Some applications on weakly pseudo-symmetric Riemannian manifolds

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Abstract

This paper deals with weakly pseudo-symmetric Riemannian manifold. In this paper, weakly pseudo-symmetric manifold $(WS)_n$ is introduced and then decomposable weakly pseudo-symmetric manifolds are examined and some theorems about them are proved. In the other part of it, the cyclic Ricci tensor of this manifold is studied.


Key words: Weakly pseudo-symmetric manifold, decomposable manifold, Ricci-associate vector field.

1 Introduction

The notions of weakly symmetric and weakly projective symmetric manifolds were introduced by Tamassy and Binh [1]. A non-flat Riemanninan manifold $V_n$ ($n > 2$) is called a weakly symmetric manifold if the curvature tensor $R_{hijk}$ satisfies the condition:

\[ R_{hijk,l} = a_l R_{hijk} + b_h R_{lijk} + c_i R_{hljk} + d_j R_{hilk} + e_k R_{hijl} \]

where $a, b, c, d, e$ are 1-forms (non-zero simultaneously) and the comma ',' denotes covariant differentiation with respect to the metric tensor of the manifold.

It may be mentioned in this connection that although the definition of a $(WS)_n$ is similar to that of a generalized pseudo-symmetric manifold studied by Chaki and Mondal [2], the defining condition of a $(WS)_n$ is weaker than that of a generalized pseudo-symmetric manifold. A reduction in generalized pseudo-symmetric manifolds has been obtained by Chaki and Mondal. But, in [3], the authors investigated and a reduction in $(WS)_n$ is obtained in a simpler form

\[ R_{hijk,l} = a_l R_{hijk} + b_h R_{lijk} + b_l R_{hljk} + d_j R_{hilk} + d_k R_{hijl} \]

The name weakly-symmetric was chosen, because if in (1.2), $a_l, b_l, d_l$ are taken as zero, then the equation (1.2) takes the form $R_{hijk,l} = 0$ and the manifold reduces to a symmetric manifold in the sense of Cartan.
In the present paper, some results on a $(PS)_n$ are established. In section III, it is shown that if a $(WS)_n$ is a decomposable manifold $V_r x V_{n-r}$ $(r > 1, n - r > 1)$, then one of the decomposition manifolds is flat and the other is a weakly symmetric manifold. In section IV, the Ricci-associate of a vector field is defined and some theorems are proved. In addition, for $B \neq 0$ or $D \neq 0$, the decomposable manifold $(WS)_n$ has a zero scalar curvature.

The last Section is concerned with $(WS)_n$ admitting a concurrent or a recurrent vector field, [4].

2 Weakly pseudo-symmetric manifolds $(WS)_n$.

In this section, we shall obtain some formulas which will be required in this study of $(WS)_n$.

Let $R_{ij}$ and $R$ denote the Ricci tensor and the scalar curvature, respectively. Then, from (1.2), we get

\[(2.1) \quad R_{ij,l} = a_{il} R_{ij} + b_i R_{lj} + d_j R_{il} + b_k R_{lijk} + d_k R_{kijl}\]

Transvecting (2.1) with $g^{ij}$, we have

\[(2.2) \quad R_{l} = a_l R + 2(b^h + d^h) R_{hl}\]

Moreover, contraction of (1.2) with $g^{ij} g^{hk}$, we get

\[(2.3) \quad \frac{R_{l}}{2} = a^h R_{hl} + d_l R + (b^h - d^h) R_{hl}\]

and multiplying (1.2) by $g^{hk} g^{jl}$, we find

\[(2.4) \quad \frac{R_{l}}{2} = a^h R_{hl} + b_l R + (d^h - b^h) R_{hl}\]

On the other hand, transvecting (1.2) by $g^{jl}$, we get

\[(2.5) \quad \frac{R_{l}}{2} = a^h R_{hl} + b_l R + (b^h - d^h) R_{hl}\]

