, Hamiltonian mechanical systems have intrinsically been described with taking into account the basis* $\{F^*, G^*, H^*\}$ *of para-quaternion structure* V^* *on the para-quaternion Kähler manifold* (M, g, V^*). *The paths of Hamilton vector field* X *on the para-quaternion Kähler manifold are the solutions Hamilton equations raised in (0.82), (0.85) and (0.89), and obtained by a canonical local basis* $\{F^*, G^*, H^*\}$ *of vector bundle* V^* *on the para-quaternion Kähler manifold* (M, g, V^*). *Lagrangian and Hamiltonian models arise to be a very important tool since they present a simple method to describe the model for mechanical systems. One can be proved that the obtained equations are* *very important to explain the rotational spatial mechanical-physical problems. Therefore, the found equations are only considered to be a first step to realize how para-quaternion geometry has been used in solving problems in different physical area. For further research, the Lagrangian and Hamiltonian mechanical equations derived here are suggested to deal with problems in electrical, magnetical and gravitational fields of quantum and classical mechanics of physics.*

CHAPTER 7

Mechanical Systems with Constraints

The purpose of this chapter is to make a contribution to the modern development of Lagrangian and Hamiltonian formalisms of classical mechanics in terms of differential-geometric methods on differentiable manifolds. So, we introduce complex and paracomplex Euler-Lagrange and Hamilton equations with constraints on the (para) Kähler manifold given in [**26, 27**].

1. Constrained Complex Mechanical Systems

Assume that (TQ, Φ_L) is symplectic manifold and $\overline{\omega} = \{\omega_1, ..., \omega_r\}$ is a system of constraints on TQ. We call to be a *constraint* on TQ to a non-zero 1-form $\omega = \wedge^a \omega_a$ on TQ, such that $\wedge^a$ are Lagrange multipliers. We call $(TQ, \Phi_L, E_L, \overline{\omega})$ a *regular Lagrangian system with constraints.* The constraints $\overline{\omega}$ are said to be classical constraints if the 1-forms $\omega_a, 1 \leq a \leq r$, are basic. Then holonomic classical constraints define foliations on the configuration manifold Q, but holonomic constraints also admit foliations on the phase space of velocities TQ. As real studies, generally a curve α satisfying the Euler Lagrange equations for Lagrangian energy E_L will not satisfy the constraints. It must be that some additional forces (or *canonical constraint forces*) act on the system in addition to the *force* dE_L for a curve α to satisfy the constraints. It is said that the quartet $(TQ, \Phi_L, E_L, \overline{\omega})$ defines a *mechanical system with constraints* if the vector field ξ given by the equations of motion

$$i_\xi \Phi_L = dE_L + \wedge^a \omega_a, \, \omega_a(\xi) = 0, \tag{1.1}$$

is a semispray. Then, it is given Euler-Lagrange equations with constraints as follows:

$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = \wedge^a (\omega_a)_i. \tag{1.2}$$

Let M be configuration manifold of real dimension m. A tensor field J on TM is called an *almost complex structure* on TM if at every point p of TM, J is endomorphism of the tangent space $T_p(TM)$ such that $J^2 = -I$. A manifold TM with fixed almost complex structure J is called *almost complex manifold.* Assume that (x_i) be coordinates of M and $(x_i, \, y_i)$ be a real coordinate system on a neighborhood U of any point p of TM. Also, let us to be $\{(\frac{\partial}{\partial x^i})_p, (\frac{\partial}{\partial y^i})_p\}$ and $\{(dx^i)_p, (dy^i)_p\}$ to natural bases over $\mathbf{R}$ of tangent space $T_p(TM)$ and cotangent space $T_p^*(TM)$ of TM, respectively.

Let TM be an almost complex manifold with fixed almost complex structure J. The manifold TM is called *complex manifold* if there exists an open covering $\{U\}$ of TM satisfying the following condition: There is a local coordinate system $(x_i,\, y_i)$ on each U, such that

$$J(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i},\; J(\frac{\partial}{\partial y_i}) = -\frac{\partial}{\partial x_i}. \tag{1.3}$$

for each point of U. Let $z_i = x_i + \mathbf{i}\, y_i, \mathbf{i} = \sqrt{-1}$, be a complex local coordinate system on a neighborhood U of any point p of TM. We define the vector fields by

$$(\frac{\partial}{\partial z^i})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^i})_p - \mathbf{i}(\frac{\partial}{\partial y^i})_p\},\; (\frac{\partial}{\partial \overline{z}^i})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^i})_p + \mathbf{i}(\frac{\partial}{\partial y^i})_p \tag{1.4}$$

and the dual covector fields

$$\left(dz^i\right)_p = \left(dx^i\right)_p + \mathbf{i}(dy^i)_p,\; \left(d\overline{z}^i\right)_p = \left(dx^i\right)_p - \mathbf{i}(dy^i)_p \tag{1.5}$$

which represent bases of the tangent space $T_p(TM)$ and cotangent space $T_p^*(TM)$ of TM, respectively. Then the endomorphism J is shown as

$$J(\frac{\partial}{\partial z_i}) = \mathbf{i}\frac{\partial}{\partial z_i},\; J(\frac{\partial}{\partial \overline{z}_i}) = -\mathbf{i}\frac{\partial}{\partial \overline{z}_i}. \tag{1.6}$$

The dual endomorphism J^* of the cotangent space $T_p^*(TM)$ at any point p of manifold TM satisfies $J^{*2} = -I$, and is defined by

$$J^*(dz_i) = \mathbf{i}dz_i,\; J^*(d\overline{z}_i) = -\mathbf{i}d\overline{z}_i. \tag{1.7}$$

A *Hermitian metric* on an almost complex manifold with almost complex structure J is a Riemannian metric g on TM such that

$$g(JX, JY) = g(X, Y), \tag{1.8}$$

for any vector fields X, Y on TM. An almost complex manifold TM with a Hermitian metric is called an *almost Hermitian manifold.* If, moreover, TM is a complex manifold, then TM is called a *Hermitian manifold.*

Let further TM be a 2m-dimensional real almost Hermitian manifold with almost complex structure J and Hermitian metric g. The triple (TM, J, g) may be named an *almost Hermitian structure.* We denote by $\chi(TM)$ the set of complex vector fields on TM and by $\wedge^1(TM)$ the set of complex 1-forms on TM. Let (TM, J, g) be an almost Hermitian structure. The 2-form defined by

$$\Phi(X, Y) = g(X, JY),\; \forall X, Y \in \chi(TM) \tag{1.9}$$

is called the *Kähler form* of (TM, J, g).

An almost Hermitian manifold is called *almost Kähler* if its Kähler form Φ is closed. If, moreover, TM is Hermitian, then TM is called a Kähler manifold.

1.1. Complex Lagrangians. Let J be an almost complex structure on the Kähler manifold and $(z^i, \overline{z}^i)$ its complex coordinates. We call to be the semispray to the vector field ξ given by

$$\xi = \xi^i \frac{\partial}{\partial z^i} + \overline{\xi}^i \frac{\partial}{\partial \overline{z}^i}, \xi^i = \dot{z}^i = \overline{z}^i, \overline{\xi}^i = \dot{\xi}^i = \ddot{z}^i = \dot{\overline{z}}^i. \tag{1.10}$$

The vector field $V = J\xi$ is called *Liouville vector field* on the Kähler manifold. We call *the kinetic energy* and *the potential energy of system* the maps given by $T, P : TM \to \mathbf{C}$ such that $T = \frac{1}{2}m_i(\overline{z}^i)^2 = \frac{1}{2}m_i(\dot{z}^i)^2$, $P = m_i\mathbf{g}h$, respectively, where m_i is mass of a mechanic system having m particles, $\mathbf{g}$ is the gravity acceleration and h is the origin distance of the a mechanic system on the Kähler manifold. Then it may be said to be *Lagrangian function* the map $L : TM \to \mathbf{C}$ such that $L = T - P$ and also *the energy function* associated L the function given by $E_L = V(L) - L$.

The vertical derivation operator i_J defined by

$$i_J\omega(Z_1, Z_2, ..., Z_r) = \sum_{i=1}^{r} \omega(Z_1, ..., JZ_i, ..., Z_r), \tag{1.11}$$

where $\omega \in \wedge^r TM$, $Z_i \in \chi(TM)$. The exterior differentiation d_J is defined by

$$d_J = [i_J, d] = i_J d - d i_J, \tag{1.12}$$

where d is the usual exterior derivation.

For almost complex structure J, the closed Kähler form is the closed 2-form given by

$$\Phi_L = -dd_J L, \tag{1.13}$$

such that

$$d_J : \mathcal{F}(TM) \to \wedge^1 TM.$$

By means of (1.1), *complex Euler-Lagrange equations* on Kähler manifold TM is found the following as:

$$\mathbf{i}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial z^i}\right) - \frac{\partial L}{\partial z^i} = 0,\ \mathbf{i}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{z}^i}\right) + \frac{\partial L}{\partial \dot{z}^i} = 0. \tag{1.14}$$

1.2. Constrained Complex Lagrangians. Let J be an almost complex structure on the Kähler manifold and $(z^i, \overline{z}^i)$ its complex coordinates. Assume to be semispray to the vector field ξ given as:

$$\xi = \xi_L + \wedge^a \omega_a = \xi^i \frac{\partial}{\partial z^i} + \overline{\xi}^i \frac{\partial}{\partial \overline{z}^i} + \wedge^a \omega_a,\ 1 \leq a \leq r, \tag{1.15}$$

The vector field determined by

$$V = J\xi_L = \mathbf{i}\xi^i \frac{\partial}{\partial z^i} - \mathbf{i}\overline{\xi}^i \frac{\partial}{\partial \overline{z}^i}, \tag{1.16}$$

is called *Liouville vector field* on the Kähler manifold TM. The closed 2-form given by $\Phi_L = -dd_J L$ such that

$$d_J = \mathbf{i}\frac{\partial}{\partial z^i}dz^i - \mathbf{i}\frac{\partial}{\partial \overline{z}^i}d\overline{z}^i : \mathcal{F}(TM) \to \wedge^1 TM. \tag{1.17}$$

is found to be

$$\begin{array}{c}\Phi_L = \mathbf{i}\frac{\partial^2 L}{\partial z^j \partial z^i}dz^i \wedge dz^j + \mathbf{i}\frac{\partial^2 L}{\partial \overline{z}^j \partial z^i}dz^i \wedge d\overline{z}^j \\ +\mathbf{i}\frac{\partial^2 L}{\partial z^j \partial \overline{z}^i}dz^j \wedge d\overline{z}^i + \mathbf{i}\frac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i}d\overline{z}^j \wedge d\overline{z}^i.\end{array} \tag{1.18}$$

