Some geometrical properties of a five-dimensional solvable Lie group

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Abstract. In this paper we consider the unimodular solvable Lie group G_n. As it is stated in [9], in 1980, Bozek has introduced G_n for the first time. In [9] Calvaruso, Kowalski and Marinosci have studied geodesics on this Lie group when it has arbitrary odd dimension. Our aim in this paper is to investigate four other geometrical properties i.e. homogeneous Ricci solitons, harmonicity of invariant vector fields, left invariant contact structures and homogeneous structures in two cases Riemannian and Lorentzian on this Lie group with dimension 5. This survey shows that, the space-like energy on the Lorentzian Lie group G_2 does not have a critical point and there is no left invariant almost complex structure on $G_2 \times \mathbb{R}$.

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1 Introduction

For any integer $n \geq 1$, the unimodular solvable Lie group G_n is as follows;

$$G_n = \begin{pmatrix} e^{u_0} & 0 & \cdots & 0 & x_0 \\ 0 & e^{u_1} & \cdots & 0 & x_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{u_n} & x_n \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

where $(x_0, x_1, \cdots, x_n, u_1, \cdots, u_n) \in \mathbb{R}^{2n+1}$ and $u_0 = -(u_1 + \cdots + u_n)$. In [9] Calvaruso, Kowalski and Marinosci have studied geodesic vectors for this Lie group. They proved that the space (G_n, g) where g is the Left-invariant Riemannian metric, admits $2n+1$ linearly independent homogeneous geodesics through the origin 0. In [13] Chavosh Khatamy introduced the tangent bundle TG_n for this Lie group and then investigated the exact form of its geodesic vectors.
In this paper we consider some other geometrical properties of this Lie group in

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dimension 5. One of these properties is Ricci solitons. As it is introduced in [7], a Ricci soliton is a pseudo-Riemannian manifold (M, g) which admits a smooth vector field X, that satisfies the following property;

$$L_X g + \rho = \lambda g \tag{1.1}$$

where L_X is the Lie derivative in the direction of X, ρ is the Ricci tensor and λ is a real number. A Ricci solition is said to be a shrinking, steady or expanding, if $\lambda > o$, $\lambda = o$ or $\lambda < o$, respectively.
In section 2, we consider the Bozek example in dimension five. In [6], Calvaruso and De Leo investigated the curvature properties of four-dimensional generalized symmetric spaces. Here we generalize their calculations for G_2 with dimension five in two Riemannian and Lorentzian cases. We show that G_2 is not a homogeneous Ricci soliton by using [7], where the authors have investigated Ricci solitons on Lorentzian Walker three manifolds. In section 3, we study harmonicity properties of invariant vector fields on G_2 using [5] and [10], where they have studied harmonicity properties of invariant vector fields on three- dimensional Lorentzian Lie groups and four dimensional generalized symmetric spaces. In section 4, we state left invariant contact structures on G_2 using [11] which has an example that presents a contact metric Lorentzian structure in the exact form on $\mathbb{R}^3$ and also [12], where the full classification of invariant contact metric structures on five dimensional Riemannian generalized symmetric spaces are obtained. In this section we also show that there does not exist a left-invariant almost complex structure on $G_2 \times \mathbb{R}$ by using the relation between contact and complex structures in [11]. Finally in section 5 we state homogeneous structures on G_2, using [8] and [1], where they have determined homogeneous structures on arbitrary sphere of Kaluza-Klein type and on homogeneous Lorentzian three-manifolds.

2 Homogeneous Ricci solitons on G_2

Bozek example states that for $n = 2$, G_2 is;

$$G_2 = \begin{pmatrix} e^{u_0} & 0 & 0 & x_0 \\ 0 & e^{u_1} & 0 & x_1 \\ 0 & 0 & e^{u_2} & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $(x_0, x_1, x_2, u_1, u_2) \in \mathbb{R}^5$ and $u_0 = -(u_1 + u_2)$. Considering the vector fields $U_\alpha = \frac{\partial}{\partial u_\alpha}, \alpha = 1, 2$ and $X_i = e^{u_i} \frac{\partial}{\partial x_i}, i = 0, 1, 2$, the set $\{X_0, X_1, X_2, U_1, U_2\}$ is a basis for the Lie algebra $\mathcal{G}$ of the Lie group G_2 and the Lie bracket is introduced as follows;

$[,]$	X_0	X_1	X_2	U_1	U_2
X_0	0	0	0	X_0	X_0
X_1	0	0	0	$-X_1$	$-X_1$
X_2	0	0	0	$-X_2$	$-X_2$
U_1	$-X_0$	X_1	X_2	0	0
U_2	$-X_0$	X_1	X_2	0	0

