Infinitesimal holomorphically projective transformations on tangent bundles with complete lift connection

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Dedicated to Professor Shigeyoshi Fujimura on his sixtieth birthday

Abstract

Let \((M, g)\) be a Riemannian manifold and \(TM\) its tangent bundle with complete lift connection and adapted almost complex structure. We determine the infinitesimal holomorphically projective transformation on \(TM\). Furthermore, if \(TM\) admits a non-affine infinitesimal holomorphically projective transformation, then \(M\) and \(TM\) are locally flat.

Key words: infinitesimal holomorphically projective transformation, complete lift connection, adapted almost complex structure.

§1. Introduction

Let \(M\) be an \(n\)-dimensional manifold and \(TM\) its tangent bundle. We denote by \(\mathfrak{T}_r^s(M)\) the set of all tensor fields of type \((r, s)\) on \(M\). Similarly, we denote by \(\mathfrak{T}_r^s(TM)\) the corresponding set on \(TM\).

Let \(\nabla\) be an affine connection on \(M\). A vector field \(V\) on \(M\) is called an infinitesimal projective transformation if there exists a 1-form \(\Omega\) on \(M\) such that

\[(L_V \nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X\]

for any \(X, Y \in \mathfrak{T}_0^1(M)\), where \(L_V\) is the Lie derivation with respect to \(V\). In this case \(\Omega\) is called the associated 1-form of \(V\). Especially, if \(\Omega = 0\), then \(V\) is called an infinitesimal affine transformation.

Next let \((M, J)\) be an almost complex manifold with affine connection \(\nabla\). A vector field \(V\) on \(M\) is called an infinitesimal holomorphically projective transformation if there exists a 1-form \(\Omega\) on \(M\) such that

\[(L_V \nabla)(X, Y) = \Omega(X)Y + \Omega(Y)X - \Omega(JX)JY - \Omega(JY)JX\]

for any \(X, Y \in \mathfrak{T}_0^1(M)\). In this case \(\Omega\) is also called the associated 1-form of \(V\). Especially, if \(\Omega = 0\), then \(V\) is the infinitesimal affine transformation. S. Ishihara [3] has

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introduced the notion of infinitesimal holomorphically projective transformation, and S. Tachibana and S. Ishihara [6] investigated infinitesimal holomorphically projective transformations on Kaehlerian manifolds. In [1] we have proved that (1) an infinitesimal holomorphically projective transformation is infinitesimal isometry on a compact Kaehlerian manifold with non-positive constant scalar curvature and (2) a compact Kaehlerian manifold $M$ with constant scalar curvature is holomorphically isometric to a complex projective space with Fubini-Study metric if $M$ admits a non-isometric infinitesimal holomorphically projective transformation.

It is well-known that there are several lift connections of $\nabla$ on $TM$ ([7, 8]). In our previous paper [2], we study the infinitesimal holomorphically projective transformation on $TM$ with horizontal lift connection and proved the following:

**Theorem A.** Let $(M, g)$ be a Riemannian manifold and $TM$ its tangent bundle with horizontal lift connection and adapted almost complex structure. A vector field $\tilde{\Omega}$ on $TM$ if and only if there exist $\varphi, \psi \in \mathfrak{X}^0(M)$, $B = (B^h)$, $D = (D^h) \in \mathfrak{X}_0^1(M)$, $A = (A^h_i)$, $C = (C^h_i)$ that satisfy

\begin{align*}
(1) & \quad (\tilde{\nabla}^h, \tilde{\nabla}^h) = (B^h + y^a A^h_a + 2\varphi y^h - y^h y^a \psi_a, D^h + y^a C^h_a + 2\psi y^h + y^h y^a \Phi_a), \\
(2) & \quad (\tilde{\nabla}^h_i, \tilde{\nabla}^h_i) = (\partial_i \psi, \partial_i \varphi) = (\psi_i, \Phi_i), \\
(3) & \quad \nabla_j \Phi_i = 0, \quad \nabla_j \psi_i = 0, \\
(4) & \quad \nabla_j A^h_i = \Phi_j \delta^h_i - \Phi_i \delta^h_j, \\
(5) & \quad \nabla_j C^h_i = \psi_i \delta^h_j - \psi_j \delta^h_i - K_{ajj}^h B^a, \\
(6) & \quad L_B \Gamma^h_{ji} = \nabla_j \nabla_i B^h + K_{ajj}^h B^a = \psi_j \delta^h_i + \psi_i \delta^h_j, \\
(7) & \quad \nabla_j \nabla_i D^h = -\Phi_i \delta^h_j - \Phi_j \delta^h_i, \\
(8) & \quad A^h_\delta K_{ajj}^h + 2\varphi K_{ajj}^h = 0, \\
(9) & \quad \psi_i K_{ajj}^h = 0,
\end{align*}

