# Deformations of Cartan framed null curves preserving the torsion

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**Abstract**

We study deformations of Cartan framed null curves in the Minkowski 3-space which preserve the torsion.

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## §1. Preliminaries

In classical differential geometry of spatial curves, it is known that a spatial curve whose binormals are the binormals of another curve, is a plane curve [4, p. 161, Ex. 14]. However, in the case of null curves, we shall show a very different situation that every Cartan framed null curve admits deformations preserving its binormal directions and torsion.

Our results may be considered as an analogue of Bäcklund transformations between constant torsion curves in Euclidean 3-space introduced by A. Calini and T. Ivey.

Let $\mathbf{E}^3_1$ be a *Minkowski* 3-*space* with the natural Lorentz metric

$$\langle \cdot, \cdot \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

in terms of natural coordinates. The vector product operation of $\mathbf{E}^3_1$ is defined by (cf. [5])

$$\mathbf{x} \times \mathbf{y} = (x_3y_2 - x_2y_3, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1),$$

for $\mathbf{x} = (x_1, x_2, x_3),\ \mathbf{y} = (y_1, y_2, y_3) \in \mathbf{E}^3_1$.

**Definition**. A parametrized curve $\gamma = \gamma(s)$ in Minkowski 3-space $\mathbf{E}^3_1$ is said to be a *null curve* if its tangent vector field is null, *i.e.*,

$$\langle \gamma', \gamma' \rangle = 0,\ \gamma' \neq 0.$$

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**Proposition**. *Let $\gamma(s)$ be a nongeodesic null curve. Then there exists a unique frame field $(A, B, C)$ such that*

$$\frac{d}{ds}(A, B, C) = (A, B, C)\begin{pmatrix} 0 & 0 & -\tau \\ 0 & 0 & -\kappa \\ \kappa & \tau & 0 \end{pmatrix}$$

*with $A = d\gamma/ds$, $\langle A, A\rangle = \langle B, B\rangle = 0$, $\langle A, B\rangle = 1$ and $C$ is defined by $C = A \times B$.* The functions $\kappa$ and $\tau$ are called the *curvature* and *torsion* of $\gamma$ respectively. The frame field $(A, B, C)$ is called the *Cartan frame* of $\gamma$. A null curve together with its Cartan frame is called a Cartan framed null curve.

*Proof.* Put $A(s) = \gamma'$. From the assumption, $\gamma'$ and $\gamma''$ are linearly independent and hence $\gamma''$ is spacelike, *i.e.*, $\langle \gamma'', \gamma''\rangle > 0$. Thus there exists a unique section (vector field) $B(s)$ of the orthogonal compliment $\gamma''(s)^{\perp}$ of $\gamma''(s)$ such that

$$\langle A(s), B(s)\rangle = 1, \quad \langle B(s), B(s)\rangle = 0.$$

We define the vector field $C(s)$ along $\gamma$ by $C(s) = A(s) \times B(s)$. Then we have $\langle A(s), C(s)\rangle = \langle B(s), C(s)\rangle = 0$ and $\langle C(s), C(s)\rangle = 1$. Moreover, there exist functions $\kappa(s)$ and $\tau(s)$ which satisfy

$$A'(s) = \gamma''(s) = \kappa(s)C(s), \quad B'(s) = \tau(s)C(s).$$

Uniqueness follows from the construction we have done. □

We call the vector fields $A, B$ and $C$ a *tangent vector field*, a *binormal vector field* and a (*principal*) *normal vector field* of $\gamma$, respectively.

**Remark**. Our expression for $dC/ds$ differs from the one given in [3] because we specify $\langle A, B\rangle = 1$ rather than $\langle A, B\rangle = -1$ [6]. Note that null geodesics are regarded as Cartan framed null curve with zero curvature.

A Cartan framed null curve with $\tau = 0$ is called a *generalized null cubic* [3]. In particular a generalized null cubic with constant $\kappa$ is actually a cubic curve.

It is easy to check that the vanishing of $\kappa$ or $\tau$ is invariant under reparametrization.