In the Riemannian case

The solvable unimodular Lie group G_2 can be equipped with the following left-invariant Riemannian metric with $a > 0$;

$$g = \sum_{i=0}^{2} e^{-2u_i}(dx_i)^2 + a \sum_{\alpha,\beta=1}^{2} du_\alpha du_\beta.$$

So the scalar product $\langle , \rangle$ on the Lie algebra $\mathcal{G}$ is;

$\langle , \rangle$	X_0	X_1	X_2	U_1	U_2
X_0	1	0	0	0	0
X_1	0	1	0	0	0
X_2	0	0	1	0	0
U_1	0	0	0	a	$\frac{a}{2}$
U_2	0	0	0	$\frac{a}{2}$	a

We can construct an orthonormal frame field $\{e_1, e_2, e_3, e_4, e_5\}$ with respect to g;

$$e_1 = X_0, \quad e_2 = X_1, \quad e_3 = X_2, \quad e_4 = \frac{U_1}{\sqrt{a}} - \frac{U_2}{\sqrt{a}}, \quad e_5 = \frac{U_1}{\sqrt{3a}} + \frac{U_2}{\sqrt{3a}}$$

and we get;

$$(2.1) \qquad [e_1, e_5] = \frac{2}{\sqrt{3a}} e_1, \qquad [e_2, e_5] = \frac{-2}{\sqrt{3a}} e_2, \qquad [e_3, e_5] = \frac{-2}{\sqrt{3a}} e_3.$$

Considering Koszul's formula $2g(\nabla_{e_i} e_j, e_k) = g([e_i, e_j], e_k) - g([e_j, e_k], e_i) + g([e_k, e_i], e_j)$ the nonzero connection components are;

$$(2.2) \qquad \begin{array}{lll} \nabla_{e_1} e_1 = \frac{-2}{\sqrt{3a}} e_5 & \nabla_{e_1} e_5 = \frac{2}{\sqrt{3a}} e_1 & \nabla_{e_2} e_2 = \frac{2}{\sqrt{3a}} e_5 \\ \nabla_{e_2} e_5 = \frac{-2}{\sqrt{3a}} e_2 & \nabla_{e_3} e_3 = \frac{2}{\sqrt{3a}} e_5 & \nabla_{e_3} e_5 = \frac{-2}{\sqrt{3a}} e_3. \end{array}$$

By using $R(X,Y)Z = \nabla_{[X,Y]}Z - \nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z$ we can determine the curvature components;

$$R(e_3, e_1)e_3 = \frac{4}{3a} e_1 \qquad R(e_5, e_1)e_5 = -\frac{4}{3a} e_1$$

$$R(e_1, e_2)e_1 = \frac{4}{3a} e_2 \qquad R(e_3, e_2)e_3 = -\frac{4}{3a} e_2$$

$$R(e_5, e_2)e_5 = -\frac{4}{3a} e_2 \qquad R(e_5, e_3)e_5 = -\frac{4}{3a} e_3.$$

Since $R(X,Y,Z,W) = g(R(X,Y)Z, W)$ we have;

$$R_{3131} = R_{1212} = \frac{4}{3a} \qquad R_{5151} = R_{3232} = R_{5252} = R_{5353} = -\frac{4}{3a}.$$

Applying the Ricci tensor formula $\rho(X,Y)=\sum_{i=1}^{5}\epsilon_i g(R(X,e_i)Y,e_i)$, we get;

$$(\rho)_{ij}=\begin{pmatrix} \frac{4}{3a} & 0 & 0 & 0 & 0\\ 0 & -\frac{4}{3a} & 0 & 0 & 0\\ 0 & 0 & -\frac{4}{3a} & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -\frac{4}{a}\end{pmatrix}$$

which is diagonal with eigenvalues $r_1=\frac{4}{3a}$, $r_2=r_3=-\frac{4}{3a}$, $r_4=0$ and $r_5=-\frac{4}{a}$.

For an arbitrary left-invariant vector field $X=\sum_{i=1}^{5}K_ie_i$ on G_2 we have;

$$\nabla_{e_1}X=\frac{-2K_1}{\sqrt{3a}}e_5+\frac{2K_5}{\sqrt{3a}}e_1 \quad \nabla_{e_2}X=\frac{2K_2}{\sqrt{3a}}e_5-\frac{2K_5}{\sqrt{3a}}e_2 \quad \nabla_{e_3}X=\frac{2K_3}{\sqrt{3a}}e_5-\frac{2K_5}{\sqrt{3a}}e_3$$

using the relation $(L_Xg)(Y,Z)=g(\nabla_YX,Z)+g(Y,\nabla_ZX)$ we have;

