The geometric and physical interpretation of fractional order derivatives of polynomial functions

M.H. Tavassoli, A. Tavassoli, M.R. Ostad Rahimi

Abstract. In this paper, after a brief mention of the definitions of fractional-order derivatives, we present a geometric interpretation of the tangent line angle of a polynomial with coefficients of fractional derivative. Then a comparison of the divergence of a gradient vector field in normal mode with the divergence of a vector field gradient fractions is performed. Finally, we show that there is a relationship between fractional derivative of polynomials at the tangent points and the order of the fractional derivative.

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Key words: Fractional calculus; fractional derivatives; polynomial functions; divergence; critical points.

1 Introduction

It is generally known that integer-order derivatives and integrals have clear physical and geometric interpretations, which significantly simplify their use for solving applied problems in various fields of science.

However, in case of fractional-order integration and differentiation, which represent a rapidly growing field both in theory and in applications to real-world problems, it is not so. Since the appearance of the idea of differentiation and integration of arbitrary (not necessary integer) order there was not any acceptable geometric and physical interpretation of these operations for more than 300 years [2, 10, 15, 5].

Fractional integration and fractional differentiation are generalizations of notions of integer-order integration and differentiation, and include \( n \)th derivatives and \( n \)-fold integrals (\( n \) denotes an integer number) as particular cases. Because of this, it would be ideal to have such physical and geometric interpretations of fractional-order operators, which will provide also a link to known classical interpretations of integer-order differentiation and integration.

Obviously, there is still a lack of geometric and physical interpretation of fractional integration and differentiation, which is comparable with the simple interpretations of their integer-order counterparts.
During the last two decades several authors have applied the fractional calculus in the field of sciences, engineering and mathematics (see [9, 4, 11, 6]). Mathematician Liouville, Riemann, and Caputo have done major work on fractional calculus, thus Fractional Calculus is a useful mathematical tool for applied sciences. Podlubny suggested a solution of more than 300 years old problem of geometric and physical interpretation of fractional integration and differentiation in 2002, for left-sided and right-sided of Riemann-Liouville fractional integrals [13, 1, 7].

J. A. Tenreiro Machado in 2003, presented a probabilistic interpretation of fractional order derivative, based on Grunwald–Letnikov definition of fractional order differentiation [8].

In this paper a new geometric interpretation for properties of polynomial’s tangent line is defined as an area of a triangle and then the relationship between this area and order of differentiation is investigated. Finally, it is shown that the area univocally increase or decrease according to the increasing of order of fractional derivative, except in the case where the order of derivative is equal to 0.5. some application of fractional derivatives in divergence of vector field gradient is also illustrated.

2 Definitions of fractional order derivatives

A number of researchers in this field have defined the fractional derivatives in different ways.[14, 12]

2.1 The Grunwald-Letnikov definition

(2.1) \[ GL_a D_x^\alpha f(x) = \lim_{h \to 0} \frac{1}{h^\alpha} \sum_{n=0}^{\left\lfloor \frac{x-a}{h} \right\rfloor} (-1)^n \binom{\alpha}{n} f(x-nh) \]

where \( a = x - nh \Rightarrow n = \frac{x-a}{h} \).

2.2 The Riemann-Liouville definition

The Riemann-liouville derivative of order \( \alpha \) and with lower limit \( a \) is defined as:

(2.2) \[ RL_a D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dx} \right)^n \int_a^x \frac{f(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau \]

where \( n \) is integer, \( \alpha \) is real number and \( (n-1) \leq \alpha < n \).

2.3 The M. Caputo (1967) definition

Caputo derivatives of order \( \alpha \) are defined as:

(2.3) \[ c_a D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}} d\tau, \]

where \( n \) is integer, \( \alpha \) is real number and \( (n-1) \leq \alpha < n \).
3 Fractional derivatives properties

3.1 Definitions of Oldham and Spanier (1974)

The scaling property of fractional derivatives is described by:

\[
\frac{d^{\alpha} f(\beta x)}{dx^\alpha} = \beta^{\alpha} \frac{d^{\alpha} f(\beta x)}{d(\beta x)^\alpha}.
\]

This makes it suitable for the study of scaling and scale invariance. There is connection between local-scaling, box-dimension of an irregular function and order of fractional derivative.

3.2 Linearity

Fractional differentiation is a linear operation:

\[
D^\alpha (\mu f(x) + \omega g(x)) = \mu D^\alpha f(x) + \omega D^\alpha g(x),
\]

where \(D^\alpha\) denotes any mutation of the fractional differentiation considered in this paper.

