Pseudo d-parallel Jacobi structure operators in non-flat complex planes

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Abstract. Real hypersurfaces of a complex space form $M_n(c)$ have been studied from many points of view. The real hypersurfaces which satisfy $(\nabla_X l)Y = \kappa\{\eta(Y)\phi AX + g(\phi AX, Y)\xi\}$, where l is the Jacobi structure operator and κ is constant, are called "real hypersurfaces with pseudo $\mathbb{D}-$parallel Jacobi structure operator". This class has been classified in [7], for the case when $M_n(c) = \mathbb{C}P^n$ $(c > 0)$ and $n \geq 3$. In the present paper, the same class is classified for the case of a complex plane $M_2(c)$ where the sectional curvature c can be positive or negative. In addition, the constant κ is now a function, therefore, a larger class is produced and classified.

M.S.C. 2010: 53B25, 53D15
Key words: Real hypersurface; structure Jacobi operator; pseudo-parallel tensor field.

1 Introduction

An n-dimensional Kaehlerian manifold of constant holomorphic sectional curvature c is called complex space form, which is denoted by $M_n(c)$. A complete and simply connected complex space form is complex analytically isometric to a projective space $\mathbb{C}P^n$ if $c > 0$, a hyperbolic space $\mathbb{C}H^n$ if $c < 0$, or a Euclidean space $\mathbb{C}^n$ if $c = 0$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ will be denoted by (ϕ, ξ, η, g). The vector field ξ is defined by $\xi = -JN$ where J is the complex structure of $M_n(c)$ and N is a unit normal vector field.

Real hypersurfaces have been studied by many authors and under several conditions ([1], [2], [13], [14]). An important class of hypersurfaces is the *Hopf Hypersurfaces*, that is real hypersurfaces satisfying $A\xi = \alpha\xi$, where A is the shape operator and $\alpha = g(A\xi, \xi)$.

Certain authors have studied real hypersurfaces under conditions which involve the Jacobi structure operator $lX = R_\xi X = R(X, \xi)\xi$ ([10], [11], [12]).

In [7], H. Lee, J. D. Pérez and Y. Jin Suh introduced the notion of *pseudo* $\mathbb{D}$-*parallel* structure Jacobi operator, that is l satisfies the following condition:

$$(\nabla_X l)Y = \kappa\{\eta(Y)\phi AX + g(\phi AX, Y)\xi\} \tag{1.1}$$

Differential Geometry - Dynamical Systems, Vol.15, 2013, pp. 105-121.

where κ is a non-zero constant, $X \in \mathbb{D}$ and $Y \in TM$. They studied the above condition in real hypersurfaces of $\mathbb{C}P^n$, $n \geq 3$ classifying them.

However, the problem remains open for the case of $\mathbb{C}H^n$, $n \geq 3$, and the case of $M_2(c)$ (both $c > 0$ and $c < 0$). In the present paper the latter case is treated in an even more generalized form: the constant κ in (1.1) is replaced by a function without any other restriction for κ. Namely we prove the following:

Main Theorem. *Let M be a real hypersurface of a complex plane $M_2(c)$, whose structure Jacobi operator satisfies condition (1.1) for some non-vanishing function κ. Then M is a Hopf hypersurface. Furthermore, we have:*

- *if $g(A\xi,\xi) \neq 0$, then M is a tube of radius $r = \frac{1}{\sqrt{|c|}} ln(2+\sqrt{3})$ around totally real geodesic $\mathbb{R}H^n(\frac{c}{4})$ of a complex hyperbolic space $\mathbb{C}H^2$;*
- *if $g(A\xi,\xi) = 0$, then the function κ is constant and equal to $-\frac{c}{4}$, and we have one of the following cases:*

1) A has two principal curvatures ($\alpha = 0$, $\lambda_1 = \lambda_2 = \frac{\sqrt{c}}{2}$) and M is a geodesic hypersphere of radius $r = \frac{\pi}{4}$ on $\mathbb{C}P^2$,

2) A has three principal curvatures ($\alpha = 0$, λ_1,λ_2), where λ_1, λ_2 are not constants and satisfy $\lambda_1\lambda_2 = \frac{c}{4}$.

2 Preliminaries

Let M_n be a Kaehlerian manifold of real dimension $2n$, equipped with an almost complex structure J and a Hermitian metric tensor G. Then for any vector fields X and Y on $M_n(c)$, the following relations hold: $J^2X = -X$, $G(JX, JY) = G(X, Y)$, $\widetilde{\nabla}J = 0$, where $\widetilde{\nabla}$ denotes the Riemannian connection of G of M_n.

Let M_{2n-1} be a real $(2n-1)$-dimensional hypersurface of $M_n(c)$, and denote by N a unit normal vector field on a neighborhood of a point in M_{2n-1} (from now on we shall write M instead of M_{2n-1}). For any vector field X tangent to M we have $JX = \phi X + \eta(X)N$, where ϕX is the tangent component of JX, $\eta(X)N$ is the normal component, and $\xi = -JN$, $\eta(X) = g(X,\xi)$, $g = G|_M$.

By properties of the almost complex structure J and the definitions of η and g, the following relations hold ([3]):

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta \circ \phi = 0, \qquad \phi\xi = 0, \quad \eta(\xi) = 1 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \qquad g(X, \phi Y) = -g(\phi X, Y). \tag{2.2}$$

The above relations define an *almost contact metric structure* on M which is denoted by (ϕ, ξ, g, η). When an almost contact metric structure is defined on M, we can define a local orthonormal basis $\{e_1, e_2, ...e_{n-1}, \phi e_1, \phi e_2, ...\phi e_{n-1}, \xi\}$, called a $\phi-basis$. Furthermore, let A be the shape operator in the direction of N, and denote by ∇ the Riemannian connection of g on M. Then, A is symmetric and the following equations are satisfied:

$$\nabla_X\xi = \phi AX, \qquad (\nabla_X\phi)Y = \eta(Y)AX - g(AX,Y)\xi \tag{2.3}$$

As the ambient space $M_n(c)$ is of constant holomorphic sectional curvature c, the equations of Gauss and Codazzi are respectively given by:

$$R(X,Y)Z = \frac{c}{4}[g(Y,Z)X - g(X,Z)Y + g(\phi Y,Z)\phi X - g(\phi X,Z)\phi Y \\ -2g(\phi X,Y)\phi Z] + g(AY,Z)AX - g(AX,Z)AY, \tag{2.4}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi]. \tag{2.5}$$

The tangent space T_pM, for every point $p \in M$, is decomposed as following: $T_pM = \mathbb{D}^\perp \oplus \mathbb{D}$, where $\mathbb{D} = ker(\eta) = \{X \in T_pM : \eta(X) = 0\}$

Based on the above decomposition, by virtue of (2.3), we decompose the vector field $A\xi$ in the following way:

$$A\xi = \alpha\xi + \beta U, \tag{2.6}$$

where $\beta = |\phi\nabla_\xi\xi|$, α is a smooth function on M and $U = -\frac{1}{\beta}\phi\nabla_\xi\xi \in ker(\eta)$, provided that $\beta \neq 0$.