$$L_Xg=\begin{pmatrix} \frac{4K_5}{\sqrt{3a}} & 0 & 0 & 0 & -\frac{2K_1}{\sqrt{3a}}\\ 0 & -\frac{4K_5}{\sqrt{3a}} & 0 & 0 & \frac{2K_2}{\sqrt{3a}}\\ 0 & 0 & -\frac{4K_5}{\sqrt{3a}} & 0 & \frac{2K_3}{\sqrt{3a}}\\ 0 & 0 & 0 & 0 & 0\\ -\frac{2K_1}{\sqrt{3a}} & \frac{2K_2}{\sqrt{3a}} & \frac{2K_3}{\sqrt{3a}} & 0 & 0\end{pmatrix}$$

In the Lorentzian case

The solvable unimodular Lie group G_2 can be equipped with the following left-invariant Lorentzian metric with $a>0$;

$$\widehat{g}=\sum_{i=0}^{2}e^{-2u_i}(dx_i)^2-a(du_1{}^2+du_2{}^2)+3adu_1du_2$$

and the scalar product $\langle,\rangle$ on the Lie algebra $\mathcal{G}$ is;

$\langle,\rangle$	$\vert$	X_0	X_1	X_2	U_1	U_2
X_0	$\vert$	1	0	0	0	0
X_1	$\vert$	0	1	0	0	0
X_2	$\vert$	0	0	1	0	0
U_1	$\vert$	0	0	0	$-a$	$\frac{3a}{2}$
U_2	$\vert$	0	0	0	$\frac{3a}{2}$	$-a$

We can construct a pseudo-orthonormal frame field $\{e_1,e_2,e_3,e_4,e_5\}$, where;

$$e_1=X_0,\quad e_2=X_1,\quad e_3=X_2,\quad e_4=\frac{U_1}{\sqrt{a}}+\frac{U_2}{\sqrt{a}},\quad e_5=\frac{U_1}{\sqrt{5a}}-\frac{U_2}{\sqrt{5a}}.$$

52 Then the metric $\widehat{g}$ is with signature $(+,+,+,+,-)$ and we have;

$$[e_1,e_4]=\frac{2}{\sqrt{a}}e_1,\qquad [e_2,e_4]=\frac{-2}{\sqrt{a}}e_2,\qquad [e_3,e_4]=\frac{-2}{\sqrt{a}}e_3. \tag{2.3}$$

Hence the connection components are;

$$\begin{array}{lll} \nabla_{e_1} e_1 = \frac{-2}{\sqrt{a}} e_4 & \nabla_{e_1} e_4 = \frac{2}{\sqrt{a}} e_1 & \nabla_{e_2} e_2 = \frac{2}{\sqrt{a}} e_4 \\ \nabla_{e_2} e_4 = \frac{-2}{\sqrt{a}} e_2 & \nabla_{e_3} e_3 = \frac{2}{\sqrt{a}} e_4 & \nabla_{e_3} e_4 = \frac{-2}{\sqrt{a}} e_3 \end{array} \tag{2.4}$$

and the curvature components can be determined as follows;

$$\begin{array}{ll} R(e_3, e_1)e_3 = \frac{4}{a} e_1 & R(e_4, e_1)e_4 = -\frac{4}{a} e_1 \\ R(e_1, e_2)e_1 = \frac{4}{a} e_2 & R(e_3, e_2)e_3 = -\frac{4}{a} e_2 \\ R(e_4, e_2)e_4 = -\frac{4}{a} e_2 & R(e_4, e_3)e_4 = -\frac{4}{a} e_3. \end{array}$$

Therefore;

$$R_{3131} = R_{1212} = \frac{4}{a} \qquad R_{4141} = R_{3232} = R_{4242} = R_{4343} = -\frac{4}{a}$$

and the Ricci tensor is;

$$(\rho)_{ij} = \begin{pmatrix} \frac{4}{a} & 0 & 0 & 0 & 0 \\ 0 & -\frac{4}{a} & 0 & 0 & 0 \\ 0 & 0 & -\frac{4}{a} & 0 & 0 \\ 0 & 0 & 0 & -\frac{12}{a} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which is diagonal with eigenvalues $r_1 = \frac{4}{a}$, $r_2 = r_3 = -\frac{4}{a}$ $r_4 = -\frac{12}{a}$ and $r_5 = 0$.