3.3 Definitions of K. S. Miller and B. Ross (1993)

\[
D^\alpha f(x) = D^{\alpha_1} D^{\alpha_2} \ldots D^{\alpha_n} f(x)
\]

\[
\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_n
\]

\[
\alpha_i < 1
\]

This definition of sequential composition is very useful concept for obtaining fractional derivative of an arbitrary order. The derivative operator can be any definition Riemann-Liouville or Caputo.

4 Geometric and physical interpretation of fractional order derivatives

Geometrical and physical interpretations of integer order derivative and integral are defined in a simple way. The fractional order derivative and fractional order integral are not yet well established in simple way. In this paper, a simple interpretation of fractional order derivative is presented, which is useful in the applications of the subject.

The fractional order derivatives of a polynomial function can be computed by the formula

\[
D^\alpha [x^\beta] = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - \alpha)} x^{\beta - \alpha},
\]
where \( \alpha \) is the order of derivative and \( 0 < \alpha < 1 \). By using the formula given in (4.1) and the property (3.2), the fractional derivative values of functions \( f(x) = x^3 \) and \( g(x) = x^4 + x^3 \) at \( x = 2 \) were computed and shown in Table 4.1 and Table 4.2 respectively.

<table>
<thead>
<tr>
<th>Table 4.1</th>
</tr>
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<tbody>
<tr>
<td>Fractional order derivatives</td>
</tr>
<tr>
<td>( D^\alpha {f}(x) )</td>
</tr>
<tr>
<td>( D^\alpha {g}(x) )</td>
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</table>

Table 4.2

| Fractional order derivatives | Fractional derivative values at \( x = 2 \) | \( \theta = \tan^{-1} m \) (in radian) | Area of triangle (\( \Delta \)) |
|---|
| \( D^\alpha \{g\}(x) \) | \( m_0 = 9.948 \) | \( \theta = 40.89 \) | \( \Delta P_1A_1B = 1.7648 \) |

Consider the function \( f(x)=x^3 \) at \( P(2,8) \) we have \( D^1 f(x)[x]=12.00 \). Now with the tangent line \( l_1 \) drawn at \( P(2,8) \) which passes through the \( X \)-axes at \( A_1 \) and with the perpendicular line from \( P(2,8) \) to \( X \)-axes at \( B(2,0) \) we have an area (\( \Delta \)) enclosed by triangle \( PA_1B=2.6667 \) (\( \Delta PA_1B=2.6667 \)). Similarly all the triangles are formed by using fractional derivative values \( m_{0.1}, m_{0.2}, \ldots, m_{0.9} \) with tangent line \( l_{0.1}, l_{0.2}, \ldots, l_{0.9} \) passing through point \( P(2,8) \). The areas of triangles are computed and the related results are shown in Table 4.1.

Similarly, the areas of triangles (\( \Delta \)) for the function \( g(x) = x^4 + x^3 \) at \( P(2,24) \) are computed and the results are shown in Table 4.2.

Figure.1 and Figure.2 show the graphs of the functions \( f(x) \) and \( g(x) \) with triangles formed by fractional derivatives of order 0.2, 0.4, 0.6 and 0.8.

From Tables 4.1 and 4.2 and from the graphs of the functions \( f(x) \) and \( g(x) \), it is observed that if the value of fractional order derivative increases, then the area of triangle decreases, and if the value of fractional order derivative decreases, then the area of triangle increases. Hence fractional order derivative values and areas of triangles are inversely proportional. Further,

\[
D^\alpha [f(x)] \propto \frac{1}{\alpha} \\
D^\alpha [g(x)] \propto \frac{1}{\alpha}
\]

infer

\[
D^\alpha [f(x)].\Delta = D^\alpha [g(x)].\Delta = \text{constant}.
\]
Interpretation of fractional order derivatives

Figure 1: Graph of the function $f(x) = x^3$ with triangles formed with fractional order derivatives.

Figure 2: Graph of function $g(x) = x^4 + x^3$ with triangles formed with fractional order derivatives.

We conclude that the product of fractional order derivative with the correspondent area is constant, so the fractional derivative produces the change in the area of the triangle enclosed by the tangent line at particular point and vertical line passing through this point and above $X$-axes with respect to fractional gradient line.

The change of area is a physical property, therefore fractional derivatives can be used to measure the changes in temperature, pressure, gradient, divergence and curl, etc.
5 Application in physical quantity divergence

Area and divergence are physical quantities. Fractional order derivative produces the change in area of a triangle mentioned in section 4. In this section we will show that fractional order derivative produces the changes in divergence of vector field.

5.1 The divergence

In physical terms, the divergence of a three dimensional vector field is the extent to which the vector field flow behaves like a source or a sink at a given point.