If the vector field $A\xi$ is expressed as $A\xi = \alpha\xi$, then ξ is called principal vector field.

Finally differentiation will be denoted by ( ). All manifolds, vector fields, e.t.c., of this paper are assumed to be connected and of class C^∞.

3 Auxiliary relations

Let $\mathcal{N} = \{p \in M : \beta \neq 0$ in a neighborhood around $p\}$. We define the open subsets $\mathcal{N}_1$ and $\mathcal{N}_2$ of $\mathcal{N}$ such that:

$\mathcal{N}_1 = \{p \in \mathcal{N} : \alpha \neq 0$ in a neighborhood around $p\}$,

$\mathcal{N}_2 = \{p \in \mathcal{N} : \alpha = 0$ in a neighborhood around $p\}$.

Then $\mathcal{N}_1 \cup \mathcal{N}_2$ is open and dense in the closure of $\mathcal{N}$.

Lemma 3.1. *Let M be a real hypersurface of a complex plane $M_2(c)$. Then the following relations hold on $\mathcal{N}_1$.*

$$AU = \left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)U + \frac{\delta}{\alpha}\phi U + \beta\xi, \qquad A\phi U = \frac{\delta}{\alpha}U + \left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right)\phi U \tag{3.1}$$

$$\nabla_\xi\xi = \beta\phi U,\ \nabla_U\xi = -\frac{\delta}{\alpha}U + \left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)\phi U, \\ \nabla_{\phi U}\xi = -\left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right)U + \frac{\delta}{\alpha}\phi U \tag{3.2}$$

$$\nabla_\xi U = \kappa_1\phi U, \quad \nabla_U U = \kappa_2\phi U + \frac{\delta}{\alpha}\xi, \quad \nabla_{\phi U}U = \kappa_3\phi U + \left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right)\xi \tag{3.3}$$

$$(3.4)\qquad \nabla_\xi \phi U = -\kappa_1 U - \beta\xi, \quad \nabla_U \phi U = -\kappa_2 U - \left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)\xi,$$
$$\nabla_{\phi U}\phi U = -\kappa_3 U - \frac{\delta}{\alpha}\xi,$$

where κ_1, κ_2, κ_3 are smooth functions on $\mathcal{N}_1$.

Proof. From (2.4) we obtain

$$(3.5)\qquad lU = \frac{c}{4}U + \alpha AU - \beta A\xi, \qquad l\phi U = \frac{c}{4}\phi U + \alpha A\phi U.$$

The inner products of lU with U and ϕU yield respectively

$$(3.6)\qquad g(AU, U) = \frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}, \quad g(AU, \phi U) = \frac{\delta}{\alpha},$$

where $\gamma = g(lU, U)$ and $\delta = g(lU, \phi U)$. So, (3.6) and $g(AU, \xi) = g(A\xi, U) = \beta$, yield the first of (3.1). Since l is symmetric with respect to metric g, the scalar products of the second of (3.5) with U and ϕU yield respectively

$$(3.7)\qquad g(A\phi U, U) = \frac{\delta}{\alpha}, \qquad g(A\phi U, \phi U) = \frac{\epsilon}{\alpha} - \frac{c}{4\alpha},$$

where $\epsilon = g(l\phi U, \phi U)$. So, (3.7) and $g(A\phi U, \xi) = g(A\xi, \phi U) = 0$, yield the second of (3.1). Combining (3.1) and (3.5), we obtain

$$(3.8)\qquad lU = \gamma U + \delta\phi U, \qquad l\phi U = \delta U + \epsilon\phi U.$$

By virtue of (2.6) and (3.1), the first of (2.3) for $X = \xi$, $X = U$ and $X = \phi U$ yields (3.2).
It is well known that:

$$(3.9)\qquad Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

The relation (3.9) for $X = \xi$, $Y = Z = U$ and $X = Z = \xi$, $Y = U$, because of (3.2), implies respectively $g(\nabla_\xi U, U) = 0 = g(\nabla_\xi U, \xi)$. So if we put $g(\nabla_\xi U, \phi U) = \kappa_1$, we have the first of (3.3). Similarly (3.9) for $X = Y = Z = U$ and $X = Y = U$, $Z = \xi$, because of (3.2) yields respectively $g(\nabla_U U, U) = 0$, $g(\nabla_U U, \xi) = \frac{\delta}{a}$. Therefore, putting $g(\nabla_U U, \phi U) = \kappa_2$, we have the second of (3.3). By use of (3.2) and (3.9) we have that $g(\nabla_{\phi U} U, U) = 0$ and $g(\nabla_{\phi U} U, \xi) = \frac{\epsilon}{\alpha} - \frac{c}{4\alpha}$. Then if we set $g(\nabla_{\phi U} U, \phi U) = \kappa_3$, we get the third of (3.3). In a similar way using (3.9) we obtain (3.4). □

The condition (1.1) for $X = Y = U$ yields

$$(\nabla_U l)U = \kappa\{\eta(U)\phi AU + g(\phi AU, U)\xi\}.$$

The above equation is further developed by making use of Lemma 3.1 and (3.8), giving the following:

$$(U\gamma)U + \kappa_2(\gamma - \epsilon)\phi U + (U\delta)\phi U - 2\kappa_2\delta - \delta\left(\frac{\beta^2}{\alpha} - \frac{c}{4\alpha}\right)\xi = -\frac{\delta\kappa}{\alpha}\xi.$$

Since the vector fields $U, \phi U$ and ξ are linearly independent, the last relation leads to

$$\delta(\beta^2 - \frac{c}{4}) = \delta\kappa, \tag{3.10}$$

$$(U\gamma) = 2\kappa_2\delta, \tag{3.11}$$

$$(U\delta) = \kappa_2(\epsilon - \gamma). \tag{3.12}$$

The condition (1.1) for $X = U$, $Y = \phi U$ yields

$$(\nabla_U l)\phi U = \kappa\{\eta(U)\phi A\phi U + g(\phi AU, \phi U)\xi\}.$$

The above equation is further developed by making use of Lemma 3.1, (3.8) and (3.12), giving the following:

$$2\delta\kappa_2\phi U + \frac{\delta^2}{\alpha}\xi + (U\epsilon)\phi U - \epsilon\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)\xi = \kappa\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)\xi.$$