**Example** (*cf.* [7]). Let $\phi$ and $\psi$ be functions satisfying $\phi' = (\psi')^2$. Then the curve

$$\gamma(s) = \left(\frac{1}{\sqrt{2}}\left(s + \frac{\phi(s)}{2}\right), \frac{1}{\sqrt{2}}\left(s - \frac{\phi(s)}{2}\right), \psi(s)\right)$$

is a Cartan framed null curve with frame

$$A = \left(\frac{1}{\sqrt{2}}\left(1 + \frac{\phi'(s)}{2}\right), \frac{1}{\sqrt{2}}\left(1 - \frac{\phi'(s)}{2}\right), \psi'(s)\right),$$

$$B = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \quad C = \left(\frac{\psi'(s)}{\sqrt{2}}, -\frac{\psi'(s)}{\sqrt{2}}, 1\right).$$

Direct computation shows that the curvature and torsion of $\gamma$ are

$$\kappa(s) = \psi''(s), \ \tau = 0.$$

Thus $\gamma$ is a generalized null cubic.

In particular, in case that $\kappa$ is constant, then

$$\phi(s) = \frac{\kappa^2}{3}s^3 + b\kappa s^2 + b^2 s + c, \quad \psi(s) = \frac{\kappa}{2}s^2 + as + b,$$

where $a, b, c$ are constants. In case that $\kappa$ is constant, clearly $\gamma$ is a cubic curve.

## §2. The main result

We shall prove the following result for Cartan framed null curves in Lorentzian geometry.

**Theorem**. *Let $(\gamma; A, B, C)(s)$ be a Cartan framed null curve. Assume that there exists a Cartan framed null curve $(\bar\gamma; \bar A, \bar B, \bar C)(\bar s)$ such that the binormal direction of $\bar\gamma$ coincides with that of $\gamma$. Then the torsions at the corresponding points coincide, i.e., $\tau(s) = \bar\tau(\bar s)$.*

*Conversely, let curve $(\gamma; A, B, C)(s)$ be a Cartan framed null curve. Then there exists a Cartan framed null curve $(\bar\gamma; \bar A, \bar B, \bar C)(\bar s)$ such that the binormal direction of $\gamma$ and $\bar\gamma$ coincide and $\tau(s) = \bar\tau(\bar s)$ .*

*Proof.* Let $\gamma = \gamma(s)$ be a Cartan framed null curve with frame $(A, B, C)$ Assume that there exists a Cartan framed null curve $\bar\gamma$ with frame $(\bar A, \bar B, \bar C)$ such that $\bar B$ is in the $B$-direction. Then $\bar\gamma$ can be parametrized as

$$\bar\gamma(\bar s(s)) = \gamma(s) + u(s)B(s) \tag{2.1}$$

for some function $u(s) \neq 0$ and some parametrization $\bar s = \bar s(s)$. Thus without loss of generality, we may assume that

$$\bar B(\bar s(s)) = a(s)B(s), \tag{2.2}$$

for some function $a(s) \neq 0$. Differentiating (2.1) by $s$, we have

$$\frac{d\bar s}{ds}\bar A(\bar s(s)) = A(s) + u'(s)B(s) + u(s)\tau(s)C(s).$$

From the conditions of a Cartan frame

$$\begin{cases} \langle \bar A(\bar s(s)), \bar A(\bar s(s))\rangle = 0, \\ \langle \bar A(\bar s(s)), \bar B(\bar s(s))\rangle = 1, \end{cases}$$

we obtain

$$2u'(s) + (u(s)\tau(s))^2 = 0 \tag{2.3}$$

and

$$a(s) = \frac{d\bar s}{ds}.$$

Thus, from (2.3), the function $u$ is completely determined by the torsion:

$$\frac{1}{u(s)} = \frac{1}{2}\int \tau(s)^2 ds + c$$

for some constant $c$. Note that $C(s) \times B(s) = -B(s)$. The principal normal $\bar{C}$ of $\bar{\gamma}$ is given by

$$\bar{C}(\bar{s}(s)) = \bar{A}(\bar{s}(s)) \times \bar{B}(\bar{s}(s)) = C(s) - u(s)\tau(s)B(s). \tag{2.4}$$