Let ξ be the semispray given by (1.15) and

$$\begin{array}{c}i_\xi \Phi_L = \mathbf{i}\xi^i \frac{\partial^2 L}{\partial z^j \partial z^i}dz^j - \mathbf{i}\xi^i \frac{\partial^2 L}{\partial z^j \partial z^i}\delta_i^j dz^i + \mathbf{i}\xi^i \frac{\partial^2 L}{\partial \overline{z}^j \partial z^i}d\overline{z}^j - \mathbf{i}\overline{\xi}^i \frac{\partial^2 L}{\partial \overline{z}^j \partial z^i}\delta_i^j dz^i \\ +\mathbf{i}\xi^i \frac{\partial^2 L}{\partial z^j \partial \overline{z}^i}\delta_i^j d\overline{z}^i - \mathbf{i}\overline{\xi}^i \frac{\partial^2 L}{\partial z^j \partial \overline{z}^i}dz^j + \mathbf{i}\overline{\xi}^i \frac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i}\delta_i^j d\overline{z}^i - \mathbf{i}\overline{\xi}^i \frac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i}d\overline{z}^j.\end{array} \tag{1.19}$$

Since the closed Kähler form Φ_L on TM is symplectic structure, we obtain

$$E_L = \mathbf{i}\xi^i \frac{\partial L}{\partial z^i} - \mathbf{i}\overline{\xi}^i \frac{\partial L}{\partial \overline{z}^i} - L \tag{1.20}$$

and hence

$$\begin{array}{rl}dE_L + \wedge^a \omega_a = & \mathbf{i}\xi^i \frac{\partial^2 L}{\partial z^j \partial z^i}dz^j - \mathbf{i}\overline{\xi}^i \frac{\partial^2 L}{\partial z^j \partial \overline{z}^i}dz^j - \frac{\partial L}{\partial z^j}dz^j \\ & +\mathbf{i}\xi^i \frac{\partial^2 L}{\partial \overline{z}^j \partial z^i}d\overline{z}^j - \mathbf{i}\overline{\xi}^i \frac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i}d\overline{z}^j - \frac{\partial L}{\partial \overline{z}^j}d\overline{z}^j + \wedge^a \omega_a.\end{array} \tag{1.21}$$

With respect to (1.1), if (1.19) and (1.21) are equalized, we conclude the equation as follows:

$$\begin{array}{c}-\mathbf{i}\xi^i \frac{\partial^2 L}{\partial z^j \partial z^i}dz^j - \mathbf{i}\overline{\xi}^i \frac{\partial^2 L}{\partial \overline{z}^j \partial z^i}dz^j + \frac{\partial L}{\partial z^j}dz^j \\ +\mathbf{i}\xi^i \frac{\partial^2 L}{\partial z^j \partial \overline{z}^i}d\overline{z}^j + \mathbf{i}\overline{\xi}^i \frac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i}d\overline{z}^j + \frac{\partial L}{\partial \overline{z}^j}d\overline{z}^j = \wedge^a \omega_a\end{array} \tag{1.22}$$

Now, let the curve $\alpha : \mathbf{C} \to TM$ be integral curve of ξ, which satisfies equations

$$\begin{array}{c}-\mathbf{i}\left[\xi^j \frac{\partial^2 L}{\partial z^j \partial z^i} + \dot{\xi}^i \frac{\partial^2 L}{\partial \dot{z}^j \partial z^i}\right]dz^j + \frac{\partial L}{\partial z^j}dz^j \\ +\mathbf{i}\left[\xi^j \frac{\partial^2 L}{\partial z^j \partial \dot{z}^i} + \dot{\xi}^j \frac{\partial^2 L}{\partial \dot{z}^j \partial \dot{z}^i}\right]d\dot{z}^j + \frac{\partial L}{\partial \dot{z}^j}d\dot{z}^j = \wedge^a \omega_a\end{array} \tag{1.23}$$

where $\omega_a = (\omega_a)_j \ dz^j + (\dot{\omega}_a)_j d\dot{z}^j$ and the dots mean derivatives with respect to the time. We infer the equations

$$\frac{\partial L}{\partial z^i} - \mathbf{i}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial z^i}\right) = \wedge^a (\omega_a)_i, \ \frac{\partial L}{\partial \dot{z}^i} + \mathbf{i}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{z}^i}\right) = \wedge^a (\dot{\omega}_a)_i. \tag{1.24}$$

Thus, by *complex Euler-Lagrange equations with constraints* we may call the equations obtained in (1.24) on Kähler manifold TM. Then the quartet $(TM, \Phi_L, \xi, \overline{\omega})$ is named *mechanical system with constraints*.

2. Constrained Paracomplex Mechanical Systems

In this section, as a contribution to the modern development of Lagrangian and Hamiltonian systems of classical mechanics, we present paracomplex analogous of some topics in the geometric theory of constraints [**9, 28, 29**].

Let (T^*Q, Φ, H) be a Hamiltonian system on symplectic manifold T^*Q with closed symplectic form Φ. Let us consider a Hamiltonian system (T^*Q, Φ, H) together with a system $\overline{\omega}$ of constraints on T^*Q. So, it is called $(T^*Q, \Phi, H, \overline{\omega})$ to be a *Hamiltonian system with constraints.* In general, a curve α satisfying the Hamiltonian equations for energy H does not satisfy the constraints. For a curve α satisfying the constraints, some additional forces must act on the system in addition to the *force dH*. So, the dynamical equations of motion become

$$i_Z\Phi = dH + \wedge^a\omega_a,\ \omega_a(Z) = 0, \tag{2.1}$$

where Z is a vector field on T^*Q. From (2.1), Hamilton equations with constraints is given by:

$$\begin{array}{l} \frac{dq^i}{dt} = (\frac{\partial H}{\partial p_i} + \wedge^a(B_a)_i), \\ \frac{dp_i}{dt} = -(\frac{\partial H}{\partial q_i} + \wedge^a(A_a)_i), \\ (A_a)_i\frac{dq^i}{dt} + (B_a)_i\frac{dp_i}{dt} = 0, \end{array} \tag{2.2}$$

where $1 \leq i \leq m$, $1 \leq a \leq s$.

It is well known that (para)Kähler manifolds play an essential role in various areas of mathematics and mathematical physics, in particular, in the theory of dynamical systems, algebraic geometry, the geometry of Einstein manifolds, quantum mechanics, quantum field theory, and in the theory of superstrings and nonlinear sigma-models, too. For example, it was shown in [**30**] that the reflector space of an Einstein self-dual non-Ricci flat 4-manifold as well as the reflector space of a paraquaternionic Kähler manifold admit both Nearly para-Kähler and almost para-Kähler structures. Wade [**31**] showed that generalized paracomplex structures are in one-to-one correspondence with pairs of transversal Dirac structures on a smooth manifold. In [**32**], it was given a representation of the quadratic Dirac equation and the Maxwell equations in terms of the three-dimensional universal complex Clifford algebra $C_{3,0}$. Baylis and Jones introduced in [**33**] that a $R_{3,0}$ Clifford algebra has enough structure to describe relativity as well as the more usual $R_{1,3}$ Dirac algebra or the $R_{3,1}$ Majorana algebra. In [**34**], Baylis represented relativistic space-time points as paravectors and applies these paravectors to electrodynamics. Tekkoyun [**9**] generalized the concept of Hamiltonian dynamics with constraints to complex case. In the above studies; although paracomplex geometry, complex mechanical systems with constraints, Lagrangian and Hamiltonian mechanics were given in a tidy and nice way, they have not dealt with constrained paracomplex mechanical systems.

2.1. Paracomplex Geometry. An *almost product structure* J on a tangent bundle TM of m-real dimensional configuration manifold M is a (1,1) tensor field J on TM such that $J^2 = I$. Here, the pair (TM, J) is called an *almost product manifold.* An *almost paracomplex manifold* is an almost product manifold (TM, J) such that the two eigenbundles TT^+M and TT^-M associated to the eigenvalues $+1$ and -1 of J, respectively, have the same rank. The dimension of an almost paracomplex manifold is necessarily even. Equivalently, a splitting of the tangent bundle TTM of tangent bundle TM, into the Whitney sum of two subbundles on $TT^{\pm}M$ of the same fiber dimension is called an *almost paracomplex structure* on TM. From physical point of view, this splitting means that a reference frame has been chosen. Obviously, such a splitting is broken under reference frame transformations. An almost paracomplex structure on a $2m$-dimensional manifold TM may alternatively be defined as a G-structure on TM with structural group $GL(n, \mathbf{R}) \times GL(n, \mathbf{R})$.

A *paracomplex manifold is* an almost paracomplex manifold (TM, J) such that G- structure defined by tensor field J is integrable. Let (x^i) and $(x^i,\ y^i)$ be a real coordinate system of M and TM, and $\{(\frac{\partial}{\partial x^i})_p, (\frac{\partial}{\partial y^i})_p\}$ and $\{(dx^i)_p, (dy^i)_p\}$ natural bases over $\mathbf{R}$ of tangent space $T_p(TM)$ and cotangent space $T_p^*(TM)$ of TM, respectively. Then, J can be denoted as

$$J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i},\ J(\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial x^i}.$$

Let $z^i = x^i + \mathbf{j}\ y^i$, $\mathbf{j}^2 = 1$, be a paracomplex local coordinate system of TM. The vector and covector fields are defined, respectively, as follows:

$$(\frac{\partial}{\partial z^i})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^i})_p - \mathbf{j}(\frac{\partial}{\partial y^i})_p\},\ (\frac{\partial}{\partial \overline{z}^i})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^i})_p + \mathbf{j}(\frac{\partial}{\partial y^i})_p\},$$

$$\left(dz^i\right)_p = \left(dx^i\right)_p + \mathbf{j}(dy^i)_p,\ \left(d\overline{z}^i\right)_p = \left(dx^i\right)_p - \mathbf{j}(dy^i)_p.$$

The above equations represent the bases of tangent space $T_p(TM)$ and cotangent space $T_p^*(TM)$ of TM, respectively. Then the following results can be easily obtained, respectively:

$$J(\frac{\partial}{\partial z^i}) = -\mathbf{j}\frac{\partial}{\partial z^i},\ J(\frac{\partial}{\partial \overline{z}^i}) = \mathbf{j}\frac{\partial}{\partial \overline{z}^i}, \tag{2.3}$$

$$J^*(dz^i) = -\mathbf{j}dz^i,\ J^*(d\overline{z}^i) = \mathbf{j}d\overline{z}^i. \tag{2.4}$$

Here, J^* stands for the dual endomorphism of cotangent space $T_p^*(TM)$ of manifold TM satisfying $J^{*2} = I$.