where $(\tilde{\nabla}^h, \tilde{\nabla}^h) := \tilde{\nabla}^a E^a + \tilde{\nabla}^a E^a = \tilde{\nabla}$, $(\tilde{\nabla}^h_i, \tilde{\nabla}^h_i) := \tilde{\nabla}^a dx^a + \tilde{\nabla}^a dy^a = \tilde{\nabla}$, $\nabla$ denotes the Riemannian connection on $M$, $\Gamma^h_{ji}$ the coefficients of $\nabla$ and $K = (K_{ajj}^h)$ the Riemannian curvature tensor of $(M, g)$ defined by $K_{ajj}^h := \partial_k \Gamma^h_{ji} - \partial_j \Gamma^h_{ki} + \Gamma^h_{ji} \Gamma^h_{ka} - \Gamma^h_{ki} \Gamma^h_{ja}$.

**Theorem B.** Let $(M, g)$ be a complete Riemannian manifold and $TM$ its tangent bundle with horizontal lift connection and adapted almost complex structure. If $TM$ admits a non-affine infinitesimal holomorphically projective transformation, then $M$ and $TM$ are locally flat.

Let $(M, g)$ be a Riemannian manifold and $\nabla$ its Riemannian connection. The horizontal lift connection of $\nabla$ does not coincide with the Riemannian connection of horizontal lift metric of $g$ to $TM$ with respect to $\nabla$. Two connections coincide if and only if $M$ is locally flat. On the other hand the complete lift connection of
∇ is the Riemannian connection of complete lift metric of \( g \). Here, since \( g \) satisfies \( \nabla g = 0 \), the complete lift metric of \( g \) coincides with the horizontal lift metric of \( g \) (see [7, 8]). Moreover, in the case of horizontal lift connection it is necessary to assume that \( M \) is complete to prove Theorem B, but not necessary in the case of complete lift connection (see Theorem 2 stated below). Therefore, in this paper we investigate the case of complete lift connection and prove the following:

**Theorem 1.** Let \((M, g)\) be a Riemannian manifold and \( TM \) its tangent bundle with complete lift connection and adapted almost complex structure. A vector field \( \tilde{V} \) is an infinitesimal homoloromorphically projective transformation with associated 1-form \( \Omega \) on \( TM \) if and only if there exist \( \varphi, \psi \in \mathcal{T}_0^1(M) \), \( A = (A_i^h), C = (C_i^h) \in \mathcal{T}_1^1(M) \) satisfying

\[
\begin{align*}
(1) & \quad (\tilde{V}^h, \tilde{V}_h) = (B^h + y^a A_a^h + 2\varphi y^h - y^b y^a \Psi_a, \quad D^h + y^a C_a^h + 2\psi y^h + y^b y^a \Phi_a), \\
(2) & \quad (\tilde{\Omega}_i, \tilde{\Omega}_i) = (\partial_i \psi, \partial_i \varphi) = (\Psi_i, \Phi_i), \\
(3) & \quad \nabla_j \Phi_i = 0, \quad \nabla_j \Psi_i = 0, \\
(4) & \quad \nabla_j A_i^h = \Phi_j \delta_i^h - \Phi_i \delta_j^h, \\
(5) & \quad \nabla_j C_i^h = \Psi_i \delta_j^h - \Psi_j \delta_i^h - K_{a ji}^h B_a, \\
(6) & \quad L_B \Gamma_{ji}^h = \nabla_j \nabla_i B^h + K_{a ji}^h B_a = \Psi_i \delta_j^h + \Psi_j \delta_i^h, \\
(7) & \quad L_D \Gamma_{ji}^h = \nabla_j \nabla_i D^h + K_{a ji}^h D_a = -\Phi_j \delta_i^h - \Phi_i \delta_j^h, \\
(8) & \quad A_k^h K_{a ji}^h + 2\varphi K_{k ji}^h = 0, \\
(9) & \quad \Psi_i K_{a ji}^h = 0, \quad \Phi_i K_{a ji}^h = 0, \\
(10) & \quad B^a \nabla_a K_{k ji}^h = K_{k ji}^h C_{a}^h - K_{a ji}^h C_{k}^a - K_{k ai}^h \nabla_j B^a - K_{k ja}^h \nabla_i B^a,
\end{align*}
\]

