# Finsler manifolds with special Berwald curvature

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**Abstract.** In this paper, we construct a new class of Finsler metrics which is an extension of the class of Berwald metrics. We prove that every complete Finsler metric in this class is Riemannian, whenever its Cartan tensor is bounded. Then we show that the class of generalized Douglas-Weyl metrics contains this new class of Finsler metrics.

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## 1 Introduction

For a Finsler metric $F = F(x, y)$, its geodesics are characterized by the system of differential equations $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients and given by following

$$G^i = \frac{1}{4}g^{il}\Big[\frac{\partial^2(F^2)}{\partial x^k \partial y^l}y^k - \frac{\partial(F^2)}{\partial x^l}\Big], \quad y \in T_xM.$$

A Finsler metric $F$ is called a Berwald metric if $G^i = \frac{1}{2}\Gamma^i_{jk}(x)y^j y^k$ is quadratic in $y \in T_xM$ for any $x \in M$ [5]. In [7], it is proved that on a Berwald space, the parallel translation along any geodesic preserves the Minkowski functionals. Therefore Berwald spaces can be viewed as Finsler spaces modeled on a single Minkowski space.

On the other hand, various interesting special forms of Cartan, Landsberg and Berwald tensors have been obtained by some Finslerists. The Finsler spaces having such special forms were called C-reducible, P-reducible, semi-C-reducible, isotropic Berwald curvature, isotropic mean Berwald curvature, and isotropic Landsberg curvature, etc [6][9][14][20][22].

In [8], Matsumoto introduced the notion of C-reducible metrics and proved that any Randers metric is C-reducible. Then Matsumoto-Hōjō proved that the converse is true [11]. A Randers metric $F = \alpha + \beta$ is just a Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ perturbed by a one form $\beta = b_i(x)y^i$ on a manifold $M$ such that $||\beta||_\alpha < 1$ (see [3][15][21]). In [10], Matsumoto-Shibata introduced the notion of semi-C-reducibility by considering the form of Cartan torsion of a non-Riemannian $(\alpha, \beta)$-metric on a manifold $M$ with dimension $n \geq 3$.

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In [6], Shen-Chen by using the structure of the Funk metric, introduced the notion of isotropic Berwald metrics. This motivates us to study special forms of Berwald curvature for other important special Finsler metrics.

In this paper, we define a new class of Finsler metrics on manifolds which their Berwald curvature satisfy in following

$$B^i{}_{jkl} = C_{jkl}\ell^i + \lambda(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}), \tag{1.1}$$

where $\ell^i = F^{-1}y^i$, $C_{ijk}$ is the Cartan tensor of $F$, $h_{ij} = g_{ij} - F^{-2}y_i y_j$ is the angular metric, $h^i_j = g^{ik}h_{kj}$ and $\lambda = \lambda(x, y)$ is a homogeneous function of degrees -1 with respect to $y$.

**Example 1.1.** Consider the following Finsler metric on the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$,

$$F(x, y) := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1 - |x|^2}, \quad y \in T_x\mathbb{B}^n = \mathbb{R}^n$$

where $|.|$ and $\langle , \rangle$ denote the Euclidean norm and inner product in $\mathbb{R}^n$, respectively. $F$ is called the Funk metric which is a Randers metric on $\mathbb{B}^n$ [17]. Since $G^i = \frac{1}{2}F$, then we have

$$B^i{}_{jkl} = C_{jkl}\ell^i + \frac{1}{2F}(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}).$$

Then the Funk metric $F$ satisfies (1.1) with $\lambda = \frac{1}{2F}$.

In this paper, we study compact Finsler manifolds $(M, F)$ which their Berwald curvatures satisfy (1.1) and prove the following.

**Theorem 1.1.** *Let $(M, F)$ be a compact Finsler manifold. Suppose that $F$ satisfies (1.1). Then $F$ is a Riemannian metric.*

The Douglas tensor is another non-Riemanian curvature defined as follows

$$D^i{}_{jkl} := \Big[G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i\Big]_{y^j y^k y^l}.$$

The Douglas tensor $\mathcal{D}$ is a non-Riemannian projective invariant, namely, if two Finsler metrics $F$ and $\bar{F}$ are projectively equivalent, $G^i = \bar{G}^i + Py^i$, where $P = P(x, y)$ is positively $y$-homogeneous of degree one, then the Douglas tensor of $F$ is same as that of $\bar{F}$ [13]. The notion of Douglas curvature was proposed by B*á*cs*ó* and Matsumoto as a generalization of Berwald curvature [1]. A Finsler metric is called a generalized Douglas-Weyl (GDW) metric if the Douglas tensor satisfy in $h^i_\alpha D^\alpha{}_{jkl|m}y^m = 0$ [12]. In [2], Bácsó-Papp show that this class of Finsler metrics is closed under projective transformation. In this paper, we prove the following.