An *almost para-Hermitian manifold* (TM, g, J) is a differentiable manifold TM endowed with an almost product structure J and a pseudo-Riemannian metric g, compatible in the sense that

$$g(JX, Y) + g(X, JY) = 0, \quad \forall X, Y \in \chi(TM).$$

An *almost para-Hermitian structure* on a differentiable manifold TM is G-structure on TM whose structural group is the representation of the paraunitary group $U(n, \mathbf{A})$ given in [**14**]. A *para-Hermitian manifold* is a manifold with an integrable almost para-Hermitian structure (g, J). 2-covariant skew tensor field Φ defined by $\Phi(X, Y) = g(X, JY)$ is so-called as *fundamental 2-form*. An almost para-Hermitian manifold (TM, g, J), such that Φ is closed, is so-called as an *almost para-Kähler manifold.*

A para-Hermitian manifold (TM, g, J) is said to be a *para-Kähler manifold* if Φ is closed. Also, by means of geometric structures, one may show that (T^*M, g, J) is a *para-Kähler manifold.*

2.2. Paracomplex Lagrangian Systems. In this subsection, some paracomplex fundamental concepts and para-Euler-Lagrange equations for classical mechanics structured on para-Kähler manifold TM introduced in [**7**] can be recalled.

Let J be an almost paracomplex structure on the para-Kähler manifold and $(z^i, \overline{z}^i)$ its coordinates. Let a second order differential equation be vector field ξ_L given by:

$$\xi_L = \xi^i \frac{\partial}{\partial z^i} + \overline{\xi}^i \frac{\partial}{\partial \overline{z}^i}, \tag{2.5}$$

Then vector field $V = J\xi_L$ is called a *para-Liouville vector field* on the para-Kähler manifold TM. The mappings given by $T, P : TM \to \mathbf{A}$ such that $T = \frac{1}{2} m_i (\dot{z}^i)^2$, $P = m_i \mathbf{g} h$ can be called as *the kinetic energy* and *the potential energy of system,* respectively, where m_i is mass of a mechanical system, $\mathbf{g}$ is the gravity and h is the distance of the mechanical system on the para-Kähler manifold to the origin. Then we call map $L : TM \to \mathbf{A}$ such that $L = T - P$ as *para-Lagrangian function* and the function given by $E_L = V(L) - L$ as *the para-energy function* associated with L.

The operator i_J induced by J and shown as

$$i_J \omega(Z_1, Z_2, ..., Z_r) = \sum_{i=1}^{r} \omega(Z_1, ..., JZ_i, ..., Z_r)$$

is said to be *vertical derivation,* where $\omega \in \wedge^r TM$, $Z_i \in \chi(TM)$. The *vertical differentiation* d_J is defined as follows:

$$d_J = [i_J, d] = i_J d - d i_J,$$

where d is the usual exterior derivation. For almost paracomplex structure J determined by (2.3), the closed para-Kähler form is the closed 2-form given by $\Phi_L = -dd_J L$ such that

$$d_J = -\mathbf{j} \frac{\partial}{\partial z^i} dz^i + \mathbf{j} \frac{\partial}{\partial \overline{z}^i} d\overline{z}^i : \mathcal{F}(TM) \to \wedge^1 TM.$$

Paracomplex-Euler-Lagrange equations on para-Kähler manifold TM are shown by

$$\mathbf{j}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial z^i}\right)+\frac{\partial L}{\partial z^i}=0,\ \mathbf{j}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \overline{z}^i}\right)-\frac{\partial L}{\partial \overline{z}^i}=0. \tag{2.6}$$

Thus, the triple (TM, Φ_L, ξ) is called a *paracomplex-mechanical system.*

2.3. Paracomplex Hamiltonian Systems. Here, we consider paracomplex-Hamilton equations for classical mechanics structured on para-Kähler manifold T^*M introduced in [**7**]. Let T^*M be any para-Kähler manifold and $(z_i, \overline{z}_i)$ its coordinates. $\{\frac{\partial}{\partial z_i}|_p, \frac{\partial}{\partial \overline{z}_i}|_p\}$ and $\{dz_i|_p, d\overline{z}_i|_p\}$ be bases over paracomplex number $\mathbf{A}$ of tangent space $T_p(TM)$ and cotangent space $T_p^*(TM)$ of TM. Assume that J^* is an almost paracomplex structure given by $J^*(dz_i) = -\mathbf{j}dz_i$, $J^*(d\overline{z}_i) = \mathbf{j}d\overline{z}_i$ and λ is a para-Liouville form given by $\lambda = J^*(\omega) = \frac{1}{2}\mathbf{j}(z_i d\overline{z}_i - \overline{z}_i dz_i)$ such that paracomplex 1-form $\omega = \frac{1}{2}(z_i d\overline{z}_i + \overline{z}_i dz_i)$ on T^*M. If $\Phi = -d\lambda$ is closed para-Kähler form, then Φ is also a para-symplectic structure on T^*M.

Let T^*M be para-Kähler manifold with closed para-Kähler form Φ. Then para-Hamiltonian vector field Z_H on T^*M with closed form Φ can be given by:

$$Z_H = -\mathbf{j}\frac{\partial H}{\partial \overline{z}_i}\frac{\partial}{\partial z_i}+\mathbf{j}\frac{\partial H}{\partial z_i}\frac{\partial}{\partial \overline{z}_i}. \tag{2.7}$$

According to (2.4), *para-Hamiltonian equations* on para-Kähler manifold T^*M are denoted by equations of

$$\frac{dz_i}{dt} = -\mathbf{j}\frac{\partial H}{\partial \overline{z}_i},\ \frac{d\overline{z}_i}{dt} = \mathbf{j}\frac{\partial H}{\partial z_i}. \tag{2.8}$$

Example 1. *A central force field $f(\rho) = A\rho^{\alpha-1}(\alpha \neq 0, 1)$ acts on a body with mass m in a constant gravitational field. Then let us find out the para-Lagrangian and para-Hamiltonian equations of the motion by assuming the body always on the vertical plane.*

The para-Lagrangian and para-Hamiltonian functions of the system are, respectively,

$$L = \frac{1}{2}m\dot{z}\dot{\overline{z}} - \frac{A}{\alpha}(\sqrt{z\overline{z}})^{\alpha} - \mathbf{j}mg\frac{(z-\overline{z})\sqrt{z\overline{z}}}{(z+\overline{z})\sqrt{1-\frac{(z-\overline{z})^2}{(z+\overline{z})^2}}},$$

$$H = \frac{1}{2}m\dot{z}\dot{\overline{z}} + \frac{A}{\alpha}(\sqrt{z\overline{z}})^{\alpha} + \mathbf{j}mg\frac{(z-\overline{z})\sqrt{z\overline{z}}}{(z+\overline{z})\sqrt{1-\frac{(z-\overline{z})^2}{(z+\overline{z})^2}}}.$$

Then, using (2.6) and (2.8), the so-called para-Lagrangian and para-Hamiltonian equations of the motion on the para-mechanical systems, can be obtained, respectively, as follows:

$$L1:\mathbf{j}\frac{\partial}{\partial t}S-S=0,\ L2:\mathbf{j}\frac{\partial}{\partial t}U+U=0,$$

such that

$$\begin{aligned} S &= -\frac{A}{2z}(\sqrt{z\overline{z}})^{\alpha}-\mathbf{j}\frac{mg(z-\overline{z})\overline{z}}{2\sqrt{z\overline{z}}(z+\overline{z})W}-\mathbf{j}\frac{mg\sqrt{z\overline{z}}}{(z+\overline{z})W} \\ &\quad +\mathbf{j}\frac{mg\sqrt{z\overline{z}}(z-\overline{z})}{(z+\overline{z})^2W}+\mathbf{j}\frac{mg\sqrt{z\overline{z}}(z-\overline{z})(-\frac{(z-\overline{z})}{(z+\overline{z})^2}+\frac{(z-\overline{z})^2}{(z+\overline{z})^3})}{(z+\overline{z})W^3}, \\ U &= -\frac{A}{2\overline{z}}(\sqrt{z\overline{z}})^{\alpha}-\mathbf{j}\frac{mg(z-\overline{z})z}{2\sqrt{z\overline{z}}(z+\overline{z})W}+\mathbf{j}\frac{mg\sqrt{z\overline{z}}}{(z+\overline{z})W} \\ &\quad +\mathbf{j}\frac{mg\sqrt{z\overline{z}}(z-\overline{z})}{(z+\overline{z})^2W}+\mathbf{j}\frac{mg\sqrt{z\overline{z}}(z-\overline{z})(\frac{(z-\overline{z})}{(z+\overline{z})^2}+\frac{(z-\overline{z})^2}{(z+\overline{z})^3})}{(z+\overline{z})W^3} \end{aligned}$$

and

$$\begin{aligned} H1 &: \frac{dz}{dt}=-\mathbf{j}(\frac{A}{2\overline{z}}(\sqrt{z\overline{z}})^{\alpha}+\mathbf{j}\frac{mg(z-\overline{z})z}{2\sqrt{z\overline{z}}(z+\overline{z})W}-\mathbf{j}\frac{mg\sqrt{z\overline{z}}}{(z+\overline{z})W} \\ &\quad -\mathbf{j}\frac{mg\sqrt{z\overline{z}}(z-\overline{z})}{(z+\overline{z})^2W}-\mathbf{j}\frac{mg\sqrt{z\overline{z}}(z-\overline{z})(\frac{(z-\overline{z})}{(z+\overline{z})^2}+\frac{(z-\overline{z})^2}{(z+\overline{z})^3})}{(z+\overline{z})W^3}), \\ H2 &: \frac{d\overline{z}}{dt}=\mathbf{j}(\frac{A}{2z}(\sqrt{z\overline{z}})^{\alpha}+\mathbf{j}\frac{mg(z-\overline{z})\overline{z}}{2\sqrt{z\overline{z}}(z+\overline{z})W}+\mathbf{j}\frac{mg\sqrt{z\overline{z}}}{(z+\overline{z})W} \\ &\quad -\mathbf{j}\frac{mg\sqrt{z\overline{z}}(z-\overline{z})}{(z+\overline{z})^2W}-\mathbf{j}\frac{mg\sqrt{z\overline{z}}(z-\overline{z})(-\frac{(z-\overline{z})}{(z+\overline{z})^2}+\frac{(z-\overline{z})^2}{(z+\overline{z})^3})}{(z+\overline{z})W^3}). \end{aligned}$$

where $W=\sqrt{1-\frac{(z-\overline{z})^2}{(z+\overline{z})^2}}$.

2.4. Constrained Paracomplex Lagrangians. In this subsection, we obtain para-Euler-Lagrange equations with constraints for classical mechanics structured on para-Kähler manifold TM.