Differentiating (2.2) by $s$ and using (2.4), we obtain

$$a(s)\bar{\tau}(\bar{s}(s))\{C(s) - u(s)\tau(s)B(s)\} = \frac{da}{ds}(s)B(s) + a(s)\tau(s)C(s).$$

Comparing the Frenet equations of both curves, we have

$$\bar{\tau}(\bar{s}(s)) = \tau(s),\ \frac{da}{ds}(s) = -a(s)u(s)\tau(s)^2. \tag{2.5}$$

Thus the torsion at the corresponding points are coincide. From (2.5), we have Since $u \neq 0$, we have $\tau = 0$. Thus $\gamma$ is a generalized null

$$a(s) = a_0 u(s)^2, \quad a_0 \in \mathbf{R}^*.$$

Differentiating (2.4), we get the following relation:

$$a(s)^2\bar{\kappa}(\bar{s}) = u(s)\tau'(s).$$

Conversely, for every Cartan framed null curve, define a null curve $\bar{\gamma}(\bar{s})$ by

$$\bar{\gamma}(\bar{s}) := \gamma(s) + u(s)B(s),\ 1/u(s) = \int \tau(s)^2 ds + 2/u_0, \tag{2.6}$$

$$\bar{s}(s) := a_0 \int u(s)^2 ds.$$

Then $\bar{\gamma}$ is a Cartan framed null curve framed by

$$\bar{A}(\bar{s}) = \frac{d}{ds}\bar{\gamma}(\bar{s}),\ \bar{B}(\bar{s}) = \frac{d\bar{s}}{ds}B(s),\ \bar{C}(\bar{s}) = \bar{A}(\bar{s}) \times \bar{B}(\bar{s}).$$

Clearly $\gamma$ and $\bar{\gamma}$ have common binormal directions. □

**Corollary**. *Let $\gamma$ be a Cartan framed null curve and $\bar{\gamma}$ is the Cartan framed null curve defined by (2.6). If $\langle A, \bar{\gamma} - \gamma\rangle$ is constant. Then $\gamma$ is a generalized null cubic. In this case $\bar{\gamma}$ is congruent to $\gamma$.*

*Proof.* The constancy of $u$ implies that $\tau = 0$ and the constancy of $B$ (See (2.1)). Thus $\bar{\gamma}$ differs from $\gamma$ only by translation by constant vector $uB$. Namely $\bar{\gamma}$ is congruent to the original curve $\gamma$. □

Our result seems to be better comparing with Bäcklund transformation for constant torsion curves investigated in [1]:

**Theorem** ([1]) *Let $\gamma(s)$ be a unit speed curve in Euclidean 3-space with non-zero constant torsion $\tau$. Denote by $(T, N, B)$ the Frenet frame field of $\gamma$ and $\kappa, \tau$ the curvature and torsion of $\gamma$, respectively (See Appendix). Then, for any constant $\lambda$ and a solution $\beta$ to the ordinary differential equation:*

$$\frac{d\beta}{ds} = \lambda \sin\beta - \kappa,$$

*the curve $\bar{\gamma}(s)$ defined by*

$$\bar{\gamma}(s) := \gamma(s) + \frac{2\lambda}{\lambda^2+\tau^2}(\cos\beta T + \sin\beta N)$$

*is a curve of constant same torsion $\tau$ with arclength parameter $s$.*

The new curve $\bar{\gamma}(s)$ is called the *Bäcklund transformation* of $\gamma$.

**Remark**. In p. 66 of [2], Duggal and Bejancu claimed that $\cdots$ "*locally, for any null curve of a* 3*-dimensional Lorentzian manifold we find a Cartan frame such that it is a generalized null cubic*".