Since the vector fields $U, \phi U$ and ξ are linearly independent, the last relation leads to

$$(\kappa + \epsilon)\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right) = \frac{\delta^2}{\alpha}, \tag{3.13}$$

$$(U\epsilon) = -2\kappa_2\delta. \tag{3.14}$$

Putting $X = \phi U$, $Y = U$ in (1.1) we obtain

$$(\nabla_{\phi U} l)U = \kappa\{\eta(\phi U)\phi AU + g(\phi A\phi U, U)\xi\}.$$

The above equation is further developed by making use of Lemma 3.1, (3.8), (3.12) and the linear independency of the vector fields U, ϕU giving the following:

$$(\kappa + \gamma)\left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right) = \frac{\delta^2}{\alpha}, \tag{3.15}$$

$$(\phi U\gamma) = 2\kappa_3\delta, \tag{3.16}$$

$$(\phi U\delta) = \kappa_3(\epsilon - \gamma). \tag{3.17}$$

Finally putting $X = Y = \phi U$ in (1.1) we get

$$(\nabla_{\phi U} l)\phi U = \kappa\{\eta(\phi U)\phi A\phi U + g(\phi A\phi U, \phi U)\xi\},$$

which, in a similar way, implies

$$-\frac{\delta c}{4} = \kappa\delta, \tag{3.18}$$

$$(\phi U\epsilon) = -2\kappa_3\delta. \tag{3.19}$$

From (3.10) and (3.18) we obtain the following lemma:

Lemma 3.2. *Let M be a real hypersurface of a complex plane $M_2(c)$ satisfying (1.1). Then on $\mathcal{N}_1$ we have $\delta = 0$.*

4 The set $\mathcal{N}_1$

We are going to use equation (2.5) for $X, Y \in \{U, \phi U, \xi\}$. For $X = U$, $Y = \xi$ we have $(\nabla_U A)\xi - (\nabla_\xi A)U = -\frac{c}{4}\phi U$. The last relation is further developed by virtue of Lemmas 3.1 and 3.2, yielding:

$$(U\alpha) = (\xi\beta)\,, \tag{4.1}$$

$$(U\beta) = \left(\xi\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)\right), \tag{4.2}$$

$$\gamma + \kappa_2\beta - (\frac{\epsilon}{\alpha} - \frac{c}{4\alpha})\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right) - \kappa_1\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right) + \kappa_1\left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right) = 0. \tag{4.3}$$

In a similar way, (2.5) for $X = \phi U$, $Y = \xi$ yields

$$(\phi U\alpha) + 3\beta(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}) - \kappa_1\beta - \alpha\beta = 0. \tag{4.4}$$

$$\begin{aligned}(\phi U\beta) + (\frac{\epsilon}{\alpha} - \frac{c}{4\alpha})\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right) + \kappa_1\left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right) - \kappa_1\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)\\ -\beta^2 - \epsilon = 0,\end{aligned} \tag{4.5}$$

$$\xi\left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right) = \kappa_3\beta. \tag{4.6}$$

Similarly, the relation (2.5) for $X = U$, $Y = \phi U$ yields

$$U\left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right) = \kappa_3\left(\frac{\gamma}{\alpha} - \frac{\epsilon}{\alpha} + \frac{\beta^2}{\alpha}\right), \tag{4.7}$$

$$\begin{aligned}\kappa_2\left(\frac{\gamma}{\alpha} - \frac{\epsilon}{\alpha} + \frac{\beta^2}{\alpha}\right) + \beta\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right) + 2\beta\left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right)\\ -\left(\phi U(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha})\right) = 0,\end{aligned} \tag{4.8}$$

We now define the subset $\mathcal{N}'_1 \subset \mathcal{N}_1$ to be the set of points $p \in \mathcal{N}_1$ such that $\gamma \neq \epsilon$ in a neighborhood around p.

Lemma 4.1. *Let M be a real hypersurface of a complex plane $M_2(c)$ satisfying (1.1). Then $\mathcal{N}'_1 = \varnothing$ and $\gamma = \epsilon$ on $\mathcal{N}_1$.*

Proof. Throughout the proof of this Lemma we work in $\mathcal{N}'_1$. By definition of $\mathcal{N}'_1$, equations (3.12), (3.17) and Lemma 3.2 yield $\kappa_2 = \kappa_3 = 0$. So, using (2.4) for $X = Z = U$, $Y = \xi$ and Lemma 3.1 we take

$$R(U,\xi)U = -\gamma\xi.$$

On the other hand, by virtue of Lemmas 3.1, 3.2, $\kappa_2 = \kappa_3 = 0$ and (4.3) we obtain

$$R(U,\xi)U = \nabla_U \nabla_\xi U - \nabla_\xi \nabla_U U - \nabla_{\nabla_U \xi - \nabla_\xi U} U = (U\kappa_1)\phi U - \gamma \xi.$$

The last two equations lead to

$$(U\kappa_1) = 0. \tag{4.9}$$

In a similar way, we calculate $R(U, \phi U)U$ first from (2.4) and then from

$$R(U,\phi U)U = \nabla_U \nabla_{\phi U} U - \nabla_{\phi U} \nabla_U U - \nabla_{\nabla_U \phi U - \nabla_{\phi U} U} U,$$

we conclude that

$$2\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right)\left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right) + \kappa_1\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha} + \frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right) + c = 0. \tag{4.10}$$

Similarly, the calculation of $R(\phi U, \xi)\phi U$ first from (2.4) and then from

$$R(\phi U,\xi)\phi U = \nabla_{\phi U} \nabla_\xi \phi U - \nabla_\xi \nabla_{\phi U} \phi U - \nabla_{\nabla_{\phi U}\xi - \nabla_\xi \phi U} \phi U$$

implies

$$(\phi U\kappa_1) = 2\beta\left(\frac{\epsilon}{\alpha} - \frac{c}{4\alpha}\right) + \kappa_1\beta. \tag{4.11}$$

Let us assume there is a point $p_1 \in \mathcal{N}'_1$ such that $\epsilon \neq \frac{c}{4}$. Then there exists a neighborhood around p_1 such that $\epsilon \neq \frac{c}{4}$ in this neighborhood. Equation (3.15) and Lemma 3.2 yield $\kappa = -\gamma$, which is combined with (3.13) and Lemma 3.2 implying $(\gamma - \epsilon)(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) = 0$. Since on $\mathcal{N}'_1$ $\gamma \neq \epsilon$ holds, then we obtain $\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha} = 0$. However the last relation, (4.8) and $\kappa_2 = 0$ imply $\epsilon = \frac{c}{4}$ which is a contradiction. Therefore there exists no point in $\mathcal{N}'_1$ such that $\epsilon \neq \frac{c}{4}$ and so in $\mathcal{N}'_1$ we have $\epsilon = \frac{c}{4}$.