Let J be an almost paracomplex structure on the para-Kähler manifold and $(z^i,\overline{z}^i)$ its coordinates. Let us take a second order differential equation to the vector field ξ given by:

$$\xi=\xi_L+\wedge^a\omega_a=\xi^i\frac{\partial}{\partial z^i}+\overline{\xi}^i\frac{\partial}{\partial \overline{z}^i}+\wedge^a\omega_a,\ 1\le a\le r, \tag{2.9}$$

The vector field $V=J\xi_L$ calculated by

$$-\mathbf{j}\xi^i\frac{\partial}{\partial z^i}+\mathbf{j}\overline{\xi}^i\frac{\partial}{\partial \overline{z}^i},$$

is *para-Liouville vector field* on the para-Kähler manifold TM. The closed 2-form expressed by $\Phi_L = -dd_J L$ is found to be:

$$\begin{aligned}\Phi_L \;\;=\;\; & -\mathbf{j}\frac{\partial^2 L}{\partial z^j \partial z^i} dz^j \wedge dz^i + \mathbf{j}\frac{\partial^2 L}{\partial \overline{z}^j \partial z^i} d\overline{z}^j \wedge dz^i \\ & -\mathbf{j}\frac{\partial^2 L}{\partial z^j \partial \overline{z}^i} dz^j \wedge d\overline{z}^i - \mathbf{j}\frac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i} d\overline{z}^j \wedge d\overline{z}^i,\end{aligned}$$

where

$$d_J = -\mathbf{j}\frac{\partial}{\partial z^i} dz^i + \mathbf{j}\frac{\partial}{\partial \overline{z}^i} d\overline{z}^i : \mathcal{F}(TM) \to \wedge^1 TM.$$

If ξ is a second order differential equation defined by (1.1), then we have (2.10)

$$\begin{aligned} i_\xi \Phi_L = & -\mathbf{j}\xi^i \tfrac{\partial^2 L}{\partial z^j \partial z^i}\delta_i^j dz^i + \mathbf{j}\xi^i \tfrac{\partial^2 L}{\partial z^j \partial z^i} dz^j + \mathbf{j}\overline{\xi}^i \tfrac{\partial^2 L}{\partial \overline{z}^j \partial z^i}\delta_i^j dz^i - \mathbf{j}\xi^i \tfrac{\partial^2 L}{\partial \overline{z}^j \partial z^i} d\overline{z}^j \\ & -\mathbf{j}\xi^i \tfrac{\partial^2 L}{\partial z^j \partial \overline{z}^i}\delta_i^j d\overline{z}^i + \mathbf{j}\overline{\xi}^i \tfrac{\partial^2 L}{\partial z^j \partial \overline{z}^i} dz^j - \mathbf{j}\overline{\xi}^i \tfrac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i}\delta_i^j d\overline{z}^i + \mathbf{j}\overline{\xi}^i \tfrac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i} d\overline{z}^j. \end{aligned}$$

Since closed para-Kähler form Φ_L on TM is para-symplectic structure, we find

$$E_L = -\mathbf{j}\xi^i \frac{\partial L}{\partial z^i} + \mathbf{j}\overline{\xi}^i \frac{\partial L}{\partial \overline{z}^i} - L$$

and hence

$$\begin{aligned} dE_L + \wedge^a \omega_a = \;& -\mathbf{j}\xi^i \tfrac{\partial^2 L}{\partial z^j \partial z^i} dz^j + \mathbf{j}\overline{\xi}^i \tfrac{\partial^2 L}{\partial z^j \partial \overline{z}^i} dz^j - \tfrac{\partial L}{\partial z^j} dz^j \\ & -\mathbf{j}\xi^i \tfrac{\partial^2 L}{\partial \overline{z}^j \partial z^i} d\overline{z}^j + \mathbf{j}\overline{\xi}^i \tfrac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i} d\overline{z}^j - \tfrac{\partial L}{\partial \overline{z}^j} d\overline{z}^j + \wedge^a \omega_a. \end{aligned} \tag{2.11}$$

According to (1.1), if (2.10) and (2.11) are equal to each other, then the following equation can be obtained:

$$\begin{aligned} & +\mathbf{j}\xi^i \tfrac{\partial^2 L}{\partial z^j \partial z^i} dz^j + \mathbf{j}\overline{\xi}^i \tfrac{\partial^2 L}{\partial \overline{z}^j \partial z^i} dz^j + \tfrac{\partial L}{\partial z^j} dz^j \\ & -\mathbf{j}\xi^i \tfrac{\partial^2 L}{\partial z^j \partial \overline{z}^i} d\overline{z}^j - \mathbf{j}\overline{\xi}^i \tfrac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i} d\overline{z}^j + \tfrac{\partial L}{\partial \overline{z}^j} d\overline{z}^j = \wedge^a \omega_a \end{aligned}$$

Now, let curve $\alpha : \mathbf{A} \to TM$ be integral curve of ξ, which satisfies equations of

$$\begin{aligned} & +\mathbf{j}\left[\xi^j \tfrac{\partial^2 L}{\partial z^j \partial z^i} + \dot{\xi}^i \tfrac{\partial^2 L}{\partial \dot{z}^j \partial z^i}\right] dz^j + \tfrac{\partial L}{\partial z^j} dz^j \\ & -\mathbf{j}\left[\xi^j \tfrac{\partial^2 L}{\partial z^j \partial \dot{z}^i} + \dot{\xi}^j \tfrac{\partial^2 L}{\partial \dot{z}^j \partial \dot{z}^i}\right] d\dot{z}^j + \tfrac{\partial L}{\partial \dot{z}^j} d\dot{z}^j = \wedge^a \omega_a, \end{aligned}$$

where the dots mean derivatives with respect to time and $\omega_a = (\omega_a)_i\, dz^i + (\dot{\omega}_a)_i\, d\dot{z}^i$.

This refers to equations of

$$\frac{\partial L}{\partial z^i} + \mathbf{j}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial z^i}\right) = \wedge^a (\omega_a)_i, \;\; \frac{\partial L}{\partial \dot{z}^i} - \mathbf{j}\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{z}^i}\right) = \wedge^a (\dot{\omega}_a)_i. \tag{2.12}$$

Thus, the equations obtained in (2.12) on para-Kähler manifold TM are so-called as *constrained paracomplex Euler-Lagrange equations*. Then the quartet $(TM, \Phi_L, \xi, \overline{\omega})$ is named *constrained paracomplex mechanical system.*

2.5. Constrained Paracomplex Hamiltonians. Here, we conclude paracomplex Hamiltonian equations with constraints on para-Kähler manifold T^*M. Similar to (0.3), the vector fields on T^*M satisfying the condition

$$i_{Za}\Phi = \omega_a,\ 1 \leq a \leq s, \tag{2.13}$$

can be represented by Z_a.

PROPOSITION 5. *Let T^*M be para-Kähler manifold with closed para-Kähler form Φ. Let us consider vector field Z_a on T^*M given by:*

$$Z_a = -\mathbf{j}(B_a)_i \frac{\partial}{\partial z_i} + \mathbf{j}(A_a)_i \frac{\partial}{\partial \overline{z}_i}. \tag{2.14}$$

PROOF. Let T^*M be para-Kähler manifold with form Φ. Consider that vector field Z_a is given by

$$Z_a = (Z_a)_i \frac{\partial}{\partial z_i} + (\overline{Z}_a)_i \frac{\partial}{\partial \overline{z}_i}.$$

From (2.13), $i_{Z_a}\Phi$ can be calculated as

$$i_{Z_a}(-d\lambda) = \mathbf{j}(\bar{Z}_a)_i dz_i - \mathbf{j}(Z_a)_i d\overline{z}_i. \tag{2.15}$$

Moreover, we set

$$\omega_a = (A_a)_i dz_i + (B_a)_i d\overline{z}_i \tag{2.16}$$

According to (2.13), if (2.15) and (2.16) are equal to each other, proof finishes. □

Now, with the case of (0.3), (2.1) and (2.13); one may easily deduce

$$Z = Z_H + \wedge^a Z_a. \tag{2.17}$$

Hence, by means of (2.8), (2.14) and (2.17) we obtain the following vector field

$$Z = -\mathbf{j}(\frac{\partial H}{\partial \overline{z}_i} + \wedge^a (B_a)_i) \frac{\partial}{\partial z_i} + \mathbf{j}(\frac{\partial H}{\partial z_i} + \wedge^a (A_a)_i) \frac{\partial}{\partial \overline{z}_i}. \tag{2.18}$$

Suppose that curve

$$\alpha : I \subset \mathbf{A} \to T^*M$$

be an integral curve of paracomplex vector field Z given by (2.18), i.e.,

$$Z(\alpha(t)) = \dot{\alpha}(t),\ t \in I.$$

In the local coordinates, for $\alpha(t) = (z_i(t), \overline{z}_i(t))$, we have

$$\dot{\alpha}(t) = \frac{dz_i}{dt} \frac{\partial}{\partial z_i} + \frac{d\overline{z}_i}{dt} \frac{\partial}{\partial \overline{z}_i}.$$

Then we reach the following equations

$$\begin{aligned} &\tfrac{dz_i}{dt} = -\mathbf{j}(\tfrac{\partial H}{\partial \overline{z}_i} + \wedge^a (B_a)_i), \\ &\tfrac{d\overline{z}_i}{dt} = \mathbf{j}(\tfrac{\partial H}{\partial z_i} + \wedge^a (A_a)_i), \\ &(A_a)_i \tfrac{dz_i}{dt} + (B_a)_i \tfrac{d\overline{z}_i}{dt} = 0, \end{aligned} \tag{2.19}$$

which are so-called as *constrained paracomplex Hamiltonian equations* on para-Kähler manifold T^*M. Here $1 \leq a \leq s$. Then the quartet $(T^*M, \Phi, H, \overline{\omega})$ is named *constrained paracomplex mechanical system.*

Conclusion 7. *Finally, considering the above, complex analogous of the geometrical and mechanical meaning of constraints given in* [**2, 29**] *may be explained as follows.*

1) *Let $\overline{\omega}$ be a system of constraints on Kähler manifold TM. Then it may be defined a distribution D on $\overline{\omega}$ as follows.*

$$D(x) = \{\, \xi \in T_x TM |\; \omega_a(\xi) = 0, \;\; \textit{for all} \;\; a, \, 1 \leq a \leq r\} \tag{2.20}$$

Thus D is $(2m-r)$ dimensional distribution on TM. In this case, a system of complex constraints $\overline{\omega}$ is called holonomic, if the distribution D is integrable; otherwise we call $\overline{\omega}$ anholonomic. Hence, $\overline{\omega}$ is holonomic if and only if the ideal ρ of $\wedge TM$ generated by $\overline{\omega}$ is a differential ideal. Obviously (1.24) holds for holonomic as well as anholonomic constraints. For a system of holonomic constraints, the motion lies on a specific leaf of the foliation defined by D.