**Theorem 1.2.** *Every Finsler metric satisfying (1.1) is a generalized Douglas-Weyl metric.*

In this paper, we use the Berwald connection and the $h$- and $v$- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively (see [18][19]).

## 2 Preliminaries

Let $M$ be a $n$-dimensional $C^\infty$ manifold. Denote by $T_xM$ the tangent space at $x \in M$, by $TM = \cup_{x\in M} T_xM$ the tangent bundle of $M$, and by $TM_0 = TM \setminus \{0\}$ the slit tangent bundle on $M$. A Finsler metric on $M$ is a function $F : TM \to [0, \infty)$ which has the following properties: (i) $F$ is $C^\infty$ on $TM_0$; (ii) $F$ is positively 1-homogeneous on the fibers of tangent bundle $TM$, and (iii) for each $y \in T_xM$, the following quadratic form $\mathbf{g}_y$ on $T_xM$ is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s \partial t}\Big[F^2(y + su + tv)\Big]|_{s,t=0}, \quad u, v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of $F_x$, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u, v)\Big]|_{t=0}, \quad u, v, w \in T_xM.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if $F$ is Riemannian [4][17].

The horizontal covariant derivatives of $\mathbf{C}$ along geodesics give rise to the Landsberg curvature $\mathbf{L}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ defined by $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^i v^j w^k$, where $L_{ijk} := C_{ijk|s}y^s$, $u = u^i \frac{\partial}{\partial x^i}|_x$, $v = v^i \frac{\partial}{\partial x^i}|_x$ and $w = w^i \frac{\partial}{\partial x^i}|_x$. The family $\mathbf{L} := \{\mathbf{L}_y\}_{y\in TM_0}$ is called the Landsberg curvature. A Finsler metric is called a Landsberg metric if $\mathbf{L}$=$\mathbf{0}$.

Given a Finsler manifold $(M, F)$, then a global vector field $\mathbf{G}$ is induced by $F$ on $TM_0$, which in a standard coordinate $(x^i, y^i)$ for $TM_0$ is given by

$$\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i},$$

where $G^i := \frac{1}{4}g^{il}[(F^2)_{x^k y^l}y^k - (F^2)_{x^l}]$, $y \in T_xM$. $\mathbf{G}$ is called the spray associated to $(M, F)$. In local coordinates, a curve $c(t)$ is a geodesic if and only if its coordinates $(c^i(t))$ satisfy $\ddot{c}^i + 2G^i(\dot{c}) = 0$.

For a tangent vector $y \in T_xM_0$, define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $\mathbf{E}_y : T_xM \otimes T_xM \to \mathbb{R}$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$ and $\mathbf{E}_y(u, v) := E_{jk}(y)u^j v^k$ where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}, \quad E_{jk} := \frac{1}{2}B^m{}_{jkm}.$$

The $\mathbf{B}$ and $\mathbf{E}$ are called the Berwald curvature and mean Berwald curvature, respectively. Then $F$ is called a Berwald metric and weakly Berwald metric if $\mathbf{B} = \mathbf{0}$ and $\mathbf{E} = \mathbf{0}$, respectively [16].

Define $\mathbf{D}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{D}_y(u, v, w) := D^i{}_{jkl}(y)u^i v^j w^k \frac{\partial}{\partial x^i}|_x$ where

$$D^i{}_{jkl} := B^i{}_{jkl} - \frac{2}{n+1}\Big[E_{jk}\delta^i_l + E_{jl}\delta^i_k + E_{kl}\delta^i_j + E_{jk,l}y^i\Big].$$

We call $\mathbf{D} := \{\mathbf{D}_y\}_{y\in TM_0}$ the Douglas curvature. A Finsler metric with $\mathbf{D} = 0$ is called a Douglas metric. The notion of Douglas metrics was proposed by Bácsó-Matsumoto as a generalization of Berwald metrics [1].

# 3 Proof of Theorem 1.1

In this section, we are going to prove the Theorem 1.1. First, we prove the following.

**Proposition 3.1.** *Let $F$ be a complete Finsler metric with bounded Cartan torsion. Suppose that $F$ satisfies (1.1). Then $F$ is a Riemannian metric*

*Proof.* By assumption, we have

$$B^i{}_{jkl} = C_{jkl}\ell^i + \lambda(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}), \tag{3.1}$$

Contracting (3.1) with $y_i$ and using

$$y_i B^i{}_{jkl} = -2L_{jkl},$$

implies that

$$L_{ijk} + \frac{1}{2}A_{jkl} = 0,$$

where $A_{ijk} := FC_{ijk}$. Let $p$ be an arbitrary point of $M$, and $y, u, v, w \in T_pM$. Let $c : (-\infty, \infty) \to M$ be the unit speed geodesic passing from $p$ and $\frac{dc}{dt}(0) = y$. Suppose that $U(t), V(t)$ and $W(t)$ are the parallel vector fields along $c$ with $U(0) = u, V(0) = v$ and $W(0) = w$. We put

$$A(t) = A\big(U(t), V(t), W(t)\big), \quad \dot{A}(t) = \dot{A}\big(U(t), V(t), W(t)\big).$$