In this case, (4.3), (4.8) and (4.10) (with $\kappa_2 = 0$) yield respectively

$$\gamma = \kappa_1\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right), \quad \phi U\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right) = \beta\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right), \tag{4.12}$$

$$-c = \kappa_1\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right).$$

From (4.12) we observe that $\kappa_1 \neq 0$ (otherwise $c = 0$ which is a contradiction). So, the differentiation of $-c = \kappa_1(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha})$ along ϕU implies

$$(\phi U\kappa_1)\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}\right) + \kappa_1\left(\phi U(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha})\right) = 0.$$

Replacing in the above equation the term $(\phi U\kappa_1)$ from (4.11)($\epsilon = \frac{c}{4}$) and by virtue of the second of (4.12), we take $\kappa_1\beta(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) = 0 \Rightarrow c = 0$ (due to (4.12)), which is a contradiction. So $\mathcal{N}'_1 = \varnothing$ and $\gamma = \epsilon$ in $\mathcal{N}_1$. □

Lemma 4.2. *Let M be a real hypersurface of a complex plane $M_2(c)$ satisfying (1.1). Then on $\mathcal{N}_1$, $\gamma \neq \frac{c}{4}$.*

Proof. Combining (4.8), with (4.3), (4.4), (4.5) we obtain

$$\left(\phi U\left(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\right)\right)=\frac{3\beta}{\alpha}\left[(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2-\frac{c}{4}\right]. \tag{4.13}$$

If $\gamma=\frac{c}{4}$ then the last relation yields $\frac{3\beta c}{4}=0$ which is a contradiction. Hence we have $\gamma\neq\frac{c}{4}$. □

Lemma 4.3. *Let satisfying (1.1). Then on $\mathcal{N}_1$, $\kappa_3=0$.*

Proof. Because of (3.3), (3.4), (4.6), (4.7) and (4.13), the well known relation $[U,\phi U]=\nabla_U\phi U-\nabla_{\phi U}U$ takes the form

$$[U,\phi U](\frac{\gamma}{\alpha}-\frac{c}{4\alpha})=$$

$$-\frac{\kappa_2\kappa_3\beta^2}{\alpha}-\kappa_3\beta(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha})-\frac{3\beta\kappa_3}{\alpha}\left[(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2-\frac{c}{4}\right]-\kappa_3\beta\,(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})$$

On the other hand (4.4), (4.5), (4.7) and (4.13) yield:

$$[U,\phi U](\frac{\gamma}{\alpha}-\frac{c}{4\alpha})=U\left(\phi U(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})\right)-\phi U\left(U(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})\right)=$$

$$\frac{3(U\beta)}{\alpha}\left[(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2-\frac{c}{4}\right]-\frac{3\beta(U\alpha)}{\alpha^2}\left[(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2-\frac{c}{4}\right]+\frac{6\kappa_3\beta^3}{\alpha^2}(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})-\frac{\beta^2}{\alpha}(\phi U(\kappa_3))$$

$$+\frac{2\kappa_3\beta}{\alpha}(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha})-\frac{2\kappa_3\beta\gamma}{\alpha}-\frac{\kappa_1\kappa_3\beta^3}{\alpha^2}-\frac{\kappa_3\beta^3}{\alpha}-\frac{3\kappa_3\beta^3\gamma}{\alpha^3}$$

$$+\frac{3\kappa_3 c\beta^3}{4\alpha^3}$$

The last equations using (4.1), (4.2) and (4.6) yield

$$\frac{3}{\alpha}\left[(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2-\frac{c}{4}\right](\xi\beta)-\frac{3\beta}{\alpha^2}\left[(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2-\frac{c}{4}\right](\xi\alpha)-\beta(\phi U\kappa_3)= \left[2c-\beta\kappa_2+\frac{\beta^2}{\alpha}\kappa_1-8(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2-\frac{5\beta^2}{\alpha}(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})\right]\kappa_3 \tag{4.14}$$

In a similar way, from the action of $[\xi,\phi U]$ on $\frac{\gamma}{\alpha}-\frac{c}{4\alpha}$ we obtain

$$\frac{3}{\alpha}[(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2-\frac{c}{4}](\xi\beta)-\frac{3\beta}{\alpha^2}[(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2-\frac{c}{4}](\xi\alpha)-\beta(\phi U\kappa_3)= [\gamma-(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2-\frac{6\beta^2}{\alpha}(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})]\kappa_3 \tag{4.15}$$

Comparing (4.14) with (4.15) and by making use of (4.3) we obtain

$$\kappa_3\left[(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2-\frac{c}{4}\right]=0$$

Let us assume there is a point p on $\mathcal{N}_1$ such that $\kappa_3\neq 0$. Then, because of the continuity of κ_3 there exists a neighborhood W(p) around p such that $\kappa_3\neq 0$. This fact and the last equation imply that $(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2=\frac{c}{4}$ on W(p). Differentiating the last equation along ξ and because of Lemma 4.2 we obtain $\xi(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})=0$. Combining the last equation with (4.6) we are led to $\kappa_3=0$, which is a contradiction. Therefore $W(p)$ is empty and $\kappa_3=0$ on $\mathcal{N}_1$. □

By virtue of (2.4) for $X = Z = \phi U$, $Y = \xi$ we obtain

$$R(\phi U, \xi)\phi U = -\gamma\xi - \beta(\frac{\gamma}{\alpha} - \frac{c}{4\alpha})U.$$

On the other hand, using Lemmas 3.1 and 4.3 we have

$$R(\phi U, \xi)\phi U = \nabla_{\phi U}\nabla_{\xi}\phi U - \nabla_{\xi}\nabla_{\phi U}\phi U - \nabla_{\nabla_{\phi U}\xi - \nabla_{\xi}\phi U}\phi U =$$

$$\Big[-(\phi U\kappa_1) + \beta(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) - \kappa_2(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) + \kappa_1\kappa_2 + \beta\kappa_1\Big]U +$$

$$\Big[-\kappa_1(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) - (\phi U\beta) - (\frac{\gamma}{\alpha} - \frac{c}{4\alpha})(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) + \kappa_1(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}) + \beta^2\Big]\xi.$$

Equalizing the above two expressions of $R(\phi U, \xi)\phi U$, we are led to

$$(\phi U\kappa_1) - 2\beta(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) + \kappa_2(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) - \kappa_1\kappa_2 - \kappa_1\beta = 0. \tag{4.16}$$

Using (3.11), (4.1), (4.2), (4.6), (4.7) and Lemmas 3.1, 4.1, 4.2, 4.3 we have

$$(U\alpha) = (\xi\beta) = 0, \qquad (U\beta) = -\frac{\beta^2}{\alpha^2}(\xi\alpha). \tag{4.17}$$