where \((\tilde{V}^h, \tilde{V}_h) := \tilde{V}^a E_a + \tilde{V}_a E_a = \tilde{V} \) and \((\tilde{\Omega}_i, \tilde{\Omega}_i) := \tilde{\Omega}_a dx^a + \tilde{\Omega}_a dy^a = \tilde{\Omega} \).

**Theorem 2.** Let \((M, g)\) be a Riemannian manifold and \( TM \) its tangent bundle with complete lift connection and adapted almost complex structure. If \( TM \) admits a non-affine infinitesimal homoloromorphically projective transformation, then \( M \) and \( TM \) are locally flat.

In the present paper everything will be always discussed in the \( C^\infty \)-category, and manifolds will be assumed to be connected and dimension \( n > 1 \).

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**§2. Preliminaries**

In this section we shall give some definitions and formulae on \( TM \) for later use (for details, see [7, 8]). Let \((M, g)\) be a Riemannian manifold, \( \nabla \) the Riemannian connection of \( g \) and \( \Gamma_{ji}^h \) the coefficients of \( \nabla \), i.e., \( \Gamma_{ji}^h \partial_h := \nabla_{ij} \partial_h \), where \( \partial_h = \frac{\partial}{\partial x^h} \) and \((x^h)\) is the local coordinates of \( M \).
Adapted frame of $TM$

We define a local frame $\{E_i, \bar{E}_i\}$ of $TM$ as follows:

\[(2.1)\quad E_i := \partial_i - y^b \Gamma^a_{ib} \partial_a \quad \text{and} \quad \bar{E}_i := \partial_i,\]

where $(x^h, y^h)$ is the induced coordinates of $TM$ derived from the local coordinates $(x^h)$ of $M$ and $\partial_i := \frac{\partial}{\partial x^i}$. This frame $\{E_i, \bar{E}_i\}$ is called the adapted frame of $TM$.

Then $\{dx^h, \delta y^h\}$ is the dual frame of $\{E_i, \bar{E}_i\}$, where $\delta y^h := dy^h + y^b \Gamma^h_{ab} dx^a$.

By the definition of the adapted frame, we have the following

**Lemma 1** The Lie brackets of the adapted frame of $TM$ satisfy the following identities:

1. $[E_j, E_i] = y^b K_{ij}^b E_a$
2. $[E_j, \bar{E}_i] = \Gamma^a_{ji} \bar{E}_a$
3. $[\bar{E}_j, E_i] = 0$
4. $[\bar{E}_j, \bar{E}_i] = 0$

Complete lift connection of $\nabla$

Let $X = X^a \partial_a$ be a vector field on $M$. Then the complete lift $X^C$ of $X$ is defined by

\[(2.2)\quad X^C := X^a E_a + y^b \nabla_b X^a E_{\bar{a}},\]

There exists a unique affine connection $\tilde{\nabla}$ on $TM$ which satisfies

\[(2.3)\quad \tilde{\nabla}_{X^C} Y^C = (\nabla_X Y)^C\]

for any $X, Y \in \mathfrak{T}_0(M)$. This affine connection is called the complete lift connection of $\nabla$ to $TM$. Then we have

\[(2.4)\quad \tilde{\nabla}_{E^j} E_i = \Gamma^a_{ji} E_a, \quad \tilde{\nabla}_{\bar{E}^j} \bar{E}_i = 0, \quad \tilde{\nabla}_{E^j} \bar{E}_i = 0.\]