From (2.3) and (2.4), we can easily see that the following expression holds

\[(2.6) \quad (b_l - d_l) R = 0\]

Similarly, using (2.4) and (2.5), we have

\[(2.7) \quad b^l R_{hl} = d^l R_{hl}\]

From (2.2) and (2.7), we obtain

\[(2.8) \quad a_l R_{,m} - a_m R_{,l} = 4(a_l b^h R_{hm} - a_m b^h R_{hl})\]

Further, we find

\[(2.9) \quad R(a_{l,m} - a_{m,l}) + (a_l R_{,m} - a_m R_{,l}) + 4((b^h R_{hl})_{,m} - (b^h R_{hm})_{,l}) = 0\]
3 Decomposable \((WS)\) \(_n\).

A Riemannian manifold \(V_n\) is said to be decomposable if it can be expressed as a product \(V_r \times V_{n-r}\) for some \(r\), i.e., if coordinates can be found so that its metric takes the form

\[
(3.1) \quad ds^2 = \sum_{a,b=1}^{r} g_{ab} dx^a dx^b + \sum_{\alpha,\beta=r+1}^{n} g_{\alpha\beta} dx^\alpha dx^\beta
\]

where \(g_{ab}\) are functions of \(x^1, x^2, \ldots, x^r (r < n)\) and \(g_{\alpha\beta}\) are functions of \(x^{r+1}, x^{r+2}, \ldots, x^n\) only; \(a, b, c,\ldots\) are taken to have range 1 to \(r\) and \(\alpha, \beta, \gamma,\ldots\) are taken to have the range \(r + 1\) to \(n\). The two parts of (3.1) are metrics of \(V_r\) and \(V_{n-r}\) which are called the decomposition manifolds.

We now suppose that \((WS)\) \(_n\) \((n > 2)\) is decomposable with \(V_r\) and \(V_{n-r}\) as decomposition manifolds. From [3], we have

\[
(3.2) \quad R_{abcd,a} = A_\alpha R_{abcd} + B_a R_{abcd} + B_\alpha R_{accd} + D_c R_{abdc} + D_d R_{abca}
\]

In view of the fact that the curvature tensor and its covariant derivative are product tensors, the above equation takes the form

\[
(3.3) \quad A_\alpha R_{abcd} = 0
\]

Also, from [3], we have

\[
(3.4) \quad R_{\alpha\beta\gamma\delta,a} = A_\alpha R_{\alpha\beta\gamma\delta} + B_\alpha R_{\alpha\beta\gamma\delta} + B_\gamma R_{\alpha\beta\gamma\delta} + D_\gamma R_{\alpha\beta\gamma\delta} + D_\gamma R_{\alpha\beta\gamma\delta}
\]

Since the curvature tensor of the space and its covariant derivative are product tensors, the equation (3.4) takes the form

\[
(3.5) \quad A_\alpha R_{\alpha\beta\gamma\delta} = 0
\]

Similarly, we get

\[
(3.6) \quad B_a R_{abcd} = 0, \quad B_\alpha R_{\alpha\beta\gamma\delta} = 0
\]

\[
(3.7) \quad D_\alpha R_{abcd} = 0, \quad D_\alpha R_{\alpha\beta\gamma\delta} = 0
\]

Since \(A_\alpha, B_a, D_\alpha\) are non-zero vectors, all its components cannot vanish. Suppose \(A_\alpha \neq 0\) for some \(\alpha\). Then, from (3.3), we get \(R_{abcd} = 0\) which means that the decomposition manifold is flat. If \(A_\alpha \neq 0\) for some \(\alpha\), then by similar argument, we have \(R_{\alpha\beta\gamma\delta} = 0\) which means that the decomposition manifold \(V_{n-r}\) is flat. Similarly, the conditions \(B_a \neq 0, D_\alpha \neq 0\) for some \(\alpha\) and \(B_a \neq 0, D_\alpha \neq 0\) for some \(\alpha\) hold, it is easy to see that the decomposition manifolds \(V_r\) and \(V_{n-r}\) are flat.