For an arbitrary left-invariant vector field $X = \sum_{i=1}^{5} K_i e_i$ on G_2 we have;

$$\nabla_{e_1} X = \frac{-2K_1}{\sqrt{a}} e_4 + \frac{2K_4}{\sqrt{a}} e_1 \qquad \nabla_{e_2} X = \frac{2K_2}{\sqrt{a}} e_4 - \frac{2K_4}{\sqrt{a}} e_2 \qquad \nabla_{e_3} X = \frac{2K_3}{\sqrt{a}} e_4 - \frac{2K_4}{\sqrt{a}} e_3$$

and the Lie derivative in the direction of X is;

$$L_X g = \begin{pmatrix} \frac{4K_4}{\sqrt{a}} & 0 & 0 & -\frac{2K_1}{\sqrt{a}} & 0 \\ 0 & -\frac{4K_4}{\sqrt{a}} & 0 & \frac{2K_2}{\sqrt{a}} & 0 \\ 0 & 0 & -\frac{4K_4}{\sqrt{a}} & \frac{2K_3}{\sqrt{a}} & 0 \\ -\frac{2K_1}{\sqrt{a}} & \frac{2K_2}{\sqrt{a}} & \frac{2K_3}{\sqrt{a}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Proposition 2.1. *The solvable unimodular Lie group G_2 is not a homogeneous Ricci soliton in both Riemannian and Lorentzian cases.*

Proof. In the Riemannian case by the Ricci soliton formula (1.1), we get the following system of differential equations;

$$\begin{cases} \frac{4K_5}{\sqrt{3a}} + \frac{4}{3a} = \lambda \\ \frac{2K_1}{\sqrt{3a}} = \frac{2K_2}{\sqrt{3a}} = \frac{2K_3}{\sqrt{3a}} = 0 \\ \lambda = 0 \\ \frac{-4K_5}{\sqrt{3a}} - \frac{4}{3a} = \lambda \\ \lambda = -\frac{4}{a} \end{cases} \tag{2.5}$$

From the first and the third equations in (2.5) we get $K_5 = -\frac{1}{\sqrt{3a}}$ and the first and the last equations in (2.5) give us $K_5 = -\frac{4}{\sqrt{3a}}$. So $a = 0$ which is a contradiction. The calculation in the Lorentzian case is similar. □

Remark 2.1. A pseudo-Riemannian manifold (M, g) is in class A if and only if the Ricci tensor is cyclic-parallel, i.e. $\nabla_X\rho(Y,Z) + \nabla_Y\rho(Z,X) + \nabla_Z\rho(X,Y) = 0$ or equivalently it is a Killing tensor , i.e. $\nabla_X\rho(X,X) = 0$ and it is in class B if and only if its Ricci tensor is a Codazzi tensor, i.e. $\nabla_X\rho(Y,Z) = \nabla_Y\rho(X,Z)$, where

$$\nabla_i\rho_{jk} = -\sum_t(\varepsilon_j B_{ijt}\rho_{tk} + \varepsilon_k B_{ikt}\rho_{tj}),$$

B_{ijk} components can be obtained by the relation $\nabla_{e_i}e_j = \sum_k \varepsilon_j B_{ijk}e_k$ and ρ_{tk} are tensor Ricci components. For more detail see [4].

Proposition 2.2. *The solvable unimodular Lie group G_2 belongs to class A in both Riemannian and Lorentzian cases.*

Proof. In the Riemannian case B_{ijk}'s are;

$$B_{115} = \frac{-2}{\sqrt{3a}} \quad B_{225} = \frac{2}{\sqrt{3a}} \quad B_{335} = \frac{2}{\sqrt{3a}}$$

so $\nabla_1\rho_{11} = \nabla_2\rho_{22} = \nabla_3\rho_{33} = \nabla_4\rho_{44} = \nabla_5\rho_{55} = 0$ as desired. In the Lorentzian case B_{ijk}'s are;

$$B_{114} = \frac{-2}{\sqrt{a}} \quad B_{224} = \frac{2}{\sqrt{a}} \quad B_{334} = \frac{2}{\sqrt{a}}$$

and in a similar manner they belong to class A. □

Here we remind the following theorem from [2].

Theorem 2.3. *A pseudo-Riemannian manifold (M^n, g) of dimension $n \geqslant 4$, is conformally flat if and only if its Weyl curvature tensor vanishes, that is*

$$\begin{aligned} R(X,Y,Z,W) = {} & \frac{1}{n-2}(g(X,Z)\rho(Y,W) + g(Y,W)\rho(X,Z) \\ & - g(X,W)\rho(Y,Z) - g(Y,Z)\rho(X,W)) \\ & - \frac{\tau}{(n-1)(n-2)}(g(X,Z)g(Y,W) - g(Y,Z)g(X,W)) \end{aligned} \tag{2.6}$$

where X, Y, Z, W are vector fields and τ is the scalar curvature.