Adapted almost complex structure on $TM$

Let $X = X^a \partial_a$ be a vector field on $M$. Then the vertical lift $X^V$ and the horizontal lift $X^H$ of $X$ with respect to $\nabla$ are defined as follows:

\[(2.5)\quad X^H := X^a E_a \quad \text{and} \quad X^V := X^a \bar{E}_a.\]

We now define a tensor field $J$ of type $(1,1)$ on $TM$ by

\[(2.6)\quad JX^H := X^V, \quad JX^V := -X^H\]

for any $X \in \mathfrak{T}_0(M)$, i.e.,

\[JE_i = \bar{E}_i \quad \text{and} \quad J\bar{E}_i = -E_i.\]

Then we obtain

\[J^2 = -I.\]
Therefore $J$ is an almost complex structure on $TM$. This almost complex structure is called the adapted almost complex structure. It is known that $J$ is integrable if and only if $M$ is locally flat.

§ 3. Proofs of Theorems

Proof of Theorem 1.

Here we prove only the necessary condition because it is easy to prove the sufficient condition. Let $\tilde{V}$ be an infinitesimal holomorphically projective transformation with the associated 1-form $\tilde{\Omega}$ on $TM$.

\begin{equation}
(L_{\tilde{V}} \tilde{\nabla})(\tilde{X}, \tilde{Y}) = \tilde{\Omega}(\tilde{X})\tilde{Y} + \tilde{\Omega}(\tilde{Y})\tilde{X} - \tilde{\Omega}(J\tilde{X})J\tilde{Y} - \tilde{\Omega}(J\tilde{Y})J\tilde{X}
\end{equation}

for any $\tilde{X}, \tilde{Y} \in \mathfrak{T}_0^1(TM)$.

From $(L_{\tilde{V}} \tilde{\nabla})(E_i, E_j) = \tilde{\Omega}_j E_i + \tilde{\Omega}_i E_j - \tilde{\Omega}_i E_j$, we obtain

\begin{equation}
\partial_j \partial_i \tilde{V}^h = -\tilde{\Omega}_j \delta_i^h - \tilde{\Omega}_i \delta_j^h
\end{equation}

and

\begin{equation}
\partial_j \partial_i \tilde{V}^h = \tilde{\Omega}_j \delta_i^h + \tilde{\Omega}_i \delta_j^h.
\end{equation}

From (3.2), there exist $\varphi \in \mathfrak{T}_0^0(M), \Psi = (\Psi_i) \in \mathfrak{T}_1^0(M), B = (B^h) \in \mathfrak{T}_0^1(M)$ and $A = (A_i^h) \in \mathfrak{T}_1^1(M)$ satisfying

\begin{equation}
\tilde{\psi} = -\varphi + y^a \Psi_a,
\end{equation}

\begin{equation}
\tilde{\Omega}_i = \partial_i \tilde{\psi} = \Psi_i
\end{equation}

and

\begin{equation}
\tilde{V}^h = B^h + y^a A_i^h + 2\varphi y^h - y^a \Psi_a y^h,
\end{equation}

where $\tilde{\psi} := -\frac{1}{n+1} \partial_a \tilde{V}^a$.

Similarly, from (3.3), there exist $\psi \in \mathfrak{T}_0^0(M), \Phi = (\Phi_i) \in \mathfrak{T}_1^0(M), D = (D^h) \in \mathfrak{T}_0^1(M)$ and $C = (C_i^h) \in \mathfrak{T}_1^1(M)$ satisfying

\begin{equation}
\tilde{\phi} = \psi + y^a \Phi_a,
\end{equation}

\begin{equation}
\tilde{\Omega}_i = \partial_i \tilde{\phi} = \Phi_i
\end{equation}

and

\begin{equation}
\tilde{V}^h = D^h + y^a C_i^h + 2\psi y^h + y^a \Phi_a y^h,
\end{equation}

where $\tilde{\phi} := \frac{1}{n+1} \partial_a \tilde{V}^a$.  