2) *From (1.1) it is obtained equalities of*

$$0 = (i_\xi \Phi)(\xi) = dE_L(\xi) = \xi(E_L), \tag{2.21}$$

Therefore, the Lagrangian energy E_L on Kähler manifold TM for a solution $\alpha(t)$ of (1.24) is conserved.

Considering the above, paracomplex analogous of the geometrical and mechanical meaning of constraints given in [**2, 9, 28, 29**] *can be explained as follows:*

3) *Let $\overline{\omega}$ be a system of constraints on para-Kähler manifold TM or T^*M. Then it may be defined a distribution D or D^* on $\overline{\omega}$ as follows:*

$$\begin{aligned} &D(x) = \{\, \xi \in T_x TM |\; \omega_a(\xi) = 0, \;\; \textit{for all} \;\; a, \, 1 \leq a \leq r\} \\ &D^*(x) = \{\, Z \in T_x T^*M) |\; \omega_a(Z) = 0, \;\; \textit{for all} \;\; a, 1 \leq a \leq s\} \end{aligned} \tag{2.22}$$

Thus D or D^* is $(2m-r)$ or $(2m-s)$-dimensional distribution on TM or T^*M. In this case, a system of paracomplex constraints $\overline{\omega}$ is paraholonomic, if the distribution D or D^* is integrable; otherwise $\overline{\omega}$ is paraanholonomic. Hence, $\overline{\omega}$ is paraholonomic if and only if the ideal ρ of $\wedge TM$ or $\wedge T^*M$ generated by $\overline{\omega}$ is a differential ideal, i.e., $d\rho \subset \rho$. Obviously, (2.12) and (2.19) hold both paraholonomic and paraanholonomic constraints. The motion for a system of paraholonomic constraints lies on a specific leaf of the foliation defined by D or D^*.

4) From (1.1) and (2.1), the following equalities can be obtained:

$$0 = (i_\xi \Phi)(\xi) = dE_L(\xi) = \xi(E_L),$$
$$0 = (i_Z \omega)(Z) = dH(Z) = Z(H). \tag{2.23}$$

So, Lagrangian energy E_L and Hamiltonian energy H of (2.12) and (2.19) for a solution $\alpha(t)$ are, respectively, conserved.

CHAPTER 8

Mechanical Systems on Distributions

As well-known Lagrangian distribution on symplectic manifolds are used in geometric quantization and a connection on a symplectic manifold is an important structure to obtain a deformation quantization [**35**].

In this chapter, by means of an almost product structure, we present Euler-Lagrange and Hamilton equations given in [**36**]. They are related to mechanical systems on the horizontal and vertical distributions of the bundles used in obtaining geometric quantization.

1. Manifolds, Bundles and Distributions

Here, we extend some definitions introduced in [**37**]. Let TM be tangent bundle of a manifold M of dimension n. Denote by x a point of M such that $\varphi(x) = (x^i)$. Given the projection $\pi : TM \to M$, $\pi(u) = x$. Let (x^i, y^i) be a real coordinate system on a neighborhood (U, φ) of any point u of TM. Then we respectively define by $(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i})$ and (dx^i, dy^i) the natural bases over R of the tangent space T_uTM and the cotangent space $T_u^*(TM)$ at the point $u \in TM$, respectively. And also $\mathcal{F}(TM)-$ and $\mathcal{F}(T^*M)-$ linear mappings (named to be almost tangent structures) $J : \chi(TM) \to \chi(TM)$ and $J^* : \chi(TM) \to \chi(TM)$ are given as follows:

$$J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i},\ J(\frac{\partial}{\partial y^i}) = 0,$$

and

$$J^*(dx^i) = dy^i, J^*(dy^i) = 0.$$

Consider that the tangent space V_u to the fibre $\pi^{-1}(x)$ in the point $u \in TM$ is locally spanned by $\{\frac{\partial}{\partial y^1}, .., \frac{\partial}{\partial y^n}\}$. The mapping given by V: $u \in TM \to V_u \subset T_uTM$ provides a regular distribution generated by the adapted basis $\{\frac{\partial}{\partial y^i}\}$. So, V is an integrable distribution on TM. And then one says that V is the vertical distribution on TM. Let N be a nonlinear connection on TM. N is characterized by v, h vertical and horizontal projectors. Assume that the vertical projector $v : \chi(TM) \to \chi(TM)$ is defined by $v(X) = X, \forall X \in \chi(VTM)$; $v(X) = 0, \forall X \in \chi(HTM)$. Similarly, the mapping given by H: $u \in TM \to H_u \subset T_uTM$ provides a regular distribution determined by the adapted basis $\{\frac{\delta}{\delta x^i}\}$. Therefore, H is an integrable distribution on TM. Finally we call to be H the horizontal distribution on TM. Suppose that there is a $\mathcal{F}(TM)-$linear mapping $h : \chi(TM) \to \chi(TM)$, for which $h^2 = h$, Ker

$h = \chi(VTM)$. If X^H and X^V are horizontal and vertical components of vector field X, respectively, then any vector field $X \in \chi(TM)$ can be uniquely given by

$$X = hX + vX = X^H + X^V$$

such that

$$X^H = X^i(\frac{\partial}{\partial x^i} - N^i_j(x,y)\frac{\partial}{\partial y^j}), \;\; X^V = X^i N^i_j(x,y)\frac{\partial}{\partial y^j}$$

where N^i_j are a local coefficients of a nonlinear connection N on TM.

A local basis adapted to the horizontal and vertical distribution denoted by HTM and VTM is $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$. Then $(dx^i, \delta y^i)$ is dual basis of $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$ basis. Let P be an almost product structure on TM. So, we have

$$\begin{array}{l} P(X) = X, \forall X \in \chi(HTM); \;\; P(X) = -X, \forall X \in \chi(VTM) \\ P^*(\omega) = \omega, \forall \omega \in \chi(HT^*M); \;\; P^*(\omega) = -\omega, \forall \omega \in \chi(VT^*M), \end{array}$$

where P^* is the dual structure of P. Also, we have

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N^i_j(x,y)\frac{\partial}{\partial y^j}.$$

and

$$\delta y^i = dy^i + N^i_j(x,y)dx^j.$$

Taking into consideration the operators h, v, P, P^*, J, J^*constructed on the distributions HTM, VTM, HT^*M, VT^*M of bundles TM and T^*M of M, one writes the following equalities:

$$\begin{array}{l} h + v = I, \;\; P = 2h - I, \;\; P = h - v, \;\; P = I - 2v, \\ JP = J, \;\; PJ = -J, \;\; J^*P^* = J^*, \;\; P^*J^* = -J^*, \\ h(\frac{\delta}{\delta x^i}) = \frac{\delta}{\delta x^i}, \;\; h(\frac{\partial}{\partial y^i}) = 0, \;\; v(\frac{\delta}{\delta x^i}) = 0, \;\; v(\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial y^i}, \\ P(\frac{\delta}{\delta x^i}) = \frac{\delta}{\delta x^i}, \;\; P(\frac{\partial}{\partial y^i}) = -\frac{\partial}{\partial y^i}, \\ P^*(dx^i) = dx^i, \;\; P^*(\delta y^i) = -\delta y^i. \end{array}$$

1.1. Lagrangian Mechanical Systems on Distributions. Here, we present Euler-Lagrange equations for classical mechanics structured by means of almost product structure P under the consideration of the basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ on distributions HTM and VTM of tangent bundle TM of manifold M. Let (x^i, y^i) be local coordinates. Also, let semispray be the vector field X given by

$$X = X^i\frac{\delta}{\delta x^i} + \dot{X}^i\frac{\partial}{\partial y^i}, \;\; \dot{X}^i = X^i N^i_j$$

where the dot indicates the derivative with respect to time t. Then the vector field given by

$$V = P(X) = X^i\frac{\delta}{\delta x^i} - \dot{X}^i\frac{\partial}{\partial y^i}$$

is called *Liouville vector field* on the bundle TM. The maps given by $\mathbf{T}, \mathbf{P}$: $TM \to \mathbf{R}$ such that $\mathbf{T} = \frac{1}{2}m_i(\dot{x}^i)^2, \mathbf{P} = m_i gh$ are called *the kinetic energy* and *the potential energy of the mechanical system,* respectively. Where m_i is the mass of a mechanical system having m particles, g is the gravity acceleration and h is the distance to the origin of a mechanical system on the tangent bundle TM. Then $L : TM \to \mathbf{R}$ is a map that satisfies the conditions: i) $L = \mathbf{T} - \mathbf{P}$ is a *Lagrangian function, ii)* the function given by $E_L = V(L) - L$ is *a Lagrangian energy.* The operator i_P shown by

$$i_P\omega(X_1, X_2, ..., X_r) = \sum_{i=1}^{r} \omega(X_1, ..., P(X_i), ..., X_r)$$

is said to be *vertical derivation,* where $\omega \in \wedge^r TM$, $X_i \in \chi(TM)$. The *vertical differentiation* d_P is given by

$$d_P = [i_P, d] = i_P d - d i_P,$$

where d is the usual exterior derivation. It is well known that the closed fundamental form is the closed 2-form given by $\Phi_L = -dd_P L$ such that

$$d_P : \mathcal{F}(TM) \to T^*M.$$

Then we have

$$\begin{aligned}\Phi_L &= -(\tfrac{\delta}{\delta x^j}dx^j + \tfrac{\partial}{\partial y^j}\delta y^j)(\tfrac{\delta L}{\delta x^i}dx^i - \tfrac{\partial L}{\partial y^i}\delta y^i) \\ &= \tfrac{\delta^2 L}{\delta x^j \delta x^i}dx^j \wedge dx^i - \tfrac{\delta(\partial L)}{\delta x^j \partial y^i}dx^j \wedge \delta y^i - \tfrac{\partial(\delta L)}{\partial y^j \delta x^i}\delta y^j \wedge dx^i + \tfrac{\partial^2 L}{\partial y^j \partial y^i}\delta y^j \wedge \delta y^i.\end{aligned}$$

and

(1.1)

$$\begin{aligned} i_X\Phi_L = &-X^i \tfrac{\delta^2 L}{\delta x^j \delta x^i}\delta_i^j dx^i + X^i \tfrac{\delta^2 L}{\delta x^j \delta x^i}dx^j + X^i \tfrac{\delta(\partial L)}{\delta x^j \partial y^i}\delta_i^j \delta y^i - \dot{X}^i \tfrac{\delta(\partial L)}{\delta x^j \partial y^i}dx^j \\ &-\dot{X}^i \tfrac{\partial(\delta L)}{\partial y^j \delta x^i}\delta_i^j dx^i + X^i \tfrac{\partial(\delta L)}{\partial y^j \delta x^i}\delta y^j + \dot{X}^i \tfrac{\partial^2 L}{\partial y^j \partial y^i}\delta_i^j \delta y^i - \dot{X}^i \tfrac{\partial^2 L}{\partial y^j \partial y^i}\delta y^j. \end{aligned}$$