Let \( x, y, z \) be a system of Cartesian coordinates on 3-dimensional space and let\( i, j, k \) be the corresponding basis of unit vector [3].

The divergence of continuous differentiable vector field \( F = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k} \) is defined to be the scalar-valued function given by

\[
\text{Div} \ F = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.
\]

Thus the divergence at a point measures how much the vector field \( F \) “spreads out” at the point. A positive divergence means that vector field has a net expansion from the point. A negative divergence means it has a net contraction into the point.

5.2 Fractional divergence

Fractional divergence of a vector filed \( F \) can be defined by using the concepts of fractional order derivatives:

\[
\nabla^\alpha F = \frac{\partial^{\alpha} F_1}{\partial x^{\alpha}} + \frac{\partial^{\alpha} F_2}{\partial y^{\alpha}} + \frac{\partial^{\alpha} F_3}{\partial z^{\alpha}},
\]

where \( \alpha \) is the order of derivative and \( 0 < \alpha < 1 \).

5.3 Comparison of two models of divergence

One might consider, e.g., \( K(x, y, z) = x^4 + y^3 + z^2 \), two points \( A(1, 1, 1) \) and \( B(4, 1, 2) \) and compare the divergence of gradient \( K \) with fractional divergence of gradient \( K \) at these points.

5.3.1 Divergence of \( \nabla K(x, y, z) \)

Let \( \nabla K = F = 4x^3 \vec{i} + 3y^2 \vec{j} + 2z \vec{k} \)

\[
\text{Div} \ F = \nabla \cdot F = \frac{\partial}{\partial x} (4x^3) + \frac{\partial}{\partial y} (3y^2) + \frac{\partial}{\partial z} (2z) \]

\[
= 12x^2 + 6y + 2.
\]

Therefore at \( A(1, 1, 1) \) we have

\[
\nabla \cdot F(1, 1, 1) = 20 \quad |F(1, 1, 1)| = \sqrt{29} = 5.3852
\]
As it can be observed, $\nabla . F > |F|$. Similarly, at $B(4, 1, 2)$, we have

$$\nabla . F(4, 1, 2) = 200$$

$$|F(4, 1, 2)| = \sqrt{65561} = 256.0488$$

and hence $\nabla . F < |F|$.

5.3.2 Fractional divergence of $\nabla K(x, y, z)$

Using the formula

$$D^\alpha [x^\beta] = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - \alpha)} x^{\beta - \alpha},$$

the fractional divergence values are computed for $\alpha = 0.1, 0.2, \ldots, 0.9$ at two points $A(1, 1, 1)$ and $B(4, 1, 2)$. As shown in table 5.1, it is observed that:

1. When $\nabla . F < |F|$ at $B(4, 1, 2)$ the amount fractional divergence of vector field $F$ is decreasing from $\nabla . F = 0.1$ to $\nabla . F = 0.9$. Fractional divergence of vector field $F$ at $\alpha = 0.1$ is very high than the divergence of $F$.

2. When $\nabla . F > |F|$ at $A(1, 1, 1)$ the amount fractional divergence of vector field $F$ is increasing from $\nabla . F = 0.1$ to $\nabla . F = 0.9$. Fractional divergence of vector field $F$ at $\alpha = 0.1$ is very low than the divergence of $F$.

It is suggested that to obtain the higher amount of vector field spread out, we can use the fractional divergence $\nabla . F$, rather than $\nabla . F$. Thus geometric and physical interpretation of fractional order derivative of a polynomial function play important role for measuring the changes in physical quantities.

6 Critical point of fractional derivatives for polynomial functions

According to the table 6.1, for values taken by fractional derivatives $D^\alpha [x^\beta]$ at different values of $\beta = 2, 3, \ldots, 10$, it can be seen that there are two different results for $x = \beta$ and $x \neq \beta$. Namely, there is a monotonically increasing or decreasing trend for the case $x \neq \beta$. But it is somehow different for the case $x = \beta$, and we will prove that in this case $\alpha = 0.5$ is a critical point for $D^\alpha [x^\beta]$.

We set

$$D (x, \beta, \alpha) = D^\alpha [x^\beta].$$
For $x = \beta$, we have

$$D(\beta, \beta, \alpha) = D(\beta, \alpha) = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + 1 - \alpha)} \beta^{\beta - \alpha},$$

and hence

$$\frac{\partial D}{\partial \alpha} = \frac{\Gamma(\beta + 1) \beta^{(\beta - \alpha)} [\psi(\beta + 1 - \alpha) - \ln(\beta)]}{\Gamma(\beta + 1 - \alpha)} = 0,$$

where

$$\psi(x) = \frac{d}{dx} \ln(\Gamma(x)) = \frac{d}{dx} \frac{\Gamma(x)}{\Gamma(x)} = \frac{\Gamma'(x)}{\Gamma(x)}.$$ 

The numerical solutions of (6.1) for different values of $\beta = 2, 3, \ldots, 10$ give the critical point $\alpha = 0.5$ in the interval $0 < \alpha < 1$.