Next, from (3.1) we have

\[(3.10) \quad (L_\Phi \bar{\nabla})(E_j, \ E_i) = \Phi_j E_i + \Phi_i E_j + \Psi_j E_i + \Psi_i E_j,\]

or

\[(L_\Phi \bar{\nabla})(E_j, \ E_i) = \Phi_j E_i + \Phi_i E_j + \Psi_j E_i + \Psi_i E_j,\]

from which, we get

\[
(\Phi_j \delta^a_i + \Phi_i \delta^a_j)E_a + (\Psi_j \delta^a_i + \Psi_i \delta^a_j)E_a
= \{(\nabla_j A_i^a + 2 \delta^a_i \partial_j \varphi) - y^b(\delta^a_b \nabla_j \Psi_i + \delta^a_i \nabla_j \Psi_b)\}E_a
+ \{(K_{bji}^a B^b + \nabla_j C_i^a + 2 \delta^a_i \partial_j \psi) + y^b(A_i^c K_{cji}^a - A_i^c K_{jbc}^a + 4 \varphi K_{bji}^a + \delta^a_i \nabla_j \Phi_i + \delta^a_i \nabla_j \Phi_b) + y^c y^b(\Psi_i K_{jeb}^a - 2 \Psi_i K_{bji}^a)\}E_a.
\]

Comparing both hands of the above equation, we obtain

\[
\Phi_i = \partial_i \varphi, \quad \nabla_j \Phi_i = 0,
\Psi_i = \partial_i \psi, \quad \nabla_j \Psi_i = 0,
\]

\[(3.12) \quad \nabla_j A_i^h = \Phi_i \delta_j^h - \Phi_j \delta_i^h,
\nabla_j C_i^h = \Psi_i \delta_j^h - \Psi_j \delta_i^h - K_{aji}^h B^a,
A_i^c K_{aji}^h = -2 \varphi K_{kji}^h, \quad \Psi_i K_{kji}^h = 0.
\]

Lastly, from \((L_\Phi \bar{\nabla})(E_j, \ E_i) = \Psi_j E_i + \Psi_i E_j - \Phi_j E_i - \Phi_i E_j,\) we obtain

\[
(\Psi_j \delta^a_i + \Psi_i \delta^a_j)E_a - (\Phi_j \delta^a_i + \Phi_i \delta^a_j)E_a
= (L_B \Gamma_{ji}^h)E_a + (L_D \Gamma_{ji}^h)
+ y^b(B^c \nabla_c K_{bji}^a - K_{bji}^c C_i^c + K_{cji}^a C_i^c + K_{bci}^a \nabla_j B^c + K_{bcj}^a \nabla_i B^c)
+ y^c y^b(\Phi_i K_{bji}^a + \Psi_i K_{bji}^a - \Phi_j K_{bji}^a - \Phi_i K_{cji}^a B^c)
+ 2 \varphi \nabla_c K_{bji}^a + 2 \varphi \nabla_j K_{cji}^a + A_i^d \nabla_d K_{kji}^a + A_i^d \nabla_j K_{dib}^a)\}E_a,
\]

from which, we get the following important information:

\[(3.13) \quad L_B \Gamma_{ji}^h = \Psi_j \delta_i^h + \Psi_i \delta_j^h.
\]

\[(3.14) \quad L_D \Gamma_{ji}^h = -\Phi_j \delta_i^h - \Phi_i \delta_j^h.
\]

(That is, \(B\) and \(D\) are infinitesimal projective transformations on \(M\).)

\[(3.15) \quad B^a \nabla_a K_{kji}^h = K_{kji}^a C_i^h - K_{a/ji}^h C_k^a - K_{kai}^h \nabla_j B^a - K_{kja}^h \nabla_i B^a.
\]

\[(3.16) \quad \Phi_i K_{kji}^h = 0.
\]

This completes the proof.
Proof of Theorem 2.

Let $\tilde{V}$ be a non-affine infinitesimal holomorphically projective transformation on $TM$. Using (3) in Theorem 1, we have $\nabla_i ||\Phi||^2 = \nabla_i ||\Psi||^2 = 0$. Therefore, $||\Phi||$ and $||\Psi||$ are constant on $M$. Suppose that $M$ is not locally flat, then $\Phi = \Psi = 0$ by virtue of (9) in Theorem 1, that is, $\tilde{V}$ is an infinitesimal affine transformation. This is a contradiction. Therefore $M$ is locally flat.

In this case, $TM$ is also locally flat, because the Riemannian curvature tensor of $\nabla$ is the complete lift of $K$ ([7, 8]).

References


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