We suppose that \(R_{\alpha\beta\gamma\delta} = 0\). Then, \(R_{abcd} \neq 0\), because, by hypothesis, \((WS)\) \(_n\) is not flat. Hence, from (3.3), (3.6) and (3.7), we get \(A_\alpha = 0, B_\alpha = 0\) and \(D_\alpha = 0\). Since \(A_\alpha, B_\alpha, D_\alpha\) are non-zero vectors, all its components cannot vanish. Hence, \(A_\alpha \neq 0, B_\alpha \neq 0, D_\alpha \neq 0\) for some \(\alpha\). Therefore, from (1.2), we obtain

\[
(3.8) \quad R_{abcd,e} = A_e R_{abcd} + B_a R_{ebcd} + B_b R_{acde} + D_c R_{abde} + D_d R_{abce}
\]

By virtue of (3.8), it follows that the part \(V_r\) is a \((WS)\)_\(r\).

We can therefore state the following theorem:
Theorem 3.1 If a (WS), is a decomposable manifold \( V^r x V_{n-r} \) \((n > 1, n - r > 1)\), then one of the decomposition manifolds is flat and the other is a weakly symmetric.

An n-dimensional Riemannian manifold \( V^n \) is said to be decomposable if in some coordinates its metric is given by

\[
(3.9) \quad ds^2 = g_{ij} dx^i dx^j = \sum_{a,b=1}^{r} \bar{g}_{ab} dx^a dx^b + \sum_{a',b'=r+1}^{n} g^*_{a'b'} dx^{a'} dx^{b'}
\]

where \( \bar{g}_{ab} \) are functions of \( x^1, x^2, \ldots, x^r \) \((r < n)\) denoted by \( \bar{x} \) and \( \bar{g}^*_{a'b'} \) are functions of \( x^{r+1}, x^{r+2}, \ldots, x^n \) denoted by \( \bar{x}^*; a, b, c, \ldots \) run from 1 to \( r \) and \( a', b', c', \ldots \) run from \( r + 1 \) to \( n \). The two parts of (3.9) are the metrics of a \( V^r \) \((r > 1)\) and a \( V^{n-r} \) \((n - r > 1)\) which are called the decomposition manifolds of \( V^n = V^r x V^{n-r} \). Throughout this paper each object denoted by a bar is assumed to be from \( \bar{g}_{ab} \) and of \( \bar{V}^r \), and each object denoted by a star is formed from \( g^*_{a'b'} \) and of \( V^{n-r} \). From (3.9), we have

\[
(3.10) \quad g_{ab} = \bar{g}_{ab}, \quad g_{a'b'} = \bar{g}^*_{a'b'}, \quad g^{ab} = \bar{g}^{ab}, \quad g^{a'b'} = g^{*a'b'}, \quad g_{aa'} = g^{aa'} = 0
\]

The only non-zero Christoffel symbols of the second kind are as follows: A comma and a dot shall denote covariant differentiation in \( V^n \) and \( V^r \), respectively. Hence, we obtain the following relations:

\[
(3.11) \quad R^x_{abcd} = \bar{R}_{abcd}, \quad R^x_{a'b'c'd'} = R^*_{a'b'c'd'}
\]

We suppose that \( V^n = V^r x V^{n-r} \) and \( B \neq 0 \). By using (1.2), we get

\[
R_{a'b'c'd'.a} = A_a R_{a'b'c'd'} + B_{a'} R_{abcd} + B_{b'} R_{ac'd'} + B_{c'} R_{adb'} + B_{d'} R_{a'bc'}
\]

in this case, we obtain

\[
(3.12) \quad B_{a'} R_{abcd} = 0
\]

