Construction of bidirectionally coupled systems using generalized synchronization method

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Abstract. This paper proposes methods for designing bidirectionally coupled systems via generalized synchronization technique. Starting from a chaotic system we are constructing synchronized bidirectionally coupled driving and response systems. Numerical simulation results are presented to prove the effectiveness of the scheme.

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1 Introduction

Pecora and Carroll [9] first introduced the concept of chaos synchronization. After that it has become an important subject in the field of non-linear science. Different types of synchronization and control methods, viz.- active control method [1], impulsive control method [12], adaptive control method [2], linear and non-linear feedback control method [8], unidirectionally and bidirectionally coupled systems [4, 5] etc., have been applied to chaos synchronization with a varying degree of success in each case.

1.1 System coupling

A $(n+m)$-dimensional dynamical system is called:

1) *decoupled* if it can be decomposed in two dynamical systems of the form

$$\dot{X} = f(X), \dot{Y} = g(Y), \tag{1.1}$$

the first being n-dimensional and the second m-dimensional.

2) *Unidirectionally coupled* if it can be decomposed in two dynamical systems of the form

$$\begin{aligned} \dot{X} &= f(X) \\ \dot{Y} &= g(Y) + k(X, Y), \end{aligned} \tag{1.2}$$

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where X is n-dimensional, Y is m-dimensional and $k(X,\ Y)$ is non-zero function of X and Y. Physically it means that in some parts of the space R^{n+m}, the behaviour of one system is influenced by the behaviour of the other, but the driving system is completely independent of the response system.

3) *Bidirectionally coupled* if it can be decomposed in two n-dimensional dynamical systems of the form

$$\begin{aligned} \dot{X} &= f(X) + k_1(X,\ Y) \\ \dot{Y} &= g(Y) + k_2(X,\ Y), \end{aligned} \tag{1.3}$$

where X is n-dimensional, Y is m-dimensional and $k_1(X,\ Y)$ and $k_2(X,\ Y)$ are non-zero functions of X and Y.

1.2 Synchronization

If the distance between the states of two dynamical systems converges to zero as the time tends to infinity, the systems are then said to be synchronized. This type of synchronization is known as identical synchronization [9]. Kocarev and Parlitz [7] introduced a new concept of synchronization known as generalized synchronization (GS). For the following systems,

$$\begin{aligned} \dot{X} &= f(X) && \leftarrow \textit{Driving system} \\ \dot{Y} &= g(Y,\ h(X)), && \leftarrow \textit{Response system} \end{aligned} \tag{1.4}$$

where X $\in\ \Re^n$, Y $\in\ \Re^m$, they developed a condition for the occurrence of generalized synchronization. According to them the system in (1.4) possesses generalized synchronization between X and Y if there exists a transformation $F\ :\ \Re^n \rightarrow \Re^m$, a manifold $M\ =\ \{\ (X,\ Y) :\ Y\ =\ F(X)\ \}$, and a set $B\ \subseteq\ \Re^n \times\ \Re^m$ with $M\ \subseteq\ B$ such that all trajectories of (1.4) starting from the basin B converges to M as time tends to infinity. If F equals identity transformation, then the generalized synchronization coincides with the identical synchronization. In a physical world, the application of generalized synchronization is more practical than those of identical synchronization because of the existence of the parameter mismatches and distortions. Authors like Rulkov *et al.* [11], Hramov *et. al* [3], Poria [10] have discussed generalized synchronization of chaos in unidirectionally coupled chaotic systems. Though most of the natural systems are bidirectionally coupled, still very few studies about synchronization of bidirectionally coupled systems are seen. In this paper, starting from a chaotic system, we have constructed bidirectionally coupled synchronized systems using generalized synchronization method in two ways. For both methods of construction numerical simulations have been performed to judge their effectiveness.

2 Designing bidirectionally coupled chaotic systems

This section develops on how to design a bidirectionally coupled chaotic system in the generalized synchronization framework.