Proposition 2.4. *The solvable unimodular Lie group G_2 is not conformally flat in both Riemannian and Lorentzian cases.*

Proof. Since the scalar curvature is $\tau = \sum_i \rho(e_i, e_i)$(see [3]. p. 43), in the Riemannian case $\tau = \frac{-16}{3a}$ (In the Lorentzian case $\tau = \frac{-16}{a}$), using (2.6) we have $R_{1212} = \frac{4}{9a} \neq \frac{4}{3a}$ ($R_{1212} = \frac{4}{3a} \neq \frac{4}{a}$). So in both cases G_2 is not conformally flat. □

3 Harmonicity of invariant vector fields on G_2

In this section we investigate the harmonicity of invariant vector fields on the Lie group G_2.

In the Riemannian case

Let $V = \sum_{i=1}^{5} K_i e_i$ be a left-invariant vector field on G_2, where $\{e_i\}$ is an orthogonal frame field, then (2.2) yields;

$$\nabla_{e_1} V = \frac{-2K_1}{\sqrt{3a}} e_5 + \frac{2K_5}{\sqrt{3a}} e_1, \quad \nabla_{e_2} V = \frac{2K_2}{\sqrt{3a}} e_5 + \frac{-2K_5}{\sqrt{3a}} e_2, \quad \nabla_{e_3} V = \frac{2K_3}{\sqrt{3a}} e_5 + \frac{-2K_5}{\sqrt{3a}} e_3$$

and with calculation $\nabla_{e_i}\nabla_{e_i} V$ and $\nabla_{\nabla_{e_i} e_i} V$ for $i = 1, \cdots, 5$;

$$\nabla_{e_1}\nabla_{e_1} V = \frac{-4}{3a}(K_1 e_1 + K_5 e_5), \qquad \nabla_{e_2}\nabla_{e_2} V = \frac{-4}{3a}(K_2 e_2 + K_5 e_5),$$

$$\nabla_{e_3}\nabla_{e_3} V = \frac{-4}{3a}(K_3 e_3 + K_5 e_5).$$

Since $\nabla_{\nabla_{e_i} e_i} V = 0$, using $\nabla^* \nabla V = \sum_{i=1}^{5} \varepsilon_i (\nabla_{e_i}\nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V)$ we get;

$$\nabla^* \nabla V = \frac{-4}{3a}(K_1 e_1 + K_2 e_2 + K_3 e_3 + 3K_5 e_5)$$

In the Lorentzian case

Let $V = \sum_{i=1}^{5} K_i e_i$ be a left-invariant vector field on G_2, where $\{e_i\}$ is an pseudo-orthogonal frame field, then (2.4) gives;

$$\nabla_{e_1} V = \frac{-2K_1}{\sqrt{a}} e_4 + \frac{2K_4}{\sqrt{a}} e_1, \quad \nabla_{e_2} V = \frac{2K_2}{\sqrt{a}} e_4 + \frac{-2K_4}{\sqrt{a}} e_2, \quad \nabla_{e_3} V = \frac{2K_3}{\sqrt{a}} e_4 + \frac{-2K_4}{\sqrt{a}} e_3.$$

Hence;

$$\nabla_{e_1}\nabla_{e_1} V = \frac{-4}{a}(K_1 e_1 + K_4 e_4), \qquad \nabla_{e_2}\nabla_{e_2} V = \frac{-4}{a}(K_2 e_2 + K_4 e_4),$$

$$\nabla_{e_3}\nabla_{e_3} V = \frac{-4}{a}(K_3 e_3 + K_4 e_4)$$

and for $i = 0, \cdots, 5$ since $\nabla_{\nabla_{e_i} e_i} V = 0$ we get;

$$\nabla^* \nabla V = \frac{-4}{a}(K_1 e_1 + K_2 e_2 + K_3 e_3 + 3K_4 e_4).$$

In both Riemannian and Lorentzian cases the following theorem is applicable, but we only prove it for the Riemannian case. The proof of the Lorentzian case is very similar.

Theorem 3.1. *Let* $V = \sum_{i=1}^{5} K_i e_i$ *be a left-invariant vector field on the Lie group* G_2*, then* V *defines a harmonic map if and only if* $V = K_4 e_4$*.*

Proof. Let $V = K_4 e_4$. Since both $\nabla^* \nabla V$ and $tr[R(\nabla.V, V).] = \sum_i \varepsilon_i R(\nabla_{e_i} V, V) e_i$ are zero, V defines a harmonic map. In the other direction, if $\nabla^* \nabla V = \frac{4}{3a}(K_1 e_1 + K_2 e_2 + K_3 e_3 + 3K_5 e_5) = 0$ and $tr[R(\nabla.V, V).] = 0$, then $V = K_4 e_4$. □