Similarly, after some calculations, we get

\[
(3.13) \quad B_a R_{a'b'c'd'} = 0
\]

Since \( B \neq 0 \), all its components cannot vanish. Hence, we consider the following two cases:

**Case** (i): Suppose that \( B_{a'} \neq 0 \) for a fixed \( a' \). Then from (3.11) and (3.12), we have

\[
\bar{R}_{abcd} = 0
\]

Transvecting the above equation by \( \bar{g}^{ad} \) and \( \bar{g}^{bc} \), we get \( \bar{R} = 0 \). From (3.11), we obtain

\[
(3.14) \quad R = R^*
\]

(ii): Suppose \( B_a \neq 0 \) for a fixed \( a \). Then, from (3.13) it follows that \( R_{a'b'c'd'} = 0 \) for all \( a'b'c'd' \). From (3.11), we have

\[
(3.15) \quad R^*_{a'b'c'd'} = 0
\]

Multiplying (3.15) with \( g^{*a'b'} \), \( g^{*b'c'} \), we get \( R^* = 0 \). From (3.14), we get

\[
(3.16) \quad R = 0
\]

for \( D \neq 0 \), we obtain the same results.
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Theorem 3.2 For $B \neq 0$ or $D \neq 0$, a decomposable manifold $(WS)_n$ has zero scalar curvature.

The remaining sections deal with non-decomposable manifolds $(WS)_n$.

4 Ricci-associate of the vector field $\lambda_i$.

Let the expression

(4.1) \[ v_i = R_{hi} \lambda^h \]

holds. Then the vector field $v_i$ shall be called the Ricci-associate of the vector. Let $\lambda = \vec{a}$, then we get

(4.2) \[ v_i = R_{hi} a^h \quad (a^i \delta_i^p = \delta^p_r) \]

If $R$ is constant, then from (2.5), (2.7) and (4.2), we get

(4.3) \[ v_i = -b_i R \]

If $R$ is zero, then using the same equations, we find $\vec{v} = 0$. Therefore, $R$ is not zero. From (2.6), we say that the vector $\vec{b}$ is not orthogonal to $\vec{d}$. Using (4.3), we have that the vector $\vec{v}$ is not orthogonal to both $\vec{b}$ and $\vec{d}$. If $R$ is a constant, then the vector field $v_i$ is collinear with the vector field $b_i$. Then we obtain

\[ a^i v_i = a^i R_{hi} a^h \]

Thus, if the Ricci form $R_{hi} a^h a^i$ is indefinite, then from the relation $\sum_r a^h a^i = g^{hi}$, we get $R = 0$. Since $R \neq 0$, the Ricci form $R_{hi} u^h u^i$ is definite, then $v_i$, $a_i$ are not orthogonal vectors. By the aid of (2.2) and (2.7), we get $a_i$ and $b_i$ are not orthogonal vectors. Therefore, any vector of $(WS)_n$ is not orthogonal to each other.

If $R$ is constant, then the vector field $v_i$ is collinear with the vector field $b_i$. We consider the manifold $(WS)_n$, ($n > 2$) of non-constant scalar curvature in which the Ricci-associate of $a_i$ is not necessarily collinear with $b_i$. Now, for $v_i = R_{hi} a^h$, we get

\[ a^i v_i = a^i R_{hi} a^h \]

Thus, if the Ricci form $R_{hi} u^h u^i$ is definite, then $a^i v_i \neq 0$, i.e., $a_i$ and $v_i$ are not orthogonal vectors. Similarly, any vector of $(WS)_n$ is not orthogonal to $v_i$ and the vectors $a_i$ and $d_i$ are not orthogonal.

From (2.6), we get $\vec{b}$ is not orthogonal to $\vec{d}$. If we put $\vec{\lambda} = \vec{b}$ or $\vec{\lambda} = \vec{d}$ in (4.1), we obtain the same results.