Since $U(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) = \xi(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) = 0$, due to Lemmas 4.1, 4.2 and (4.6), (4.7), the equality $[U, \xi](\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) = 0$ holds. However, the same Lie bracket is calculated from (3.2) and (3.3) as $[U, \xi](\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) = (\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha} - \kappa_1)\phi U(\frac{\gamma}{\alpha} - \frac{c}{4\alpha})$. So the two expressions of $U(\frac{\gamma}{\alpha} - \frac{c}{4\alpha})$ yield

$$\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha} - \kappa_1\right)\phi U(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) = 0. \tag{4.18}$$

Lemma 4.4. *Let M be a real hypersurface of a complex plane $M_2(c)$ satisfying (1.1). Then on $\mathcal{N}_1$ the relation $\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha} = \kappa_1$ holds.*

Proof. If there existed a point $p' \in \mathcal{N}_1$ such that $\frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha} \neq \kappa_1$ in a neighborhood W_1 of p', then (4.18) would give $\phi U(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) = 0$. Developing this equation with the aid of Lemmas 3.1, 4.2, 4.3 and relation (3.16), we result to

$$(\phi U\alpha) = 0. \tag{4.19}$$

(4.19) is combined with (4.4) and Lemma 4.1, giving

$$\kappa_1 = 3(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) - \alpha. \tag{4.20}$$

(4.20) is combined with (4.3), (4.13), (4.19), (3.16) and Lemmas 4.1, 4.3, giving

$$\kappa_2 = -\frac{1}{\beta}(\gamma - \frac{c}{4}) + 4(\frac{\gamma}{\alpha} - \frac{c}{4\alpha})\frac{\beta}{\alpha} - \beta. \tag{4.21}$$

So, replacing with (4.20), (4.21) in (4.16), and by making use of (3.16), (4.19), Lemma 4.3 we arrive to

$$(\beta^2 - \alpha^2)\left(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}\right) + \frac{c\alpha}{2} - \frac{2\beta^2 c}{\alpha} = 0.$$

Differentiating the above relation along ϕU (because of (3.17), (4.19), Lemma 4.3), it is proved

$$(\phi U\beta)\left[\left(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}\right)-\frac{2c}{\alpha}\right]=0. \tag{4.22}$$

Let $W_2 \subseteq W_1$ be the set of points $p \in W_1$ where $(\phi U\beta) \neq 0$ in a neighborhood around p. So, in W_2 (4.22) implies $(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})-\frac{2c}{\alpha}=0 \Rightarrow (\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2=\frac{2c^2}{\alpha^2}$. Combining this relation with (3.16), (4.13), (4.19) and Lemma 4.3 we obtain $\alpha^2=8c \Rightarrow (U\alpha)=(\xi\alpha)=0$. Therefore (4.17) gives $[U,\xi]\beta=U(\xi\beta)-\xi(U\beta)=0$. The same Lie bracket is also calculated from Lemma 3.1 as $[U,\xi]\beta=(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}-\kappa_1)(\phi U\beta)$ which means $(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}-\kappa_1)(\phi U\beta)=0$. Since $\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}-\kappa_1 \neq 0$ it follows that $(\phi U\beta)=0$ which is a contradiction, since we have assumed $(\phi U\beta)\neq 0$. This means that W_2 is empty and in W_1 we have $(\phi U\beta)=0$.

In this case (4.5) is combined with (4.20) giving

$$-\gamma+(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2-2\frac{\beta^2}{\alpha}(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})=0. \tag{4.23}$$

However from (3.16), (4.13), (4.19) and Lemma 4.3 we obtain $(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^2=\frac{c}{4}$ which is combined with (4.23) and Lemma 4.2, resulting to $\alpha^2+2\beta^2=0$ which is a contradiction. Therefore W_1 is empty and we conclude there exists no point $p' \in \mathcal{N}_1$ such that $\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}\neq\kappa_1$ in a neighborhood of p'. This means that $\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}-\kappa_1=0$ holds in $\mathcal{N}_1$. □

Lemma 4.5. *Let M be a real hypersurface of a complex plane $M_2(c)$ satisfying (1.1). Then $\mathcal{N}_1$ is empty.*

Proof. From Lemma 4.4 we have $\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha}=\kappa_1$. In this case, (4.3) and Lemma 4.1 yield

$$\kappa_2=-\frac{\gamma}{\beta}+\frac{1}{\beta}(\frac{\gamma}{\alpha}-\frac{c}{4\alpha}+\frac{\beta^2}{\alpha})^2. \tag{4.24}$$

Moreover, from (3.11), (3.16) and Lemma 4.3, we have $[\phi U,U]\gamma=(\phi U(U\gamma))-(U(\phi U\gamma))=0$. The same Lie bracket is calculated from Lemma 3.1 as $[\phi U,U]\gamma=\left[2(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})+\frac{\beta^2}{\alpha}\right](\xi\gamma)$. The previous two relations yield

$$\left[2(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})+\frac{\beta^2}{\alpha}\right](\xi\gamma)=0.$$

If there was point in $\mathcal{N}_1$ such that $(\xi\gamma)\neq 0$ then from the above equation it would be $2(\gamma-\frac{c}{4})+\beta^2=0$. Differentiation of this equation along ξ, due to (4.6), (4.17) Lemmas 4.2, 4.3, would lead to $(\xi\gamma)=0$, which is a contradiction.

Therefore it must be $(\xi\gamma)=0$. So, from (4.6), (4.7), and Lemma 4.3 we obtain

$$(U\alpha)=(U\beta)=(\xi\alpha)=(\xi\beta)=0. \tag{4.25}$$

In addition, (3.16) with (4.13) and Lemma 4.3 give

$$(\phi U\alpha)=-3\beta\Big[(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})-\frac{c}{4}(\frac{\gamma}{\alpha}-\frac{c}{4\alpha})^{-1}\Big] \tag{4.26}$$

Also $\kappa_1 = \frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}$, (4.5) and Lemma 4.1 yield

$$(\phi U\beta) = \gamma - (\frac{\gamma}{\alpha} - \frac{c}{4\alpha})^2 + \frac{\beta^4}{\alpha^2} + \beta^2. \tag{4.27}$$

By virtue of (4.4), (4.26) $\kappa_1 = \frac{\gamma}{\alpha} - \frac{c}{4\alpha} + \frac{\beta^2}{\alpha}$ and Lemma 4.1 we get

$$(\frac{\gamma}{\alpha} - \frac{c}{4\alpha})^2 + \frac{\beta^2}{\alpha}(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) + \gamma = c. \tag{4.28}$$

The differentiation of (4.28) along ϕU, in combination with Lemmas 3.1, 4.3 and (3.16), (4.13), (4.26), (4.27), leads to

$$4(\frac{\gamma}{\alpha} - \frac{c}{4\alpha})^2(\frac{\gamma}{\alpha} - \frac{c}{4\alpha}) + (\frac{\gamma}{\alpha} - \frac{c}{4\alpha})(-\frac{3c}{2} + 2\gamma + \frac{2\beta^4}{\alpha^2} + 2\beta^2) +$$

$$\frac{6\beta^2}{\alpha}(\frac{\gamma}{\alpha} - \frac{c}{4\alpha})^2 - \frac{3\beta^2 c}{2\alpha} = 0.$$