Because the closed 2-form Φ_L is in the symplectic structure, one obtains

$$E_L = V(L) - L = X^i \frac{\delta L}{\delta x^i} - \dot{X}^i \frac{\partial L}{\partial y^i} - L$$

and hence

$$\begin{aligned} dE_L = &X^i \tfrac{\delta^2 L}{\delta x^j \delta x^i}dx^j - \dot{X}^i \tfrac{\delta(\partial L)}{\delta x^j \partial y^i}dx^j - \tfrac{\delta L}{\delta x^j}dx^j \\ &+X^i \tfrac{\partial(\delta L)}{\partial y^j \delta x^i}\delta y^j - \dot{X}^i \tfrac{\partial^2 L}{\partial y^j \partial y^i}\delta y^j - \tfrac{\partial L}{\partial y^j}\delta y^j \end{aligned} \tag{1.2}$$

By means of (0.1), (1.1), (1.2) we find

$$\begin{aligned} &-X^i \tfrac{\delta^2 L}{\delta x^j \delta x^i}dx^j - \dot{X}^i \tfrac{\partial(\delta L)}{\partial y^j \delta x^i}dx^j + \tfrac{\delta L}{\delta x^j}dx^j \\ &+X^i \tfrac{\delta(\partial L)}{\delta x^j \partial y^i}\delta y^j + \dot{X}^i \tfrac{\partial^2 L}{\partial y^j \partial y^i}\delta y^j + \tfrac{\partial L}{\partial y^j}\delta y^j = 0. \end{aligned}$$

Taking a curve $\alpha : \mathbf{R} \to TM$ being an integral curve of X, i.e. $X(\alpha(t)) = \frac{d\alpha(t)}{dt}$, then we introduce the equations given by

$$\frac{d}{dt}(\frac{\delta L}{\delta x^i}) - \frac{\delta L}{\delta x^i} = 0, \; \frac{d}{dt}(\frac{\partial L}{\partial y^i}) + \frac{\partial L}{\partial y^i} = 0. \tag{1.3}$$

Thus the equations obtained by (1.3) are shown to be *Euler-Lagrange equations* on HTM horizontal and VTM vertical distributions, and then the triple (TM, Φ_L, X) is named to be a *mechanical system* with taking into account almost product structure P especially and the basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ on the distributions HTM and VTM.

2. Hamiltonian Mechanical Systems on Distributions

Now, here we obtain Hamiltonian equations for classical mechanics constructed on the distributions HT^*M and VT^*M. By P^*, λ and ω we denote an almost product structure, a Liouville form and a 1-form on T^*M, respectively. Then we can write

$$\omega = \frac{1}{2}(y^i dx^i + x^i \delta y^i)$$

and

$$\lambda = P^*(\omega) = \frac{1}{2}(y^i dx^i - x^i \delta y^i).$$

As ϕ is a closed 2- form on T^*M, then ϕ_H is also a symplectic structure on T^*M. If Hamiltonian vector field X_H is given by

$$X_H = X^i \frac{\delta}{\delta x^i} + Y^i \frac{\partial}{\partial y^i},$$

then we have

$$\phi_H = -d\lambda = -\delta y^i \wedge dx^i$$

and

$$i_{X_H}\phi = -Y^i dx^i + X^i \delta y^i. \tag{2.1}$$

Besides, the differential of Hamiltonian energy is

$$dH = \frac{\delta H}{\delta x^i} dx^i + \frac{\partial H}{\partial y^i} \delta y^i. \tag{2.2}$$

By (0.3), (2.1), (2.2), one finds

$$X_H = \frac{\partial H}{\partial y^i} \frac{\delta}{\delta x^i} - \frac{\delta H}{\delta x^i} \frac{\partial}{\partial y^i}. \tag{2.3}$$

Consider that a curve

$$\alpha : I \subset \mathbf{R} \to T^*M$$

be an integral curve of the Hamiltonian vector field X_H, i.e.,

$$X_H(\alpha(t)) = \frac{d\alpha(t)}{dt}, \ t \in I. \tag{2.4}$$

Then we can write the equations

$$\alpha(t) = (x^i(t), y^i(t))$$

and

$$\frac{d\alpha(t)}{dt} = \frac{dx^i}{dt} \frac{\delta}{\delta x^i} + \frac{dy^i}{dt} \frac{\partial}{\partial y^i}, \tag{2.5}$$

in the local coordinates. Using (2.3), (2.4), (2.5), one gets the result equations as follows:

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial y^i}, \ \frac{dy^i}{dt} = -\frac{\delta H}{\delta x^i}. \tag{2.6}$$

Thus, the equations (2.6) are named to be *Hamilton equations* on the horizontal distributionHT^*M and vertical distribution VT^*M, and then the triple (T^*M, ϕ_H, X_H) is seen to be a *Hamiltonian mechanical system* with the use of almost product structure P^* and basis $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ on the distributions HT^*M and VT^*M.

CONCLUSION 8. *Lagrangian and Hamiltonian dynamics have intrinsically been described with almost product structure and taking into account the basis* $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$ *and dual basis* $(dx^i, \delta y^i)$ *on distributions of tangent and cotangent bundles* TM *and* T^*M *of manifold* M*. As is well known, geometry of Lagrangians and Hamiltonians introduces a model for relativity, Gauge theory, electromagnetism, quantum mechanics, analytical mechanics and classical fields theory. These geometrical models determine the characteristics properties of these physical fields. Therefore we say that the equations (1.3) and (2.6) especially can be used in the above fields.*

CHAPTER 9

Bi-Para Mechanical Systems on Lagrangian Distributions

Some works in paracomplex geometry are used for mathematical models. These works can be the papers numbered as [**14, 38, 39, 40**] at the end of this document. The first reference is a well-known survey about paracomplex geometry. In the second reference the authors study the paraholomorphic functions and manifolds modelled over the paracomplex numbers. The last reference is the classical paper about paracomplex structures of Kaneyuki and Kozai. As known, Lagrangian foliations on symplectic manifolds are used in geometric quantization and a connection on a symplectic manifold is an important structure to obtain a deformation quantization. A para-Kähler manifold M is said to be endowed with an almost bi-para-Lagrangian structure (a bi-para-Lagrangian manifold) if M has two transversal Lagrangian distributions (involutive transversal Lagrangian distributions) D_1 and D_2 [**35**].

In this chapter, equations related to bi-para-mechanical systems on the bi-Lagrangian manifold given in [**41**] and used in obtaining geometric quantization have been presented.

1. Bi-Para-Complex Geometry

An almost bi-para-complex structure on a differentiable manifold is given by two tensor fields F and P of type (1, 1) verifying $F^2 = P^2 = 1$, $F \circ P + P \circ F = 0$ (see [**35**]). The name of bi-para-complex manifold is due to the existence of two almost paracomplex structures on M, the tensor fields F and P. Note that $P \circ F$ is an almost complex structure.

If the G-structure defined by the almost bi-para-complex structure is integrable then for every point $p \in M$ there exists an open neighborhood U of p and local coordinates $(U; x^1, ..., x^n, y^1, ..., y^n)$ such that

$$\begin{aligned} F(\partial/\partial x^i) &= \partial/\partial y^i, F(\partial/\partial y^i) = \partial/\partial x^i, \\ P(\partial/\partial x^i) &= \partial/\partial x^i, P(\partial/\partial y^i) = -\partial/\partial y^i, \forall i = 1, ..., n, \end{aligned}$$

(see [**42**]). The existence of these kind of local coordinates on M permit to construct holomorphic local coordinates, $(U; z^1, ..., z^n), z^k = x^k + \mathbf{i}y^k$, $k = 1, ..., n$, or paraholomorphic local coordinates, $(U; z^1, ..., z^n), z^k = x^k + \mathbf{j}y^k, k = 1, ..., n$, where $\mathbf{i}^2 = -1$ and $\mathbf{j}^2 = 1$ (see [**38, 40**]).

A para-Kähler manifold (M, g, J) always posses two transversal distributions defined by the eigenspaces associated to the $+1$ and -1 eigenvalues of J. Moreover, these distributions are involutive Lagrangian distributions if one considers the symplectic form Φ defined by

$$\Phi(X, Y) = g(JX, Y), \forall X, Y \in \chi(M).$$

Let $(x^i,\ y^i)$ be a real coordinate system on a neighborhood U of any point p of M, and let $\{(\frac{\partial}{\partial x^i})_p, (\frac{\partial}{\partial y^i})_p\}$ and $\{\left(dx^i\right)_p, \left(dy^i\right)_p\}$ be natural bases over R of the tangent space $T_p(M)$ and the cotangent space $T_p^*(M)$ of M, respectively. Then the definitions can be given by

$$J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i},\ J(\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial x^i}.$$

Let $z^i = x^i + \mathbf{j}y^i$, $\mathbf{j}^2 = 1$, also be a para-complex local coordinate system on a neighborhood U of any point p of M. The vector fields can then be shown:

$$(\frac{\partial}{\partial z^i})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^i})_p - \mathbf{j}(\frac{\partial}{\partial y^i})_p\},\ (\frac{\partial}{\partial \overline{z}^i})_p = \frac{1}{2}\{(\frac{\partial}{\partial x^i})_p + \mathbf{j}(\frac{\partial}{\partial y^i})_p\}.$$

And the dual covector fields are:

$$\left(dz^i\right)_p = \left(dx^i\right)_p + \mathbf{j}(dy^i)_p,\ \left(d\overline{z}^i\right)_p = \left(dx^i\right)_p - \mathbf{j}(dy^i)_p,$$

which represent the bases of the tangent space $T_p(M)$ and cotangent space $T_p^*(M)$ of M, respectively. Then the following expression can be found

$$J(\frac{\partial}{\partial z^i}) = -\mathbf{j}\frac{\partial}{\partial z^i},\ J(\frac{\partial}{\partial \overline{z}^i}) = \mathbf{j}\frac{\partial}{\partial \overline{z}^i}.$$

The dual endomorphism J^* of the cotangent space $T_p^*(M)$ at any point p of manifold M satisfies that $J^{*2} = I$, and is defined by

$$J^*(dz^i) = -\mathbf{j}dz^i,\ J^*(d\overline{z}^i) = \mathbf{j}d\overline{z}^i.$$