Hence, according with results of table 6.1 for $D^{\alpha}[f(x)] = D^{\alpha}[x^\beta]$ and also from the produced triangles of Section 4, we conclude that:

1. If $x > \beta$, then the value of the fractional derivative $(m)$ decreases and the area of the triangle $(\triangle)$ increases.

2. If $x < \beta$, then the value of the fractional derivative $(m)$ increases and the area of the triangle $(\triangle)$ decreases.

3. If $x = \beta$ then the value of the fractional derivative increases from $\alpha = 0.1$ to $\alpha = 0.5$ and decreases from $\alpha = 0.5$ to $\alpha = 1.0$, and conversely the area of the triangle primarily decreases and then increases.
Interpretation of fractional order derivatives

Table 6.1 Fractional derivative values for polynomial function $f(x) = x^k$

<table>
<thead>
<tr>
<th>$D^\alpha f(x)$</th>
<th>$\beta = 2$</th>
<th>$\beta = 4$</th>
<th>$\beta = 6$</th>
<th>$\beta = 8$</th>
<th>$\beta = 10$</th>
</tr>
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<tbody>
<tr>
<td>$D^0 f(x)$</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td>$D^1 f(x)$</td>
<td>1.2948</td>
<td>1.5564</td>
<td>1.7416</td>
<td>1.8506</td>
<td>2.0102</td>
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<tr>
<td>$D^2 f(x)$</td>
<td>1.5990</td>
<td>1.7945</td>
<td>2.0058</td>
<td>2.3120</td>
<td>2.5421</td>
</tr>
<tr>
<td>$D^3 f(x)$</td>
<td>1.9045</td>
<td>2.0865</td>
<td>2.3010</td>
<td>2.6729</td>
<td>2.8380</td>
</tr>
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<td>$D^4 f(x)$</td>
<td>2.2101</td>
<td>2.3678</td>
<td>2.5906</td>
<td>2.8530</td>
<td>3.0293</td>
</tr>
<tr>
<td>$D^5 f(x)$</td>
<td>2.5142</td>
<td>2.7102</td>
<td>2.9077</td>
<td>3.1442</td>
<td>3.3051</td>
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<td>$D^6 f(x)$</td>
<td>2.8182</td>
<td>3.0041</td>
<td>3.2011</td>
<td>3.4317</td>
<td>3.6007</td>
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<tr>
<td>$D^7 f(x)$</td>
<td>3.1212</td>
<td>3.2229</td>
<td>3.4044</td>
<td>3.6330</td>
<td>3.7976</td>
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<td>$D^8 f(x)$</td>
<td>3.4240</td>
<td>3.5000</td>
<td>3.6600</td>
<td>3.8800</td>
<td>4.0000</td>
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<td>$D^9 f(x)$</td>
<td>3.7268</td>
<td>3.7857</td>
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<td>4.2400</td>
</tr>
<tr>
<td>$D^{10} f(x)$</td>
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<td>4.0703</td>
<td>4.2000</td>
<td>4.4000</td>
<td>4.5200</td>
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<td>$D^{11} f(x)$</td>
<td>4.3325</td>
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<td>$D^{12} f(x)$</td>
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<td>4.6458</td>
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<td>4.9500</td>
<td>5.1200</td>
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<td>$D^{13} f(x)$</td>
<td>4.9381</td>
<td>4.9381</td>
<td>5.0350</td>
<td>5.2300</td>
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<td>$D^{14} f(x)$</td>
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<td>5.6500</td>
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<tr>
<td>$D^{15} f(x)$</td>
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<td>5.5437</td>
<td>5.6150</td>
<td>5.7900</td>
<td>5.9200</td>
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<td>5.8465</td>
<td>5.8465</td>
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<td>6.3500</td>
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<td>6.4521</td>
<td>6.5000</td>
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<td>6.7600</td>
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<tr>
<td>$D^{19} f(x)$</td>
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<td>6.7549</td>
<td>6.8000</td>
<td>6.9400</td>
<td>7.0600</td>
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<tr>
<td>$D^{20} f(x)$</td>
<td>7.0577</td>
<td>7.0577</td>
<td>7.1000</td>
<td>7.2400</td>
<td>7.3600</td>
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References


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