In the above equation, the term $(\frac{\gamma}{\alpha} - \frac{c}{4\alpha})^2$ is replaced from (4.28) and we obtain

$$-\frac{4\beta^2}{\alpha}(\frac{\gamma}{\alpha} - \frac{c}{4\alpha})^2 + (\frac{\gamma}{\alpha} - \frac{c}{4\alpha})(\frac{5c}{2} - 2\gamma - \frac{4\beta^4}{\alpha^2} + 2\beta^2) \tag{4.29}$$

$$-\frac{6\beta^2\gamma}{\alpha} + \frac{9\beta^2 c}{2\alpha} = 0.$$

In equation (4.29) the term $(\frac{\gamma}{\alpha} - \frac{c}{4\alpha})^2$ is replaced from (4.28) giving

$$\gamma = \frac{5c}{4}. \tag{4.30}$$

Now, (4.28) and (4.30) result to

$$\alpha^2 + 4\beta^2 = -4c, \Rightarrow c < 0. \tag{4.31}$$

So by virtue of (4.30) and (4.31), equations (4.24), (4.26) and (4.27) are written respectively as

$$\kappa_2 = -\frac{\beta}{4} - \frac{3c}{2\beta}, \qquad (\phi U\alpha) = \frac{3\alpha\beta}{4} - \frac{3\beta c}{\alpha}, \qquad (\phi U\beta) = \frac{3c}{2} + \frac{3\beta^2}{4}. \tag{4.32}$$

The third of (4.32) gives

$$(\phi U\beta) - \frac{3c}{2} = \frac{3\beta^2}{4} > 0 \Rightarrow (\phi U\beta) > \frac{3c}{2}.$$

By virtue of the second of (4.32), (4.31) and $(\phi U\beta) > \frac{3c}{2}$, equation (4.31) is differentiated along ϕU giving:

$$0 = \alpha(\phi U\alpha) + 4\beta(\phi U\beta) > \alpha(\phi U\alpha) + 6\beta c = -3\beta^3 \Rightarrow$$

$$\beta > 0.$$

Since $\beta > 0$ and $c < 0$ (due to (4.31)), equation (4.31) is rewritten as

$$\beta^2 + \beta c + c = -\frac{\alpha^2}{4} + \beta c < 0. \tag{4.33}$$

From (4.33), we observe that $f(\beta) = \beta^2 + \beta c + c$ is always negative for every β. However the discriminant of $f(\beta)$ is $c^2 - 4c > 0$, due to (4.31), which is a contradiction. Therefore the set $\mathcal{N}_1$ is empty and the lemma is proved. $\square$

Lemma 4.6. *Let M be a real hypersurface of a complex plane $M_2(c)$ satisfying (1.1). Then, $\mathcal{N} = \varnothing$.*

Proof. From Lemma 4.5 we have $\alpha = 0$ in $\mathcal{N}$. Then (2.4), combined with (2.6), yields

$$lX = \frac{c}{4}[X - \eta(X)\xi] - g(X,U)\beta^2 U, \quad lU = (\frac{c}{4} - \beta^2)U, \quad l\phi U = \frac{c}{4}\phi U. \tag{4.34}$$

Condition (1.1) for $X = Y = U$ yields $(\nabla_U l)U = \kappa\{g(\phi AU, U)\xi\}$, which is further analyzed with the aid of (4.34) and $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, giving

$$-2\beta(U\beta)U - \beta^2\nabla_U U + (\frac{c}{4} + \kappa)g(AU, \phi U)\xi = 0. \tag{4.35}$$

The inner products of (4.35) with U, ϕU and ξ (using also the rule $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ and (2.6)) imply respectively

$$(U\beta) = 0, \qquad g(\nabla_U U, \phi U) = 0, \qquad (\frac{c}{4} + \kappa - \beta^2)g(AU, \phi U) = 0. \tag{4.36}$$

Similarly, putting $X = \phi U$, $Y = U$ in (1.1) we obtain $(\nabla_{\phi U} l)U = \kappa\{g(\phi A\phi U, U)\xi\}$, which is further analyzed with the aid of (4.34) and $Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, giving

$$-2\beta(\phi U\beta)U - \beta^2\nabla_{\phi U} U + (\frac{c}{4} + \kappa)g(A\phi U, \phi U)\xi = 0. \tag{4.37}$$

The inner products of the (4.37) with ϕU and U result respectively to

$$g(\nabla_{\phi U} U, \phi U) = 0, \qquad (\phi U\beta) = 0. \tag{4.38}$$

Finally, putting $X = Y = \phi U$ in (1.1) we obtain $(\nabla_{\phi U} l)\phi U = \kappa\{g(\phi A\phi U, \phi U)\xi\}$, which is further analyzed with the aid of (4.34) and (4.38), giving $(\frac{c}{4} + \kappa)g(AU, \phi U) = 0$. Combining the last relation with (4.36) we have $g(AU, \phi U) = g(U, A\phi U) = 0$. This equality using $\beta = g(A\xi, U) = g(AU, \xi)$, $g(A\phi U, \xi) = g(\phi U, A\xi) = 0$, leads to the following decompositions:

$$AU = \lambda U + \beta\xi, \qquad A\phi U = \mu\phi U, \tag{4.39}$$

where $\mu = g(A\phi U, \phi U)$. (2.3), (2.6), (4.36) and (4.39) are used to develop $(\nabla_U A)\xi - (\nabla_\xi A)U = -\frac{c}{4}\phi U$-which holds due to (2.5). Therefore after the development we end up to:

$$\beta\nabla_U U - \lambda\mu\phi U - (\xi\lambda)U - \lambda\nabla_\xi U - (\xi\beta)\xi - \beta^2\phi U + A\nabla_\xi U = -\frac{c}{4}\phi U.$$

The inner product of the above relation with ϕU, combined with (2.6), (4.34), (4.36) and (4.39) results to

$$-\lambda\mu + (\mu - \lambda)g(\nabla_\xi U, \phi U) - \beta^2 + \frac{c}{4} = 0. \tag{4.40}$$