Definition 2.1. Let us consider the following chaotic systems,

$$\begin{aligned} \dot{X} &= f(X,\ Y,\ t) \leftarrow \textit{Master system} \\ \dot{Y} &= g(X,\ Y,\ t) \leftarrow \textit{Slave system}, \end{aligned} \tag{2.1}$$

where $X = (x_1,\ x_2, \dots,\ x_n)^t$, $Y = (y_1,\ y_2, \dots,\ x_n)^t$. For a constant invertible matrix

$$D = \begin{pmatrix} d_{11} & d_{12} & \dots & d_{1n} \\ d_{21} & d_{22} & \dots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \dots & d_{nn} \end{pmatrix}, \tag{2.2}$$

if $lim_{t \to \infty} \parallel X - DY \parallel = 0$, then the two systems given in (2.1) are said to be in a *state of generalized synchronization.*

2.1 The first method of synchronization

For the chaotic system $\dot{X} = AX + f(X,\ t)$, let us take the drive and response systems as

$$\dot{X} = AX + f(X,\ t) + V_1 \tag{2.3}$$

and

$$\dot{Y} = AY + g(Y,\ t) + V_2, \tag{2.4}$$

where

$$\begin{aligned} V_1 &= Dg(Y,\ t) + (DA - AD)Y \\ V_2 &= D^{-1}f(X,\ t) + D^{-1}BK(X - DY). \end{aligned} \tag{2.5}$$

Here $X \in \Re^n$, $Y \in \Re^n$, A is $n \times n$ matrix, f and g are both $n \times 1$ matrices. Taking $e = X - DY$, one gets from(2.3) and (2.4),

$$\dot{e} = (A - BK)\, e, \tag{2.6}$$

where K is $1 \times n$ feedback matrix and B is $n \times 1$ suitable matrix [6]. If all the eigenvalues of the matrix $A - BK$ have negative real parts, then $\lim_{t \to \infty} \parallel X - DY \parallel = 0$ and the generalized synchronization is achieved between (2.3) and (2.4), with V_1 and V_2 being given by (2.5).

2.2 The second method of synchronization

For a chaotic system, $\dot{X} = AX + f(X,\ t)$, let us consider a driving system in the form

$$\dot{X} = AX + f(X,\ t) + h_1(X,\ Y), \tag{2.7}$$

where $X \in \Re^n$, A is a $n \times n$, f and h_1 are $n \times 1$ matrices. Let the response system coupled with (2.7) be

$$\dot{Y} = AY + g(Y, t) + h_2(X, Y) + U, \tag{2.8}$$

where $Y \in \Re^n$, g and h_2 are both $n \times 1$ matrices. Error between the systems (2.7) and (2.8) can be defined as $e = X - DY$, where D is non-singular constant matrix. Thus the error dynamical system of(2.7) and (2.8) becomes

$$\dot{e} = Ae \tag{2.9}$$

provided

$$U = D^{-1}[f(X, t) + h(X, y)] - g(Y, t) - h_2(X, t) - AY + D^{-1}ADY. \tag{2.10}$$

If real parts of all the eigenvalues of A are negative, then the system (2.9) is asymptotically stable at the origin and hence the systems (2.7) and (2.8) are in the state of generalized synchronization.

3 Application of synchronization techniques

As an application, let us consider the Shimizu-Morioka chaotic dynamical system [4]

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - \lambda y - xz \\ \dot{z} &= -\alpha z + x^2. \end{aligned} \tag{3.1}$$

The system is chaotic for the values of the positive parameters $\lambda = 0.605$ and $\alpha = 0.549$.

3.1 Technique I

The system of equations (3.1) can equivalently be written as $\dot{X} = AX + f(X, t)$, where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\alpha \end{pmatrix}, \quad f(X, t) = \begin{pmatrix} x_1 + x_2 \\ -x_1 x_3 \\ x_1^2 \end{pmatrix}.$$

Using the first method, the driving and the response systems of the forms of (2.3) and (2.4) are constructed as follows:

$$\begin{aligned} \dot{x_1} &= x_2 + d_{11}(y_1 + y_2) + d_{12}\{y_1 - y_1 y_3 + (1-\lambda)y_2\} + d_{13}\{y_1^2 + (1-\alpha)y_3\} \\ \dot{x_2} &= x_1 - \lambda x_2 - x_1 x_3 + d_{21}(y_1 + y_2) - d_{22} y_1 y_3 + d_{23} y_1^2 + \\ &\quad (d_{22} - d_{11} - d_{21} + \lambda d_{21})y_1 - d_{12} y_2 + \{(\lambda - \alpha)d_{23} - d_{13}\}y_3 \\ \dot{x_3} &= -\alpha x_3 + x_1^2 + d_{31}(y_1 + y_2) - d_{32} y_1 y_3 + d_{33} y_1^2 + \{(\alpha - 1)d_{31} + d_{32}\}y_1 + \\ &\quad (\alpha - \lambda)d_{32} y_2 \end{aligned} \tag{3.2}$$

where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},\ Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},\ D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$$