They considered the following procedure:

Define a new frame field $\bar{F} = (\bar{A}(s), \bar{B}(s), \bar{C}(s))$ by

$$\bar{A}(s) := A(s),\ \bar{B}(s) := -\frac{f(s)}{2}A(s) + B(s) + f(s)C(s),\ \bar{C}(s) := C(s) - f(s)A(s),$$

where $f$ is a solution to

$$\frac{df}{ds} + \frac{\kappa(s)}{2}f(s)^2 + \tau(s) = 0.$$

Then the new frame has zero torsion $\bar{\tau} = 0$. However this frame $\bar{F}$ is not a Cartan frame because of uniqueness of Cartan frame. We can check that $\bar{F}$ is not the Cartan frame by straightforward computation. In fact, the new frame satisfies

$$\frac{d}{ds}(\bar{A}, \bar{B}, \bar{C}) = (\bar{A}, \bar{B}, \bar{C}) \begin{pmatrix} \bar{k}_0 & 0 & -\bar{\tau} \\ 0 & -\bar{k}_0 & -\bar{\kappa} \\ \bar{\kappa} & \bar{\tau} & 0 \end{pmatrix}$$

with

$$\bar{k}_0 = f(s)\kappa(s), \quad \bar{\kappa}(s) = \kappa(s), \quad \bar{\tau} = 0.$$

Thus $(\gamma, \bar{F})$ is not a generalized null cubic.

## §3. Appendix

In this Appendix, we give a proof of the following classical result:

**Theorem**. *Let $\gamma = \gamma(s)$ be a curve parametrized by the arclength parameter in Euclidean* 3*-space. If the binormals of $\gamma$ are the binormals of another curve, then $\gamma$ is a plane curve.*

*Proof.* Let us denote by $(T, N, B)$ the Frenet frame field of $\gamma$. Namely, $T = \gamma'$, $N$ is the principal normal vector field and $B$ is the binormal vector field. Assume that $\bar{\gamma} = \bar{\gamma}(\bar{s})$ is a curve whose binormal direction coincides with that of $\gamma$. We denote by $(\bar{T}, \bar{N}, \bar{B})$ the Frenet frame field of $\bar{\gamma}$; then $\bar{B}(\bar{s}) = \pm B(s)$.

The curve $\bar{\gamma}$ is parametrized by $s$ as

$$\bar{\gamma}(\bar{s}(s)) = \gamma(s) + u(s)B(s) \tag{3.7}$$

for some function $u(s) \neq 0$ and parametrization $\bar{s} = \bar{s}(s)$. Differentiating (3.7) by $s$, we get

$$\bar{\gamma}' = T - u\tau N + u'B.$$

Since the binormal direction of $\bar{\gamma}$ coincides with that of $\gamma$, $\langle \bar{\gamma}', B \rangle = 0$. Thus we have $u' = 0$. Hence $u$ is constant. Note that the arclength parameter $\bar{s}$ of $\bar{\gamma}$ is related to $s$ by

$$\frac{d\bar{s}}{ds} = \sqrt{1 + (u\tau)^2}.$$

The Frenet frame field of $\bar{\gamma}$ is given by

$$\bar{T} = \frac{T - u\tau N}{\sqrt{1 + (u\tau)^2}}, \ \bar{N} = \pm \frac{u\tau T + N}{\sqrt{1 + (u\tau)^2}}, \ \bar{B} = \pm B.$$

Since $\bar{B} = \pm B$, by computing $B'$ and $\bar{B}'$, we have

$$\bar{\tau}\bar{N} = \pm \frac{\tau}{\sqrt{1 + (u\tau)^2}} N.$$

Comparing the both hand sides of this equation, we have

$$\tau\bar{\tau}u = 0, \ \tau = \pm\bar{\tau}.$$

Thus we have $\tau = \bar{\tau} = 0$, and hence $\bar{s} = s$, and $\gamma$ is a plane curve. Note that since $B$ and $u$ are constant, $\bar{\gamma}$ differs from $\gamma$ only by translation, and therefore $\bar{\gamma}$ is congruent to the original curve. □

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