Proposition 3.2. *In both Riemannian and Lorentzian cases the left-invariant vector field* $V = \sum_{i=1}^{5} K_i e_i$ *is an invariant harmonic vector field on the Lie group* G_2 *if and only if* $K_5 = K_4 = 0$*.*

Proof. Since in the Riemannian case $\nabla^* \nabla V = \frac{-4}{3a} V + (\frac{4}{3a} K_4 e_4 + \frac{-8}{3a} K_5 e_5)$ and in the Lorentzian case $\nabla^* \nabla V = \frac{-4}{a} V + (\frac{4}{a} K_5 e_5 + \frac{-8}{a} K_4 e_4)$, using $\nabla^* \nabla V = \lambda V$, we can complete the proof. □

Let (M, g) be a compact pseudo-Riemannian manifold and g^s be the Sasaki metric on the tangent bundle TM, then the energy of a smooth vector field $V : (M, g) \to (TM, g^s)$ on G_2 is;

$$E(V) = \frac{n-1}{2} vol(M, g) + \frac{1}{2} \int_M \parallel \nabla V \parallel^2 dv \tag{3.1}$$

(see [5]). Since G_2 is not compact we suppose that D is its relatively compact domain and calculate the energy of $V|_D$.

Proposition 3.3. *Let* V *be a a smooth left-invariant vector field on* G_2*, the energy of* $V|_D$ *in the Riemannian case is;*

$$E_D(V) = (2 + \frac{2 \parallel V \parallel^2}{3a} + \frac{4K_5^2}{3a} - \frac{2}{3a} K_4^2) volD$$

and in the Lorentzian case is

$$E_D(V) = (2 + \frac{2 \parallel V \parallel^2}{a} + \frac{4K_4^2}{a} + \frac{2}{a} K_5^2) volD$$

where $E_D(V)$ *denotes the energy of* $V|_D$*.*

Proof. In the Lorentzian case we have;

$$\parallel \nabla V \parallel^2 = \sum_{i=1}^{5} \varepsilon_i g(\nabla_{e_i} V, \nabla_{e_i} V) = \frac{4K_1^2}{a} + \frac{4K_2^2}{a} + \frac{4K_3^2}{a} + \frac{12K_4^2}{a}.$$

By replacing $\parallel V \parallel = K_1^2 + K_2^2 + K_3^2 + K_4^2 - K_5^2$ in the relation (3.1) we can complete the proof. We can prove the Riemannian case in a similar manner. □

Recall that for a Lorentzian Lie group G_2, a left-invariant vector field V is spatially harmonic if and only if $\widehat{X}_V = \delta V$, where $\delta \in \mathbb{R}$ and for

$$divV = \sum_i g(\nabla_{e_i} V, e_i)$$

and

$$(\nabla V)^t(\nabla_V V) = \sum_i \varepsilon_i g(\nabla_V V, \nabla_{e_i} V) e_i$$

$\widehat{X}_V$ is;

$$\widehat{X}_V = -\nabla^* \nabla V - \nabla_V \nabla_V V - divV.\nabla_V V + (\nabla V)^t(\nabla_V V). \tag{3.2}$$

(see[10]). Also a time-like vector field is called a unit time-like vector field when its norm is equal to -1.

Proposition 3.4. *Let V be a unit time-like vector field on the Lorentzian Lie group G_2, then V is not spatially harmonic.*

Proof. For a unit time-like vector field V, we have;

$$\begin{aligned}\nabla_V V &= K_1(\tfrac{-2K_1}{\sqrt{a}} e_4 + \tfrac{2K_4}{\sqrt{a}} e_1) + K_2(\tfrac{2K_2}{\sqrt{a}} e_4 + \tfrac{-2K_4}{\sqrt{a}} e_2) + K_3 \tfrac{2K_3}{\sqrt{a}} e_4 + \tfrac{-2K_4}{\sqrt{a}} e_3) \\ &= \tfrac{2K_1K_4}{\sqrt{a}} e_1 - \tfrac{2K_2K_4}{\sqrt{a}} e_2 - \tfrac{2K_3K_4}{\sqrt{a}} e_3 + (\tfrac{-2K_1^2}{\sqrt{a}} + \tfrac{2K_2^2}{\sqrt{a}} + \tfrac{2K_3^2}{\sqrt{a}}) e_4,\end{aligned}$$