and

$$g(Y,\ t) = \begin{pmatrix} y_1 + y_2 \\ -y_1 y_3 \\ y_1^2 \end{pmatrix}.$$

Also,

$$(3.3)\quad \begin{aligned} \dot{y_1} &= y_2 + \beta_{11}(x_1 + x_2) - \beta_{12} x_1 x_3 + \beta_{13} x_1^2 \\ &\quad + \{\Sigma_{i=1}^{3} k_i (x_i - d_{i1} y_1 - d_{i2} y_2 - d_{i3} y_3)\} \Sigma_{j=1}^{3} \beta_{1j} b_j \\ \dot{y_2} &= y_1 - \lambda y_2 - y_1 y_3 + \beta_{21}(x_1 + x_2) - \beta_{22} x_1 x_3 + \beta_{23} x_1^2 \\ &\quad + \{\Sigma_{i=1}^{3} k_i (x_i - d_{i1} y_1 - d_{i2} y_2 - d_{i3} y_3)\} \Sigma_{j=1}^{3} \beta_{2j} b_j \\ \dot{y_3} &= -\alpha y_3 + y_1^2 + \beta_{31}(x_1 + x_2) - \beta_{32} x_1 x_3 + \beta_{33} x_1^2 \\ &\quad + \{\Sigma_{i=1}^{3} k_i (x_i - d_{i1} y_1 - d_{i2} y_2 - d_{i3} y_3)\} \Sigma_{j=1}^{3} \beta_{3j} b_j, \end{aligned}$$

where $B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, $K = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}^T$, $D^{-1} = \begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix}$. The error dynamics of this system as described by (2.6) is

$$(3.4)\quad \begin{aligned} \dot{e_1} &= -(1 + b_1 k_1) e_1 - b_1 k_2 e_2 - b_1 k_3 e_3 \\ \dot{e_2} &= (1 - b_2 k_1) e_1 - (\lambda + b_2 k_2) e_2 - b_2 k_3 e_3 \\ \dot{e_3} &= -b_3 k_1 e_1 - b_3 k_2 e_2 - (\alpha + b_3 k_3) e_3, \end{aligned}$$

where

$$(3.5)\quad \begin{aligned} e_1 &= x_1 - d_{11} y_1 - d_{12} y_2 - d_{13} y_3 \\ e_2 &= x_2 - d_{21} y_1 - d_{22} y_2 - d_{23} y_3 \\ e_3 &= x_3 - d_{31} y_1 - d_{32} y_2 - d_{33} y_3. \end{aligned}$$

3.2 Technique II

System of equations (3.1) can alternatively be written as $\dot{X} = AX + f(X,\ t)$, where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},\ A = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -\alpha \end{pmatrix},\ f(X,\ t) = \begin{pmatrix} x_1 + 2x_2 \\ (1-\lambda)x_2 - x_1 x_3 \\ x_1^2 \end{pmatrix}.$$

Let us now consider the synchronized driving and response systems as given by (2.7) and (2.8). Here we take

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},\ g(Y,\ t) = \begin{pmatrix} y_1 + 2y_2 \\ (1-\lambda)y_2 - y_1 y_3 \\ y_1^2 \end{pmatrix},$$

$$h_1(X,\ Y)\ =\ \begin{pmatrix} y_1\ +\ 2y_2 -\ x_1 - 2x_2 \\ (1-\lambda)(y_2 -\ x_2) - y_1y_3 +\ x_1x_3 \\ y_1^2 - x_1^2 \end{pmatrix},$$

$$h_2(X,\ Y)\ =\ \begin{pmatrix} x_1 +\ 2x_2 \\ (1-\lambda)x_2 -\ x_1x_3 \\ x_1^2 \end{pmatrix}.$$

Using this technique, the driving and the response systems of the forms (2.7) and (2.8) are constructed as follows

$$\begin{aligned} \dot{x_1}\ &= -\,x_1 - x_2 + y_1 + 2y_2 \\ \dot{x_2}\ &= x_1 - x_2 + (1-\lambda)y_2 - y_1y_3 \\ \dot{x_3}\ &= -\,\alpha x_3 + y_1^2, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \dot{y_1}\ =\ & \beta_{11}\{y_1 + 2y_2 - (\Sigma d_{1j}y_j + \Sigma d_{2j}y_j)\} \\ & + \beta_{12}\{(1-\lambda)y_2 - y_1y_3 + \Sigma d_{1j}y_j - \Sigma d_{2j}y_j\} \\ & + \beta_{13}\{y_1^2 - \alpha\Sigma d_{3j}y_j\} \\ \dot{y_2}\ =\ & \beta_{21}\{y_1 + 2y_2 - (\Sigma d_{1j}y_j + \Sigma d_{2j}y_j)\} \\ & + \beta_{22}\{(1-\lambda)y_2 - y_1y_3 + \Sigma d_{1j}y_j - \Sigma d_{2j}y_j\} \\ & + \beta_{23}\{y_1^2 - \alpha\Sigma d_{3j}y_j\} \\ \dot{y_3}\ =\ & \beta_{31}\{y_1 + 2y_2 - (\Sigma d_{1j}y_j + \Sigma d_{2j}y_j)\} \\ & + \beta_{32}\{(1-\lambda)y_2 - y_1y_3 + \Sigma d_{1j}y_j - \Sigma d_{2j}y_j\} \\ & + \beta_{33}\{y_1^2 - \alpha\Sigma d_{3j}y_j\}. \end{aligned} \tag{3.7}$$