Let V^A be a commutative group $(V, +)$ endowed with a structure of unitary module over the ring A of para-complex numbers. Let V^R denote the group $(V, +)$ endowed with the structure of real vector space inherited from the restriction of scalars to R. The vector space V^R will then be called the real vector space associated to V^A. Setting

$$J(u) = ju,\ P^+(u) = e^+u,\ P^-(u) = e^-u,\ u \in V^A,$$

the expressions

$$\begin{aligned}
&J^2 = 1_V,\ P^{+2} = P^+,\ P^{-2} = P^-,\ P^+ \circ P^- = P^- \circ P^+ = 0\\
&P^+ + P^- = 1_V,\ P^+ - P^- = J,\\
&P^- = (1/2)(1_V - J),\ P^+ = (1/2)(1_V + J),\\
&j^2 = 1,\ e^{+2} = e^+,\ e^{-2} = e^-,\ e^+ \circ e^- = e^- \circ e^+ = 0,\\
&e^+ + e^- = 1,\ e^+ - e^- = j, e^- = (1/2)(1 - j),\ e^+ = (1/2)(1 + j).
\end{aligned}$$

can be written. Also, it is found that

$$J(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial y^i} = \mathbf{j}\frac{\partial}{\partial x^i},\ J(\frac{\partial}{\partial y^i}) = \frac{\partial}{\partial x^i} = \mathbf{j}\frac{\partial}{\partial y^i}\,,$$
$$P^{\mp}(\frac{\partial}{\partial z^i}) = -e^{\mp}\frac{\partial}{\partial z^i},\ P^{\mp}(\frac{\partial}{\partial \overline{z}^i}) = e^{\mp}\frac{\partial}{\partial \overline{z}^i},$$
$$P^{*\mp}(dz^i) = -e^{\mp}dz^i, P^{*\mp}(d\overline{z}^i) = e^{\mp}d\overline{z}^i.$$

If the manifold $(M, g, J = P^+ - P^-)$ satisfies the following conditions simultaneously then the manifold is an almost para-Hermitian manifold. The first condition can be written as follows:

$$g(X, Y) + g(X, Y) = 0 \Leftrightarrow g(X, Y) = 0,\ \forall X, Y \in \chi(D_1). \tag{1.1}$$

Because P^+ and P^- are the projections over D_1 and D_2 respectively, then $(P^+ - P^-)(X) = P^+X - P^-X = P^+X = X$, $(P^+ - P^-)(Y) = P^+Y - P^-Y = P^+Y = Y$. Analogously we can write the second condition as follows

$$g(X, Y) + g(X, Y) = 0 \Leftrightarrow g(X, Y) = 0,\ \forall X, Y \in \chi(D_2). \tag{1.2}$$

Let $X = X_1 + X_2, Y = Y_1 + Y_2$ be vector fields on M such that $X_1, Y_1 \in D_1$ and $X_2, Y_2 \in D_2$. Then

$$\begin{aligned} g(JX, Y) &= g(JX_1 + JX_2, Y_1 + Y_2) = g(X_1 - X_2, Y_1 + Y_2) \\ &= g(X_1, Y_1) - g(X_2, Y_1) + g(X_1, Y_2) - g(X_2, Y_2) \\ &= -g(X_2, Y_1) + g(X_1, Y_2), \end{aligned}$$

$$\begin{aligned} g(X, JY) &= g(X_1 + X_2, JY_1 + JY_2) = g(X_1 + X_2, Y_1 - Y_2) \\ &= g(X_1, Y_1) + g(X_2, Y_1) - g(X_1, Y_2) - g(X_2, Y_2) \\ &= g(X_2, Y_1) - g(X_1, Y_2), \end{aligned}$$

and hence $g(JX, Y) + g(X, JY) = -g(X_2, Y_1) + g(X_1, Y_2) + g(X_2, Y_1) - g(X_1, Y_2) = 0$, for all vector fields X, Y on M. If the conditions (1.1) and (1.2) are true then D_1 and D_2 Lagrangian distributions respect to the 2- form $\Phi(X, Y) = g(JX, Y)$.Therefore, if the almost paracomplex structure J is integrable then (M, g, J) is para-Kähler manifold, or equivalently, (M, Φ, D_1, D_2) is a bi-Lagrangian manifold.

2. Bi-Para-Lagrangians

In this section, bi-para-Euler-Lagrange equations and a bi-para-mechanical system can be obtained for classical mechanics structured under the consideration of the basis $\{e^+, e^-\}$ on bi-Lagrangian manifold.

Let (P^+, P^-) be an almost bi-para-complex structure on the bi-Lagrangian manifold, and $(z^i, \overline{z}^i)$ be its paracomplex structures. Let semispray be the vector field ξ given by

$$\xi = e^+(\xi^{i+}\frac{\partial}{\partial z^i} + \overline{\xi}^{i+}\frac{\partial}{\partial \overline{z}^i}) + e^-(\xi^{i-}\frac{\partial}{\partial z^i} + \overline{\xi}^{i-}\frac{\partial}{\partial \overline{z}^i});$$
$$z^i = z^{i+}e^+ + z^{i-}e^-;\ \dot{z}^i = \dot{z}^{i+}e^+ + \dot{z}^{i-}e^- = \xi^{i+}e^+ + \xi^{i-}e^-;$$
$$\overline{z}^i = \overline{z}^{i+}e^+ + \overline{z}^{i-}e^-;\ \dot{\overline{z}}^i = \dot{\overline{z}}^{i+}e^+ + \dot{\overline{z}}^{i-}e^- = \overline{\xi}^{i+}e^+ + \overline{\xi}^{i-}e^-;$$

where the dot indicates the derivative with respect to time t. The vector field denoted by $V = (P^+ - P^-)(\xi)$ and given by

$$(P^+ - P^-)(\xi) = e^+(-\xi^{i+}\frac{\partial}{\partial z^i} + \overline{\xi}^{i+}\frac{\partial}{\partial \overline{z}^i}) + e^-(\xi^{i-}\frac{\partial}{\partial z^i} - \overline{\xi}^{i-}\frac{\partial}{\partial \overline{z}^i})$$

is called *bi-para-Liouville vector field* on the bi-Lagrangian manifold. The maps given by $T, P : M \to A$ such that $T = \frac{1}{2}m_i(\overline{z}^i)^2 = \frac{1}{2}m_i(\dot{z}^i)^2, P = m_i gh$ are called *the kinetic energy* and *the potential energy of the system,* respectively. Here m_i, g and h stand for mass of a mechanical system having m particles, the gravity acceleration and distance to the origin of a mechanical system on the bi-Lagrangian manifold, respectively. Then $L : M \to A$ is a map that satisfies the conditions; i) $L = T - P$ is a *bi-para-Lagrangian function, ii)* the function given by $E_L = V(L) - L$ is *a bi-para-energy function.*

The operator $i_{(P^+ - P^-)}$ induced by $P^+ - P^-$ and shown by

$$i_{P^+ - P^-}\omega(Z_1, Z_2, ..., Z_r) = \sum_{i=1}^{r} \omega(Z_1, ..., (P^+ - P^-)Z_i, ..., Z_r)$$

is said to be *vertical derivation,* where $\omega \in \wedge^r M$, $Z_i \in \chi(M)$. The *vertical differentiation* $d_{(P^+ - P^-)}$ is defined by

$$d_{(P^+ - P^-)} = [i_{(P^+ - P^-)}, d] = i_{(P^+ - P^-)}d - di_{(P^+ - P^-)}$$

where d is the usual exterior derivation. For an almost para-complex structure $P^+ - P^-$, the closed para-Kähler form is the closed 2-form given by $\Phi_L = -dd_{(P^+ - P^-)}L$ such that

$$d_{(P^+ - P^-)} = e^+B - e^-B : \mathcal{F}(M) \to \wedge^1 M$$

where

$$B = -\frac{\partial}{\partial z^i}dz^i + \frac{\partial}{\partial \overline{z}^i}d\overline{z}^i.$$

Then

$$\Phi_L = e^+C - e^-C$$

where

$$\begin{aligned} C \;\; = \;\; & \frac{\partial^2 L}{\partial z^j \partial z^i}dz^j \wedge dz^i - \frac{\partial^2 L}{\partial z^j \partial \overline{z}^i}dz^j \wedge d\overline{z}^i \\ & + \frac{\partial^2 L}{\partial \overline{z}^j \partial z^i}d\overline{z}^j \wedge dz^i - \frac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i}d\overline{z}^j \wedge d\overline{z}^i. \end{aligned}$$

Let ξ be the second order differential equation satisfying **Eq.** (0.1) and

$$\begin{aligned} i_\xi \Phi_L = {} & e^+[\xi^{i+}\tfrac{\partial^2 L}{\partial z^j \partial z^i}\delta_i^j dz^i - \xi^{i+}\tfrac{\partial^2 L}{\partial z^j \partial z^i}dz^j - \xi^{i+}\tfrac{\partial^2 L}{\partial z^j \partial \overline{z}^i}\delta_i^j d\overline{z}^i + \overline{\xi}^{i+}\tfrac{\partial^2 L}{\partial z^j \partial \overline{z}^i}dz^j \\ & -\xi^{i+}\tfrac{\partial^2 L}{\partial \overline{z}^j \partial z^i}d\overline{z}^j + \overline{\xi}^{i+}\tfrac{\partial^2 L}{\partial \overline{z}^j \partial z^i}\delta_i^j dz^i - \overline{\xi}^{i+}\tfrac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i}\delta_i^j d\overline{z}^i + \overline{\xi}^{i+}\tfrac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i}d\overline{z}^j] \\ & +e^-[-\xi^{i-}\tfrac{\partial^2 L}{\partial z^j \partial z^i}\delta_i^j dz^i + \xi^{i-}\tfrac{\partial^2 L}{\partial z^j \partial z^i}dz^j + \xi^{i-}\tfrac{\partial^2 L}{\partial z^j \partial \overline{z}^i}\delta_i^j d\overline{z}^i - \overline{\xi}^{i-}\tfrac{\partial^2 L}{\partial z^j \partial \overline{z}^i}dz^j \\ & +\xi^{i-}\tfrac{\partial^2 L}{\partial \overline{z}^j \partial z^i}d\overline{z}^j - \overline{\xi}^{i-}\tfrac{\partial^2 L}{\partial \overline{z}^j \partial z^i}\delta_i^j dz^i + \overline{\xi}^{i-}\tfrac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i}\delta_i^j d\overline{z}^i - \overline{\xi}^{i-}\tfrac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i}d\overline{z}^j]. \end{aligned}$$