$$\begin{aligned}\nabla_V \nabla_V V &= \nabla_V \{\tfrac{2K_1K_4}{\sqrt{a}} e_1 - \tfrac{2K_2K_4}{\sqrt{a}} e_2 - \tfrac{2K_3K_4}{\sqrt{a}} e_3 + (\tfrac{-2K_1^2}{\sqrt{a}} + \tfrac{2K_2^2}{\sqrt{a}} + \tfrac{2K_3^2}{\sqrt{a}}) e_4\} \\ &= K_1(\tfrac{-4K_1^2+4K_2^2+4K_3^2}{a}) e_1 + K_2(\tfrac{4K_1^2-4K_2^2-4K_3^2}{a}) e_2 \\ &+ K_3(\tfrac{4K_1^2-4K_2^2-4K_3^2}{a}) e_3 + K_4(\tfrac{-4K_1^2-4K_2^2-4K_3^2}{a}) e_4 \ ,\end{aligned}$$

$$divV = \sum_{i=1}^{5} g(\nabla_{e_i} V, e_i) = -\frac{2K_4}{\sqrt{a}},$$

$$\begin{aligned}(\nabla V)^t(\nabla_V V) &= K_1(\tfrac{4K_4^2+4K_1^2-4K_2^2-4K_3^3}{a}) e_1 + K_2(\tfrac{4K_4^2-4K_1^2+4K_2^2+4K_3^2}{a}) e_2 \\ &+ K_3(\tfrac{4K_4^2-4K_1^2+4K_2^2+4K_3^2}{a}) e_3,\end{aligned}$$

using the relation (3.2), we get;

$$\begin{aligned}\widehat{X}_V &= (\tfrac{4+8K_1^2-8K_2^2-8K_3^2+8K_4^2}{a}) K_1 e_1 + (\tfrac{4-8K_1^2+8K_2^2+8K_3^2}{a}) K_2 e_2 + (\tfrac{4-8K_1^2+8K_2^2+8K_3^2}{a}) K_3 e_3 + \\ &+ (\tfrac{12+8K_2^2+8K_3^2}{a}) K_4 e_4 = \tfrac{4}{a} V + (\tfrac{8K_1^2-8K_2^2-8K_3^2+8K_4^2}{a}) K_1 e_1 + (\tfrac{-8K_1^2+8K_2^2+8K_3^2}{a}) K_2 e_2 \\ &+ (\tfrac{-8K_1^2+8K_2^2+8K_3^2}{a}) K_3 e_3 + (\tfrac{8+8K_2^2+8K_3^2}{a}) K_4 e_4 - \tfrac{4}{a} K_5 e_5.\end{aligned}$$

Therefore, V is spatially harmonic if and only if we have the following system of equations;

$$
(3.3) \qquad \begin{cases} K_5 = 0 \\ K_1^2 + K_4^2 = K_2^2 + K_3^2 & or \quad K_1 = 0 \\ K_1^2 = K_2^2 + K_3^2 & or \quad K_2 = 0 \\ K_1^2 = K_2^2 + K_3^2 & or \quad K_3 = 0 \\ K_4 = 0 \end{cases}
$$

Since V is unite time-like, $K_1^2 + K_2^2 + K_3^2 + K_4^2 - K_5^2 = -1$. On the other hand (3.3) gives us $K_4 = K_5 = 0$ and hence $K_1^2 + K_2^2 + K_3^2 = -1$. Now if $K_1 = 0$ or $K_1^2 + K_4^2 = K_2^2 + K_3^2$ occur, there is a contradiction(because K_i's are real constants). □

For the Lorentzian Lie group G_2 consideration of the space like energy of its unit time-like vector field is meaningful. As it is mentioned in [5] the space-like energy of the unit time-like vector field V on the Lorentzian manifold M is the integral of the square norm of the restriction of ∇V to the distribution $V^{\perp}$. If V is a critical point of the space-like energy, then it is spatially harmonic. So we have the following corollary.

Corollary 3.5. *The space-like energy of the Lorentzian Lie group G_2 does not have a critical point.*

4 Left invariant contact structures on G_2

An almost contact structure on a $(2n+1)$-dimensional smooth manifold M consists of a triple (φ, ξ, η), where φ is a $(1,1)$-tensor, ξ is a nowhere vanishing vector field and η is a 1-form, such that

$$\eta(\xi) = 1, \quad \varphi^2 = -id + \eta \otimes \xi,$$

and φ has rank 2n(see [12]). If the 1-form η satisfies $\eta \wedge (d\eta)^n \neq o$ then η is called the contact form.