The error dynamical system, corresponding to the constructed bidirectionally coupled drive and response systems, is $\dot{e}\ =\ Ae$, i.e,

$$\begin{aligned} \dot{e_1}\ &= -\,e_1 - e_2 \\ \dot{e_2}\ &= e_1 - e_2 \\ \dot{e_3}\ &= -\,\alpha e_3, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} e_1\ &= x_1 - d_{11}y_1 - d_{12}y_2 - d_{13}y_3 \\ e_2\ &= x_2 - d_{21}y_1 - d_{22}y_2 - d_{23}y_3 \\ e_3\ &= x_3 - d_{31}y_1 - d_{32}y_2 - d_{33}y_3. \end{aligned} \tag{3.9}$$

4 Results and discussions

Numerical simulations are performed to show the effectiveness of the proposed techniques. Numerical simulation is carried out with $D = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$, $B = (1,2,3)^T, K = (2,1,4)$ and $\lambda = 0.605, \alpha = 0.549$. The time evolution of the synchronization errors

$e = (e_1, e_2, e_3)^T$ are plotted in Fig. 1 and Fig. 3 for Technique I and Technique II respectively.

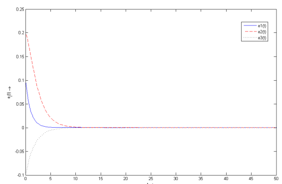

Fig. 1.

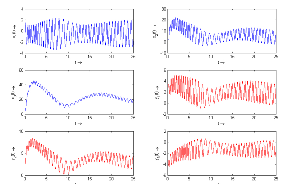

Fig. 2.

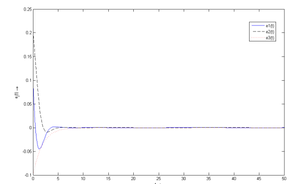

Fig. 3.

The initial synchronization errors are taken as $(e_1(0), e_2(0), e_3(0)) = (0.1, 0.2, -0.1)$, in each case. Figures show that the errors tend to zero as time goes to infinity which establishes the achievement of synchronization between the constructed drive and response systems using our techniques.

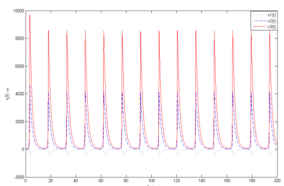

Fig. 4.

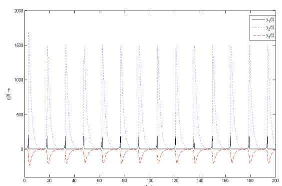

Fig. 5.

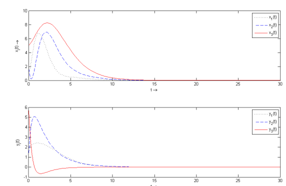

Fig. 6.

Time evolution of the state variables $x_i (i = 1, 2, 3)$, for the drive system, and $y_i (i = 1, 2, 3)$, for the response system, are plotted in Fig. 2 for Technique I, taking initial values of the state variables $(x_1(0), x_2(0), x_3(0)) = (2.2, 2.1, 2)$ and $(y_1(0), y_2(0), y_3(0)) = (2.1, 2.2, -2)$. A similar approach gives Figs. 4, 5 and 6 for Technique II. In this case, noticeable behaviour of the trajectories of the state variables for both the driving and the response systems are observed. Fig. 4 and 5 correspond to the initial values of the state variables $(x_1(0), x_2(0), x_3(0)) = (3, 2, 5)$ and $(y_1(0), y_2(0), y_3(0)) = (4, 1, 6)$ whereas Fig 6 corresponds to the initial condition $(x_1(0), x_2(0), x_3(0)) = (1, 2, 5)$ and $(y_1(0), y_2(0), y_3(0)) = (2, 1, 6)$.

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