In a similar way, the relation $(\nabla_{\phi U}A)\xi - (\nabla_\xi A)\phi U = \frac{c}{4}U$ is analyzed with the aid of (4.38), (4.39), giving

$$\beta\nabla_{\phi U}U + \beta\mu\xi + \lambda\mu U - (\xi\mu)\phi U - \mu\nabla_\xi\phi U + A\nabla_\xi\phi U = \frac{c}{4}U, \tag{4.41}$$

whose inner product with ξ because of (2.3) and (2.6) yields

$$g(\nabla_\xi U, \phi U) = 3\mu. \tag{4.42}$$

Replacing with (4.42) in (4.40) we obtain

$$3\mu^2 - 4\lambda\mu - \beta^2 + \frac{c}{4} = 0. \tag{4.43}$$

On the other hand, the inner product of (4.41) with U, because of (4.42), leads to

$$3\mu^2 - 2\lambda\mu - \beta^2 - \frac{c}{4} = 0.$$

So, the above relation and (4.43) give

$$\lambda\mu = \frac{c}{4}, \qquad \lambda, \mu \neq 0. \tag{4.44}$$

Finally, relation $(\nabla_U A)\phi U - (\nabla_{\phi U}A)U = -\frac{c}{2}\xi$ is developed by virtue of (4.38) and (4.39) giving

$$(U\mu)\phi U + \mu\nabla_U\phi U - A\nabla_U\phi U - (\phi U\lambda)U - \lambda\nabla_{\phi U}U +$$
$$\beta\mu U + A\nabla_{\phi U}U = -\frac{c}{2}\xi.$$

The inner product of the above equation with U, because of (4.36), (4.38), (4.39) yields

$$\lambda + 2\mu - (\phi U\lambda) = 0.$$

However, (4.43) and (4.44) yield $3\mu^2 - \frac{3c}{4} - \beta^2 = 0$ which is differentiated along ϕU (see also (4.38), (4.44)) giving $(\phi U\mu) = 0$. Relation $(\phi U\mu) = 0$ and (4.44) give $(\phi U\lambda) = 0$. Combining the last relation with $\lambda + 2\mu - (\phi U\lambda) = 0$ we get

$$\lambda + 2\mu = 0.$$

From the above equation and (4.44) we obtain

$$\mu^2 = -\frac{c}{8} \tag{4.45}$$

On the other hand, condition (1.1) for $X = U$, $Y = \xi$, with $l\xi = 0$, (2.1) and (2.3) infer $-l\phi AU = \kappa\phi AU$. Analyzing this equation with the aid of (4.34) we are led to

$$\frac{c}{4} + \kappa = 0.$$

The above relation, (4.37) and (4.38) yield

$$\nabla_{\phi U}U = 0 \Rightarrow g(\nabla_{\phi U}U, \xi) = 0 \Rightarrow g(\nabla_{\phi U}\xi, U) = 0 \Rightarrow g(U, \phi A\phi U) = 0,$$

which by virtue of (4.39) yields $\mu = 0$, a contradiction due to (4.45). Therefore the set $\mathcal{N}$ is empty. □

5 Proof of main theorem

From Lemma 4.6 in the hypersurface M, we have $\beta = 0$. Therefore M is Hopf i.e. $A\xi = \alpha\xi$. According to [9] the function α must be constant.

Let H_1 be the set of points $p \in M$ such that $A\xi = \alpha\xi$, $(\alpha \neq 0)$ in a neighborhood around p, and H_2 be the set of points $q \in M$ such that $A\xi = 0$, in a neighborhood around q. Then $H_1 \cup H_2$ is open and dense in the closure of M.

At every point of H_1 there exists a ϕ-basis $\{e, \phi e, \xi\}$ such that, the vector fields Ae, $A\phi e$ are decomposed as follows:

$$Ae = \lambda_1 e, \qquad A\phi e = \lambda_2 \phi e, \qquad A\xi = \alpha\xi, \tag{5.1}$$

where λ_1, λ_2 are functions. Also equation (2.4) gives

$$lX = \frac{c}{4}[X - \eta(X)\xi] + \alpha AX - \alpha^2\eta(X)\xi, \tag{5.2}$$

$$le = \frac{c}{4}e + \alpha Ae, \qquad l\phi e = \frac{c}{4}\phi e + \alpha A\phi e.$$

By making use of (2.5) for $X = e$, $Y = \phi e$ we obtain $\nabla_e A\phi e - A\nabla_e \phi e - \nabla_{\phi e}Ae + A\nabla_{\phi e}e = -\frac{c}{2}\xi$, whose inner product with ξ (combined with (5.1), (2.3) and (3.9)) results to

$$\alpha(\lambda_1 + \lambda_2) - 2\lambda_1\lambda_2 = -\frac{c}{2}. \tag{5.3}$$

Similarly, (2.5) for $X = e$, $Y = \xi$ yields $\nabla_e A\xi - A\nabla_e\xi - \nabla_e A\xi + A\nabla_e\xi = -\frac{c}{4}\phi e$, whose inner product with ϕe (combined with (5.1), (2.3) and (3.9)) results to

$$\alpha\lambda_1 - \lambda_1\lambda_2 - (\lambda_1 - \lambda_2)g(\nabla_\xi e, \phi e) = -\frac{c}{4}. \tag{5.4}$$

Finally, (2.5) for $X = \phi e$, $Y = \xi$ yields $\nabla_{\phi e}A\xi - A\nabla_{\phi e}\xi - \nabla_{\phi e}A\xi + A\nabla_{\phi e}\xi = -\frac{c}{4}\phi e$, whose inner product with e (combined with (5.1), (2.3) and (3.9)) results to

$$\alpha\lambda_2 - \lambda_1\lambda_2 - (\lambda_1 - \lambda_2)g(\nabla_\xi e, \phi e) = \frac{c}{4}. \tag{5.5}$$

Combining (5.4) and (5.5) we obtain $\alpha(\lambda_1 - \lambda_2) = -\frac{c}{2}$. The last equation and (5.3) result to

$$\lambda_2(\lambda_1 - \alpha) = 0. \tag{5.6}$$

Let $H_1' \subseteq H_1$ be the set of points $p' \in H_1$ such that $\lambda_1 - \alpha \neq 0$ in a neighborhood around p'. Therefore $\lambda_2 = 0$ and from (5.1) and (5.3) there exist 3 constant principal curvatures: α, $-\frac{c}{2\alpha}$ and 0.

• $\mathbb{C}P^2$. According to Takagi [14] (see also [9]), the only possible three-dimensional hypersurface with three constant distinct principal curvatures is type B, where $\alpha = 2cotr$ and the other eigenvalues are $cot(r-\frac{\pi}{4})$ and $-tan(r-\frac{\pi}{4})$. Therefore it must be $cot(r-\frac{\pi}{4}) = 0$, $-tan(r-\frac{\pi}{4}) = -\frac{c}{2\alpha}$ or $cot(r-\frac{\pi}{4}) = -\frac{c}{2\alpha}$, $-tan(r-\frac{\pi}{4}) = 0$, which both lead to contradictions.

• $\mathbb{C}H^2$. Based on the list of eigenvalues ([1], [8], [9]), the only way to have zero as an eigenvalue is to have a tube of radius $r = 0$ which is impossible ($r > 0$).