Theorem 4.1. *The Lie group G_2 does not admit a left-invariant contact structure in both Riemannian and Lorentzian cases.*

Proof. Let $\{e^1, \cdots, e^5\}$ be the dual to the basis $\{e_1, \cdots, e_5\}$. In the Riemannian case using (2.1), we get;

$$de^1 = \frac{-2}{\sqrt{3a}} e^1 \wedge e^5, \quad de^2 = \frac{2}{\sqrt{3a}} e^2 \wedge e^5, \quad de^3 = \frac{2}{\sqrt{3a}} e^3 \wedge e^5 \quad, de^4 = 0, \quad de^5 = 0.$$

and in the Lorentzian case using (2.3), we obtain;

$$de^1 = \frac{-2}{\sqrt{a}} e^1 \wedge e^4, \quad de^2 = \frac{2}{\sqrt{a}} e^2 \wedge e^4, \quad de^3 = \frac{2}{\sqrt{a}} e^3 \wedge e^4 \quad, de^4 = 0, \quad de^5 = 0.$$

Hence for all indices $i, j = 1, \cdots, 5$ in both cases $de^i \wedge de^j = 0$. So for any left-invariant differential 1-form $\eta = \sum_{i=1}^{5} c_i e^i$ since $d\eta \wedge d\eta = 0$, the Lie group G_2 does not carry a left-invariant contact structure, where $c_1, \cdots, c_5$ are real constants. □

Since an almost contact structure (φ, ξ, η) on a manifold M^{2n+1} admits an almost complex structure on $M^{2n+1} \times \mathbb{R}$, by the definition $J(X, f\frac{d}{dt}) = (\varphi X - f\xi, \eta(X)\frac{d}{dt})$, we have the following corollary.

Corollary 4.2. *There does not exist any left-invariant almost complex structure on $G_2 \times \mathbb{R}$.*

5 Homogeneous structure on G_2

A homogeneous pseudo-Riemannian structure on a connected pseudo-Riemannian manifold (M, g) is a tensor field T of type $(1, 2)$ such that the connection $\widehat{\nabla} = \nabla - T$ satisfies;

$$\widehat{\nabla} g = 0, \qquad \widehat{\nabla} R = 0, \qquad \widehat{\nabla} T = 0$$

where ∇ is the Levi-Civita connection of g and R is its Ricci curvature tensor field. More exactly, T is the solution of the following Ambrose-Singer equations;

$$g(T_X Y, Z) + g(Y, T_X Z) = 0, \tag{5.1}$$

$$(\nabla_X R)_{YZ} = [T_X, R_{YZ}] - R_{T_X Y Z} - R_{Y T_X Z}, \tag{5.2}$$

$$(\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y}. \tag{5.3}$$

For more detail see [1]. Since G_2 is the special linear group, it is connected (see[14]. p.15). Hence it makes sense to define a homogeneous structure on it.

Proposition 5.1. *A homogeneous Riemannian structure on the five-dimensional Lie group G_2 is;*

$$T = \frac{-4}{\sqrt{3a}} e^1 \otimes (e^1 \wedge e^5) + \frac{4}{\sqrt{3a}} e^2 \otimes (e^2 \wedge e^5) + \frac{4}{\sqrt{3a}} e^3 \otimes (e^3 \wedge e^5),$$

and a homogeneous Lorentzian structure on Lorentzian Lie group G_2 is;

$$T = \frac{-4}{\sqrt{a}} e^1 \otimes (e^1 \wedge e^4) + \frac{4}{\sqrt{a}} e^2 \otimes (e^2 \wedge e^4) + \frac{4}{\sqrt{a}} e^3 \otimes (e^3 \wedge e^4).$$

Proof. Let $T_{e_i} := \frac{1}{2} \sum_{jk} T_{ij}^k e_j \wedge e_k$, where $e_j \wedge e_k(X) = g(e_j, X)e_k - g(e_k, X)e_j$. Then for $i, j, k, s = 1, \cdots, 5$ the first equation of Ambrose-Singer equations (5.1) implies that $T_{ij}^k = -T_{ik}^j$ and $T_{i1}^1 = T_{i2}^2 = T_{i3}^3 = T_{i4}^4 = T_{i5}^5 = 0$. If we replace this relation in (5.2), we get $\nabla_{e_i} R(e_j, e_k)e_j = T_{e_i} R(e_j, e_k)e_j$ or $T_{e_i} e_k = \nabla_{e_i} e_k$ that implies $T_{15}^1 - T_{11}^5 = T_{22}^5 - T_{25}^2 = T_{33}^5 - T_{35}^3 = \frac{4}{\sqrt{3a}}$ and since $T_{15}^1 = -T_{11}^5, T_{22}^5 = -T_{25}^2, T_{33}^5 = -T_{35}^3$, we have $-T_{15}^1 = T_{25}^2 = T_{35}^3 = \frac{-2}{\sqrt{3a}}$. By (5.3) it can be shown that the other components are zero. The homogeneous Lorentzian structure can be obtained in a similar way. □

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