Since the closed para-Kähler form Φ_L on M is the para-symplectic structure, E_L is written as follows:

$$E_L = e^+(-\xi^{i+}\frac{\partial L}{\partial z^i} + \overline{\xi}^{i+}\frac{\partial L}{\partial \overline{z}^i}) + e^-(\xi^{i-}\frac{\partial L}{\partial z^i} - \overline{\xi}^{i-}\frac{\partial L}{\partial \overline{z}^i}) - L$$

and thus

$$\begin{aligned} dE_L = & -\xi^{i+}\frac{\partial^2 L}{\partial z^j \partial z^i}dz^j e^+ + \overline{\xi}^{i+}\frac{\partial^2 L}{\partial z^j \partial \overline{z}^i}dz^j e^+ - \xi^{i+}\frac{\partial^2 L}{\partial \overline{z}^j \partial z^i}d\overline{z}^j e^+ \\ & + \overline{\xi}^{i+}\frac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i}d\overline{z}^j e^+ - \frac{\partial^2 L}{\partial z^j}dz^i + \xi^{i-}\frac{\partial^2 L}{\partial z^j \partial z^i}dz^j e^- - \overline{\xi}^{i-}\frac{\partial^2 L}{\partial z^j \partial \overline{z}^i}dz^j e^- \\ & + \xi^{i-}\frac{\partial^2 L}{\partial \overline{z}^j \partial z^i}d\overline{z}^j e^- - \overline{\xi}^{i-}\frac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i}d\overline{z}^j e^- - \frac{\partial^2 L}{\partial \overline{z}^j}d\overline{z}^i. \end{aligned}$$

With the use of **Eq.** (0.1), the following expression can be obtained:

$$\begin{aligned} & \xi^{i+}\frac{\partial^2 L}{\partial z^j \partial z^i}dz^j e^+ - \xi^{i+}\frac{\partial^2 L}{\partial z^j \partial \overline{z}^i}d\overline{z}^j e^+ + \overline{\xi}^{i+}\frac{\partial^2 L}{\partial \overline{z}^j \partial z^i}dz^j e^+ - \overline{\xi}^{i+}\frac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i}d\overline{z}^j e^+ \\ & -\xi^{i-}\frac{\partial^2 L}{\partial z^j \partial z^i}dz^j e^- + \xi^{i-}\frac{\partial^2 L}{\partial z^j \partial \overline{z}^i}d\overline{z}^j e^- - \overline{\xi}^{i-}\frac{\partial^2 L}{\partial \overline{z}^j \partial z^i}dz^j e^- + \overline{\xi}^{i-}\frac{\partial^2 L}{\partial \overline{z}^j \partial \overline{z}^i}d\overline{z}^j e^- \\ & + \frac{\partial^2 L}{\partial z^j}dz^i + \frac{\partial^2 L}{\partial \overline{z}^j}d\overline{z}^i = 0. \end{aligned}$$

If a curve denoted by $\alpha : A \to M$ is considered to be an integral curve of ξ, then the equations given in the following are

$$\begin{aligned} (e^+ - e^-)(\left[\xi^{i+}\frac{\partial}{\partial z^{ji}} + \overline{\xi}^{i+}\frac{\partial}{\partial \overline{z}^j}\right]e^+ + \left[\xi^{i-}\frac{\partial}{\partial z^j} + \overline{\xi}^{i-}\frac{\partial}{\partial \overline{z}^j}\right]e^-)(\frac{\partial L}{\partial z^i})dz^j \\ (e^+ - e^-)(-\left[\xi^{i+}\frac{\partial}{\partial z^{ji}} + \overline{\xi}^{i+}\frac{\partial}{\partial \overline{z}^j}\right]e^+ - \left[\xi^{i-}\frac{\partial}{\partial z^j} + \overline{\xi}^{i-}\frac{\partial}{\partial \overline{z}^j}\right]e^-)(\frac{\partial L}{\partial \overline{z}^i})d\overline{z}^j \\ + \frac{\partial L}{\partial z^j}dz^j + \frac{\partial L}{\partial \overline{z}^j}d\overline{z}^j = 0. \end{aligned}$$

Then the following equations are found:

$$(2.1) \qquad (e^+ - e^-)\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial z^i}\right) + \frac{\partial L}{\partial z^i} = 0, \quad (e^+ - e^-)\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \overline{z}^i}\right) - \frac{\partial L}{\partial \overline{z}^i} = 0.$$

Thus the equations obtained in **Eq.** (2.1) are seen to be a *bi-para-Euler-Lagrange equations* on the distributions D_1 and D_2, and then the triple (M, Φ_L, ξ) is seen to be a *bi-para-mechanical system* with taking into account the basis $\{e^+, e^-\}$ on the bi-Lagrangian manifold (M, Φ, D_1, D_2).

3. Bi-Para-Hamiltonians

Here, bi-para-Hamilton equations and bi-para-Hamiltonian mechanical system for classical mechanics structured on the bi-Lagrangian manifold (M, Φ, D_1, D_2) are derived.

Let $(z_i, \overline{z}_i)$ be paracomplex coordinates. Let $\{\frac{\partial}{\partial z_i}|_p, \frac{\partial}{\partial \overline{z}_i}|_p\}$ and $\{dz_i|_p, d\overline{z}_i|_p\}$ be bases over para-complex number A of tangent space $T_p(M)$ and cotangent space $T_p^*(M)$ of M, respectively. Assume that an almost bi-para-complex structure, a bi-para-Liouville form and a bi-para-complex 1-form on the distributions D_1 and D_2 are shown by $P^{*+} - P^{*-}$, λ and ω, respectively. Then $\omega = \frac{1}{2}[(z_i d\overline{z}_i + \overline{z}_i dz_i)e^+ + (z_i d\overline{z}_i + \overline{z}_i dz_i)e^-]$ and $\lambda = (P^{*+} - P^{*-})(\omega) = \frac{1}{2}[(z_i d\overline{z}_i - \overline{z}_i dz_i)e^+ - (z_i d\overline{z}_i - \overline{z}_i dz_i)e^-]$. It is concluded that if Φ is a closed para-Kähler form on the bi-Lagrangian manifold, then Φ is also a para-symplectic structure on the bi-Lagrangian manifold.

Consider that bi-para-Hamiltonian vector field Z_H associated with bi-para-Hamiltonian energy H is given by

$$Z_H = (Z_i\frac{\partial}{\partial z_i} + \overline{Z}_i\frac{\partial}{\partial \overline{z}_i})e^+ + (Z_i\frac{\partial}{\partial z_i} + \overline{Z}_i\frac{\partial}{\partial \overline{z}_i})e^-.$$

Then

$$\Phi = -d\lambda = (e^+ - e^-)(d\overline{z}_i \wedge dz_i),$$

and

$$i_{Z_H}\Phi = \Phi(Z_H) = (\overline{Z}_i dz_i - Z_i d\overline{z}_i)e^+ + (-\overline{Z}_i dz_i + Z_i d\overline{z}_i)e^-. \tag{3.1}$$

Moreover, the differential of bi-para-Hamiltonian energy is obtained as follows:

$$dH = (\frac{\partial H}{\partial z_i}dz_i + \frac{\partial H}{\partial \overline{z}_i}d\overline{z}_i)e^+ + (\frac{\partial H}{\partial z_i}dz_i + \frac{\partial H}{\partial \overline{z}_i}d\overline{z}_i)e^-. \tag{3.2}$$

By means of **Eq.**(0.1), using **Eq.** (3.1) and **Eq.** (3.2), the bi-para-Hamiltonian vector field is found to be

$$Z_H = (-\frac{\partial H}{\partial \overline{z}_i}\frac{\partial}{\partial z_i} + \frac{\partial H}{\partial z_i}\frac{\partial}{\partial \overline{z}_i})e^+ + (\frac{\partial H}{\partial \overline{z}_i}\frac{\partial}{\partial z_i} - \frac{\partial H}{\partial z_i}\frac{\partial}{\partial \overline{z}_i})e^-. \tag{3.3}$$

Suppose that a curve

$$\alpha : I \subset A \to M$$

be an integral curve of the bi-para-Hamiltonian vector field Z_H, i.e.,

$$Z_H(\alpha(t)) = \dot{\alpha}(t),\ t \in I. \tag{3.4}$$

In the local coordinates, it is obtained that

$$\alpha(t) = (z_i(t), \overline{z}_i(t))$$

and

$$\dot{\alpha}(t) = (\frac{dz_i}{dt}\frac{\partial}{\partial z_i} + \frac{d\overline{z}_i}{dt}\frac{\partial}{\partial \overline{z}_i})e^+ + (\frac{dz_i}{dt}\frac{\partial}{\partial z_i} + \frac{d\overline{z}_i}{dt}\frac{\partial}{\partial \overline{z}_i})e^-. \tag{3.5}$$

Under the consideration of **Eqs.** (3.3), (3.4), (3.5), the following results can be obtained:

$$\frac{dz_i}{dt} = -(e^+ - e^-)\frac{\partial H}{\partial \overline{z}_i},\ \frac{d\overline{z}_i}{dt} = (e^+ - e^-)\frac{\partial H}{\partial z_i}. \tag{3.6}$$

Hence, the equations obtained in **Eq.** (3.6) are seen to be *bi-para Hamilton equations* on the bi-Lagrangian manifold (M, Φ, D_1, D_2), and then the triple (M, Φ, Z_H) is seen to be a *bi-para-Hamiltonian mechanical system* with the use of basis $\{e^+, e^-\}$ on the bi-Lagrangian manifold (M, Φ, D_1, D_2).

CONCLUSION 9. *This chapter has shown to exist physical proof of the mathematical equality given by* $M = D_1 \oplus D_2$. *Also, formalisms of Lagrangian and*

Hamiltonian mechanics have intrinsically been described with taking into account the basis $\{e^{+}, e^{-}\}$ *on the bi-Lagrangian manifold* (M, Φ, D_1, D_2). *Bi-para-Lagrangian and bi-para-Hamiltonian models arise to be a very important tool since they present a simple method to describe the model for bi-para-mechanical systems. In solving problems in classical mechanics, the bi-para-complex mechanical system will then be easily usable model. With the use of the corresponding approach, thus, a differential equation resulted in mechanics is seen to have a non-trivial solution. J. W. Moffat's theory using paracomplex geometry in gravitational field of physics has been a controversial one. Since physical phenomena, as well-known, do not take place all over the space, a new model for dynamical systems on subspaces is needed. Therefore, equations (2.1) and (3.6) are only considered to be a first step to realize how bi-para-complex geometry has been used in solving problems in different physical area. For further research, bi-para-complex Lagrangian and Hamiltonian vector fields derived here are suggested to deal with problems in electrical, magnetical and gravitational fields of physics.*

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