This leads to the following theorem:

Theorem 4.1 In a $(WS)_n$, ($n > 2$) with constant scalar curvature, if the vector fields $a_i$ form an orthogonal ennuple, then the Ricci-associate of $a_i$ are not orthogonal to the any vector of $(WS)_n$. If in $(WS)_n$, ($n > 2$) with non-constant scalar curvature, the Ricci form $R_{hi} u^h u^i$ is definite, then the any vector of $(WS)_n$ is not orthogonal to each other.
Now, we consider the manifold \((WS)_n\) with non-constant scalar curvature. Then \(R\) will be another non-zero vector field. A vector field \(u_i\) is called closed if \(u_{i,m} - u_{m,i} = 0\).

We enquire if the Ricci-associate of \(a_i\) and \(a_i\) can be both closed. In virtue of \(v_i = R_{hi}a^h\), the equations (2.8) and (2.9) can be expressed as follows:

\begin{equation}
(4.4) \quad a_lR_{,m} - a_mR_{,l} = 4(a_lv_m - a_mv_l)
\end{equation}

and

\begin{equation}
(4.5) \quad R(a_{i,m} - a_{m,i}) + (a_lR_{,m} - a_mR_{,l}) + 4(v_{l,m} - v_{m,l}) = 0
\end{equation}

From (4.5), it follows that

\begin{equation}
(4.6) \quad a_lR_{,m} - a_mR_{,l} = 0
\end{equation}

and therefore, using (4.4), we get

\begin{equation}
(4.7) \quad a_lv_m - a_mv_l = 0
\end{equation}

Now, from (4.6) and (4.7), it follows that the vector field \(R_{,i}\) is collinear with both the vector fields \(a_i\) and \(v_i\). We can therefore state as follows:

**Theorem 4.2** In the manifold \((WS)_n\) with non-constant scalar curvature, the Ricci-associate of \(a_i\) and \(a_i\) cannot be both closed, unless the vector field \(R_{,i}\) is collinear with both \(a_i\) and its Ricci-associate.

### 5 The manifold \((WS)_n\) \((n > 3)\) with cyclic Ricci tensor.

A Riemannian manifold is said to be cyclic Ricci tensor if the condition

\begin{equation}
(5.1) \quad R_{ij,k} + R_{jk,i} + R_{ki,j} = 0
\end{equation}

holds. According to [5], from (5.1), we have \(R = const\). Hence, using (2.5) and (2.7), we get

\begin{equation}
(5.2) \quad a^bR_{hl} + b_lR = 0, \quad R = const \neq 0
\end{equation}

In this section, we suppose that a \((WS)_n\) \((n > 2)\) has cyclic Ricci tensor. Using (2.1) and I. Bianchi Identity, we get

\begin{equation}
(5.3) \quad R_{ij,k} + R_{jk,i} + R_{ki,j} = (a_i + b_i + d_i)R_{jk} + (a_j + b_j + d_j)R_{ki} + (a_k + b_k + d_k)R_{ij}
\end{equation}

With the help of (2.6), the equation (5.3) can be written as

\begin{equation}
(5.4) \quad A^*_iR_{jk} + A^*_jR_{ki} + A^*_kR_{ij} = 0
\end{equation}

where \(A^*_i = a_i + 2b_i\). Transvecting (5.4) with \(A^*_i\), we obtain

\begin{equation}
(5.5) \quad A^{*i}A^*_iR_{jk} + A^{*i}A^*_jR_{ki} + A^{*i}A^*_kR_{ij} = 0
\end{equation}
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where \( A^*_i = a^i + 2b^i \). By the aid of the equations (2.2), (2.5), (2.7) and (5.2), we can easily obtain that

\[
A^*_i A^*_j = \frac{-A^*_i R}{2}
\]

Then, using (5.5) and (5.6), we have

\[
A^*_i A^*_j R_{jk} = A^*_j A^*_i R
\]