Therefore in both $\mathbb{C}P^2$ and $\mathbb{C}H^2$ we have a contradiction and $H_1' = \varnothing$.

We have proved that in H_1, $\alpha = \lambda_1$ holds. So, due to (5.3) we have two constant distinct principal curvatures: α of multiplicity 2 and $\lambda_2 = \frac{c}{2\alpha} + \alpha$ of multiplicity 1. Based on [8], [13] this can only happen when M is a real hypersurface of type (B) in CH^2, that is a tube of radius $r = \frac{1}{\sqrt{|c|}} ln(2+\sqrt{3})$ around totally real geodesic $RH^n(\frac{c}{4})$. At every point of H_2, there exists a ϕ-basis $\{e, \phi e, \xi\}$ too, such that, the vector fields Ae, $A\phi e$ are decomposed as following:

$$Ae = \mu_1 e, \qquad A\phi e = \mu_2 \phi e, \qquad A\xi = 0, \tag{5.7}$$

where λ_1, λ_2 are functions. Also equation (2.4) gives

$$lX = \frac{c}{4}[X - \eta(X)\xi], \qquad le = \frac{c}{4}e, \qquad l\phi e = \frac{c}{4}\phi e. \tag{5.8}$$

By virtue of (3.9) it is shown that $\nabla_\xi e \perp \{\xi, \phi e\}$. Therefore we have

$$\nabla_\xi e = n_1 \phi e, \qquad n_1 = g(\nabla_\xi e, \phi e).$$

In a similar way, from (3.9) and (2.3) it is proved that $\nabla_e e \perp \{\xi, e\}$, $\nabla_{\phi e} e \perp e$, $g(\nabla_{\phi e} e, \xi) = \mu_2$.

So we have the following covariant derivatives:

$$\nabla_\xi e = n_1 \phi e, \qquad \nabla_e e = n_2 \phi e, \qquad \nabla_{\phi e} e = n_3 \phi e + \mu_2 \xi, \tag{5.9}$$

where n_1, n_2, n_3 are functions on H_2.

Using the above derivatives and the second of (2.3) we also have

$$\nabla_\xi \phi e = -n_1 e, \qquad \nabla_e \phi e = -n_2 e - \mu_1 \xi, \qquad \nabla_{\phi e} \phi e = -n_3 e. \tag{5.10}$$

Using condition (1.1) for $X = e$, $Y = \phi e$ and $X = \phi e$, $Y = e$, and by virtue of (5.8), (5.9), (5.10), we obtain respectively

$$(\frac{c}{4} + \kappa)\mu_1 = 0, \qquad (\frac{c}{4} + \kappa)\mu_2 = 0.$$

From the above relations we conclude that $\kappa = -\frac{c}{4}$, otherwise we would have $\mu_1 = \mu_2 = 0$ which is a contradiction.

Equation (2.5) for $X = e$, $Y = \phi e$ yields $(\nabla_e A)\phi e - (\nabla_{\phi e} A)e = -\frac{c}{2}\xi$. The last relation is further analyzed by virtue of (5.7), (5.9) and (5.10) giving

$$(e\mu_2) = n_3(\mu_1 - \mu_2), \quad (\phi e \mu_1) = n_2(\mu_1 - \mu_2), \quad \mu_1\mu_2 = \frac{c}{4}. \tag{5.11}$$

In a similar way, from (2.5) we take $(\nabla_e A)\xi - (\nabla_\xi A)e = -\frac{c}{4}\phi e$, which is further developed with the aid of (5.7), (5.9) and (5.10), giving

$$(\xi\mu_1) = 0 \qquad n_1(\mu_1 - \mu_2) = 0. \tag{5.12}$$

Again from (2.5) we have $(\nabla_{\phi e} A)\xi - (\nabla_\xi A)\phi e = \frac{c}{4}e$, which yields

$$(\xi\mu_2) = 0. \tag{5.13}$$

Next we make use of (2.4) for $X = Z = e$, $Y = \xi$ and obtain $R(e,\xi)e = -\frac{c}{4}e$. On the other hand it is $R(e,\xi)e = \nabla_e\nabla_\xi e - \nabla_\xi\nabla_e e - \nabla_{[e,\xi]}e$. So , equalizing the two expressions of $R(e,\xi)e$ we get

$$\nabla_e\nabla_\xi e - \nabla_\xi\nabla_e e - \nabla_{[e,\xi]}e = -\frac{c}{4}e.$$

The last equation is developed with the aid of (2.3), (5.7), (5.9), (5.10), resulting to

$$(en_1) - (\xi n_2) = (\mu_1 - n_1)n_3. \tag{5.14}$$

Similarly, the calculation of $R(\phi e,\xi)e$ yields

$$\nabla_{\phi e}\nabla_\xi e - \nabla_\xi\nabla_{\phi e} e - \nabla_{[\phi e,\xi]}e = 0.$$

The above relation yields

$$(\phi e n_1) - (\xi n_3) = (n_1 - \mu_2)n_2. \tag{5.15}$$

Finally, (2.4) gives $R(e,\phi e)e = -(c + \mu_1\mu_2)\phi e$ which which eventually yields

$$(en_3) - (\phi e n_2) + n_2^2 + n_3^2 + n_1(\mu_1 + \mu_2) = -(c + \mu_1\mu_2). \tag{5.16}$$

We are going to distinguish two cases: $\mu_1 = \mu_2$ and $\mu_1 \neq \mu_2$.

If $\mu_1 = \mu_2$ then from (5.11) and (5.12)-or (5.13)-we have two distinct constant principal curvatures $\alpha = 0$ and $\mu_1 = \mu_2 = \frac{\sqrt{c}}{2}$, $c > 0$. Based on [13] M is a geodesic hypersphere of radius $r = \frac{\pi}{4}$.

If $\mu_1 \neq \mu_2$ then (5.12) implies $n_1 = 0$. If at least one of μ_1, μ_2 was constant, then (5.11) and (5.14) would give $n_2 = n_3 = 0$. Then the last relation combined with (5.6) and the third of (5.11) would result to $c = 0$ which is a contradiction. This means that the functions μ_1, μ_2 must not be constant. □

Remark. A hypersurface of type (B) mentioned in the main theorem, can be considered of many points of view. Based on [8] we can classify them with respect to its principal foliations and geodesics. In addition, we can find necessary and sufficient conditions on real hypersurfaces satisfying $A\xi = \alpha\xi$, in [4], [5], [6].

Acknowledgements. The author would like to thank Professor Young Jin Suh for his email response. Also Professor Philippos J. Xenos for his careful reading of the manuscript and suggestions to develop this article. Finally, the author expresses his sincere gratitude to the referee and the editor, for examining the article and making valuable comments and corrections that led to the article's final form.

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