Therefore

$$L(t) = \dot{A}(t).$$

By definition, we have the following ODE,

$$\dot{A}(t) + \frac{1}{2}A(t) = 0,$$

which its general solution is

$$A(t) = A(0)e^{-\frac{t}{2}}.$$

Using $||A|| < \infty$, and letting $t \to +\infty$ or $t \to -\infty$, we have

$$A(0) = A(u, v, w) = 0.$$

So $A = 0$ and then $F$ is a Riemannian metric. □

**Remark 3.1.** The completeness condition in the above theorem can not be replaced by positively complete or negatively complete. For example the Funk metric on $\mathbb{B}^n \subseteq \mathbb{R}^n$ satisfies all conditions of the above proposition but completeness, more precisely Funk metric is a non-Riemannian positively complete Finsler metric.

**Proof of Theorem 1.1**: Since every compact Finsler manifold is complete with bounded Cartan tensor, then by Proposition 3.1 we get the proof. □

By (3), we can conclude the following.

**Corollary 3.2.** *Every Landsberg metric satisfying (1.1) is a Riemannian metric.*

# 4 Proof of Theorem 1.2

In this section, we are going to prove the Theorem 1.2.

**Lemma 4.1.** ([12]) Let $(M, F)$ be a Finsler metric. Then $F$ is a GDW-metric if and only if

$$D^i{}_{jkl|s}y^s = T_{jkl}y^i,$$

for some tensor $T_{jkl}$ on manifold $M$.

**Proposition 4.2.** *Let $F$ be a non-Riemannian Finsler metric satisfies (1.1). Then $F$ is a Douglas metric if and only if $\lambda = \frac{1}{2F}$.*

*Proof.* Suppose that $F$ satisfies (1.1). Then

$$B^i{}_{jkl} = C_{jkl}\ell^i + \lambda(h^i_j h_{kl} + h^i_k h_{jl} + h^i_l h_{jk}), \tag{4.1}$$

Taking a trace of (4.1) yields

$$2E_{jk} = (n+1)\lambda h_{jk}. \tag{4.2}$$

Thus

$$B^i{}_{jkl} = C_{jkl}\ell^i + \frac{2}{n+1}(E_{jk}h^i_l + E_{kl}h^i_j + E_{jl}h^i_k).$$

On the other hand, we have

$$h_{ij,k} = 2C_{ijk} - F^{-2}(y_j h_{ik} + y_i h_{jk}),$$

which implies that

$$2E_{jk,l} = (n+1)\lambda_{,l}h_{jk} + (n+1)\lambda\Big[2C_{jkl} - F^{-2}(y_k h_{jl} + y_j h_{kl})\Big]. \tag{4.3}$$

The Douglas tensor is given by

$$D^i{}_{jkl} = B^i{}_{jkl} - \frac{2}{n+1}\{E_{jk}\delta^i{}_l + E_{kl}\delta^i{}_j + E_{lj}\delta^i{}_k + E_{jk,l}y^i\}. \tag{4.4}$$

Putting (4.1), (4.2) and (4.3) in (4.4) yields

$$D^i{}_{jkl} = (F^{-1} - 2\lambda)C_{jkl}y^i - (\lambda y_l F^{-2} + \lambda_{,l})h_{jk}y^i. \tag{4.5}$$

For the Douglas curvature, we have $D^i{}_{jkl} = D^i{}_{jlk}$. Then by (4.5), we conclude that

$$\lambda y_l F^{-2} + \lambda_{,l} = 0. \tag{4.6}$$

From (4.5) and (4.6) we deduce

$$D^i{}_{jkl} = (F^{-1} - 2\lambda)C_{jkl}y^i. \tag{4.7}$$

By (4.7), we get the proof. □

**Proof of Theorem 1.2**: The Douglas tensor of $F$ is given by

$$D^i{}_{jkl} = (F^{-1} - 2\lambda)C_{jkl}y^i. \tag{4.8}$$

Taking a horizontal derivation of (4.8) implies that

$$D^i{}_{jkl|s}y^s = -2(\lambda' C_{jkl} + \lambda L_{jkl})y^i. \tag{4.9}$$

where $\lambda' = \lambda_{|m}y^m$. By Lemma 4.1, $F$ is a GDW-metric with

$$T_{jkl} = -2(\lambda' C_{jkl} + \lambda L_{jkl}).$$

This completes the proof. □

# 5 Conclusion

By considering the special form of Berwald curvature of Funk metrics, it has been constructed an extension of the class of Berwald metrics which is a subclass of the class of generalized Douglas-Weyl metrics. It has been showed that every complete Finsler metric with bounded Cartan torsion in this class is Riemannian.

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