In this case, transvecting (5.7) with \( A^*_i \) and using the equation (5.6), we get

\[
\frac{3}{2} A^*_i A^*_j R = 0
\]

i.e.,

\[
A^*_i A^*_i = 0 \quad \text{or} \quad A^*_j = 0
\]

Suppose that \( A^*_i A^*_i = 0 \) and the metric of this manifold is positive definite. In this case, we obtain

\[
A^*_i A^*_i = (\vec{a}_i + 2\vec{b}_i)(\vec{a}^i + 2\vec{b}^i) = (\vec{a} + 2\vec{b})^2
\]

Therefore, \( A^*_i A^*_i \neq 0 \). From (5.9), we have

\[
A^*_i = 0
\]

If \( A^*_i = 0 \), then from (5.7), we have \( R_{jk} = 0 \). By the aid of (5.2), we get \( b_l = 0 \). (since \( R \neq 0 \)). Using (2.2) and (2.6), we obtain \( a_i = d_i = 0 \), i.e., in the manifold \((WS)_n\) with a positive metric, if \( A^*_i = 0 \), then this manifold is a symmetric manifold. For this reason, the condition \( A^*_j \neq 0 \) must be hold. Therefore, by the aid of (5.8), we say that \( R = 0 \). From (5.3), we obtain

\[
A^*_i A^*_j R_{jk} = A^*_j A^*_i A^*_i R_{ki} + A^*_k A^*_i A^*_i R_{ij} = 0
\]

where \( A^*_i = a^i + b^i + d^i \). In this case, with the help of (2.2), (2.4) and (2.7), we get \( A^*_i R_{hl} = 0 \). It is easy to see that the equation (5.12) can be changed into the form

\[
A^*_i A^*_i R_{jk} = 0
\]

then, we get

\[
A^*_i A^*_i = (a^h + b^h + d^h)(a_h + b_h + d_h) = (\vec{a} + \vec{b} + \vec{d})^2
\]

If the metric of the manifold is positive definite, then \( A^*_i A^*_i \neq 0 \). Therefore, we have \( R_{jk} \) which means that the manifold is an Einstein manifold with zero scalar curvature.

It is known [6] that an n-dimensional \((n > 2)\) Einstein manifold has constant curvature. Then, an Einstein manifold \((WS)_3\) is locally symmetric. Nevertheless, the condition \( R_{hijk,l} \neq 0 \) is satisfied.

In view of these results, it follows that an Einstein \((WS)_n(n > 2)\) does not exist, i.e., \( R \neq 0 \), the manifold \((WS)_n\) can not be an Einstein manifold with a cyclic positive definite Ricci tensor.

Thus, we can state the following theorem:

**Theorem 5.1** If a manifold \((WS)_n(n > 2)\) with positive definite metric has cyclic Ricci tensor, then this manifold is an Einstein manifold with zero scalar curvature.
(WS)$_n$ admitting a concurrent or a recurrent vector field.

This section consists of two parts, the first deals with a (WS)$_n$ admitting a concurrent vector field $u^i$, [4], given by

\begin{equation}
(6.1) \quad u^i_j = p\delta^i_j
\end{equation}

where $p$ is a non-zero constant and the second deals with a (WS)$_n$ admitting a recurrent vector field $u^i$, [4], given by

\begin{equation}
(6.2) \quad u^i_j = \beta_j u^i
\end{equation}

where $\beta_j$ is a non-zero covariant vector.

Part I.

According to [5], we have

\begin{equation}
(6.3) \quad u^h R_{hijk} = 0, \quad u^h R_{hkk} = 0, \quad p R_{lij} + u^h R_{hiijk,l} = 0
\end{equation}

Transvecting (1.2) with $u^h$, we get

$$u^h R_{hijk,l} = a_l u^h R_{hijk} + b_h u^h R_{lij} + b_j u^h R_{hjik} + d_j u^h R_{hihk} + d_k u^h R_{hijl}$$

In virtue of (6.3), the above equation takes the form

$$R_{lij}(p + b_h u^h) = 0$$

Since $R_{lij} \neq 0$, we get

\begin{equation}
(6.4) \quad p + u^h b_h = 0
\end{equation}

i.e., the condition $u^h b_h = -p \neq 0$ holds.

Multiplying (2.1) by $u^k$ and $u^l$, respectively, and using the symmetric properties of $R_{hijk}$ and from (6.3), we get

\begin{equation}
(6.5) \quad u^k d_k = -p \neq 0, \quad u^l R_{hijk,l} = a_l u^l R_{hijk}
\end{equation}

From (6.5), we have

\begin{equation}
(6.6) \quad u^l R_{l} = a_l u^l R
\end{equation}

Using (2.5), (2.7), (6.3)$_2$ and (6.6), we get

\begin{equation}
(6.7) \quad (a_l - 2 b_l) u^l R = 0
\end{equation}

i.e.,

\begin{equation}
(6.8) \quad a_l u^l = 2 b_l u^l \quad \text{or} \quad R = 0
\end{equation}
We suppose that the condition $R \neq 0$ holds, by the aid of (6.4), (6.5) and (6.8), we say that: If the manifold $(WS)_n$ admits a concurrent vector field $u^i$ given by (6.1), then $u_i$ is not orthogonal to the vectors of $(WS)_n$.

Suppose that $R = 0$, from (2.1) and (6.3), we obtain

\begin{equation}
(6.9) \quad u^i R_{ij,l} = u^i b_i R_{ij}
\end{equation}

Using (6.1) and (6.3), (6.9) takes the form

\begin{equation}
(6.10) \quad (p - u^i b_i) R_{ij} = 0
\end{equation}

Then, with the help of (6.4) and (6.10), we find $R_{ij} = 0$, which means that the manifold is an Einstein manifold with zero scalar curvature.

This leads to the following theorem:

**Theorem 6.1** If, in a $(WS)_n$ which admits a concurrent vector field $u^i$ given by (6.1), $R$ is non-zero then $u_i$ is not orthogonal to the vectors of $(WS)_n$. Otherwise, the manifold is an Einstein manifold with zero scalar curvature.

**Part II.**

If the recurrent vector field $u^i$ given by (6.2) is non-null, then we have

\begin{equation}
(6.11) \quad u^h R_{hijk, l} = 0, \quad u^h R_{hijk} = 0
\end{equation}

Using the expressions (1.2) and (6.11), we have $u^h b_h R_{hijk} = 0$. Since $R_{hijk} \neq 0$, it is easy to see that

\begin{equation}
(6.12) \quad u^h b_h = 0
\end{equation}

By the aid of (1.2) and (6.11) and remembering the symmetric properties of $R_{hijk}$, we get

\begin{equation}
(6.13) \quad u^k d_k = 0
\end{equation}

Using (6.11), we have

\begin{equation}
(6.14) \quad u^h R_{hk, l} = 0, \quad u^h R_{hk} = 0
\end{equation}

By the aid of (2.3), (2.7), (6.13) and (6.14), $u^l R_{l} = 0$ holds. Then, from (2.2), we have $a_l u^l R = 0$. Let us suppose that $R \neq 0$. Hence, we get

\begin{equation}
(6.15) \quad a_l u^l = 0
\end{equation}

Differentiating (6.11)2 covariantly and using (6.2), we get $\beta_i u^l = 0$, i.e., $\beta_i$ is orthogonal to $u^l$.

Hence, we can state the theorem as follows:

**Theorem 6.2** If a $(WS)_n$ (with $R \neq 0$) admits a non-null recurrent vector field $u^i$ given by (6.2), then the associated vector field of recurrence $\beta_i$ is orthogonal to $u_i$ and $u_i$ is orthogonal to the vectors of $(WS)_n$. 
References


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