A basic algebraic proof of a theorem of Milnor

Z. Chen and M. Liu

Abstract. Milnor has classified the flat metrics on Lie groups. In this paper, we give a basis algebraic proof of Milnor's classification.

M.S.C. 2010: 17B60, 53C30, 81R05.
Key words: Left invariant metric; Flat metric; Affine structure; Left-symmetric algebra.

1 Introduction

In [9], Milnor studied the curvatures of left invariant metrics on Lie groups. For a Lie group G with a left invariant Riemannian metric g, Milnor considered the left invariant Riemannian connection ∇ associated to g, i.e., ∇ satisfies the "symmetry" condition (torsion free)

$$\nabla_x y - \nabla_y x = [x, y], \;\; \forall\, x, y \in \mathfrak{g}, \tag{1.1}$$

and the compatibility condition (the parallel translation perseveres the metric g ([10]))

$$g(\nabla_x y, z) + g(y, \nabla_x z) = 0, \;\; \forall x, y \in \mathfrak{g}, \tag{1.2}$$

and in addition, the flatness of the connection ∇ or the metric g corresponds to

$$R_{xy}z = \nabla_{[x,y]}z - \nabla_x\nabla_y z + \nabla_y\nabla_x z = 0, \;\; \forall x, y, z \in \mathfrak{g}, \tag{1.3}$$

that is, the Riemannian curvature tensor R is zero. In particular, Milnor gave the classification of the flat metrics (see Theorem 1.5 in [9]).

On the other hand, one can try to find a left invariant Riemannian metric g associated to a left invariant connection ∇ satisfying Eqs. (1.1) and (1.3) such that the compatibility (1.2) holds. In fact, the manifolds (not necessarily Riemannian) or Lie groups with a connection ∇ satisfying Eqs. (1.1) and (1.3) have already been studied independently. Such structures are called affine manifolds or the affine structures on Lie groups ([2, 6, 7, 8] etc.). Like most of the geometric structures on Lie groups, the study of left invariant affine structures on a Lie group can be given through the corresponding structures on its Lie algebra. The left-invariant affine structures on a Lie group G bijectively correspond to the left-symmetric algebra structures on the Lie algebra $\mathfrak{g}$ of G and the correspondence is given by

$$\nabla_x y = xy, \;\; \forall x, y \in \mathfrak{g}. \tag{1.4}$$

Differential Geometry - Dynamical Systems, Vol.16, 2014, pp. 92-97.

Left-symmetric algebras also appear in many other fields in mathematics and mathematical physics (see [4] and the references therein). Thus it is natural to study the left invariant Riemannian metrics on a Lie group G by considering the left invariant affine structures on G with a compatible Riemannian metric in terms of their corresponding left-symmetric algebras. By this view, there is a new proof of Milnor's classification theorem in [5].

But the proof of proposition 3.4 in [5] (i.e. Theorem 3.5 in this paper) is based on the study on affine representations in [3]. In this paper, we give a basic algebraic proof of Theorem 3.5, hence a basic algebraic proof of Milnor's classification theorem.

2 Preliminaries on left-symmetric algebras

For self-contained, we recall some basic facts on left-symmetric algebras (cf. [6, 8], etc.).

Definition 2.1. Let A be a vector space over a field $\mathbb{F}$ with a bilinear product $(x, y) \mapsto xy$. Then A is called a left-symmetric algebra if for any $x, y, z \in A$, the associator

$$(x, y, z) = (xy)z - x(yz) \tag{2.1}$$

is symmetric in x, y, that is,

$$(x, y, z) = (y, x, z), \quad \text{or equivalently} \quad (xy)z - x(yz) = (yx)z - y(xz). \tag{2.2}$$

Let A be a left-symmetric algebra. Then A is called *trivial* if all products are zero. For any $x, y \in A$, let $L(x)$ and $R(x)$ denote the left and right multiplication operator respectively, that is, $L(x)(y) = xy,\ R(x)(y) = yx$.

Proposition 2.1. *Let A be a left-symmetric algebra.*

(1) *The commutator*

$$[x, y] = xy - yx, \quad \forall x, y \in A, \tag{2.3}$$

defines a Lie algebra $\mathfrak{g}(A)$, which is called the sub-adjacent Lie algebra of A and A is also called the compatible left-symmetric algebraic structure on the Lie algebra $\mathfrak{g}(A)$.

(2) *Let $L : A \to gl(A)$ be a linear map defined by $x \mapsto L(x)$ (for every $x \in A$). Then L gives a regular representation of the Lie algebra $\mathfrak{g}(A)$, that is,*

$$[L(x), L(y)] = L([x, y]), \quad \forall x, y \in A. \tag{2.4}$$

Proposition 2.2. *If a real or complex Lie algebra $\mathfrak{g}$ has a compatible left-symmetric algebra structure, then $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$.*

Definition 2.2. Let A be a left-symmetric algebra. A bilinear form $f : A \times A \to \mathbb{F}$ is called left invariant if f satisfies

$$f(xy, z) + f(y, xz) = 0, \quad \forall x, y, z \in A. \tag{2.5}$$

3 A basic algebraic proof of a theorem of Milnor

In this section, let A be a real left-symmetric algebra and f be a positive definitive symmetric left invariant bilinear form on A. For any subspace V in A, set

$$V^{\perp} = \{x \in A \mid f(x, y) = 0, \ \forall\, y \in V\}. \tag{3.1}$$

Recall that the adjoint F^* of a linear transformation F on A with f is defined by the formula

$$f(F(x), y) = f(x, F^*(y)), \ \ \forall\, x, y \in A. \tag{3.2}$$

The transformation F is self-adjoint if $F^* = F$ and skew-adjoint if $F^* = -F$. So Eq. (2.5) is equivalent to the fact that $L(x)$ is skew-adjoint for any $x \in A$. For any Lie algebra $\mathfrak{g}$, $\mathrm{ad}x$ is the linear transformation given by $\mathrm{ad}x(y) = [x, y]$ for any $x, y \in \mathfrak{g}$.

Lemma 3.1 ([5]). $[A, A]^{\perp} = \{x \in A \mid R(x) = R(x)^*\}$.

Proof. For any $x, y, z \in A$, $x \in [A, A]^{\perp}$ if and only if $f(x, [y, z]) = 0$, if and only if $f(x, yz) - f(x, zy) = 0$, if and only if $f(R(x)y, z) = f(R(x)z, y)$, if and only if $R(x) = R(x)^*$. □

Lemma 3.2 ([5]). $[A, A]^{\perp} = \{x \in A \mid R(x) = 0\}$.

Proof. For any $x \in [A, A]^{\perp}$ and $y \in A$, by Lemma 3.1, we know that

$$f(xx, y) = f(R(x)x, y) = f(x, R(x)y) = f(x, yx) = 0.$$

Hence $xx = 0$ due to the positive definiteness of f. For any $x \in [A, A]^{\perp}$, $R(x)$ is diagonalizable over the real number field $\mathbb{R}$ since it is self-adjoint. Let $\lambda \in \mathbb{R}$ be an arbitrary eigenvalue of $R(x)$ and $y \in A$ be a non-zero eigenvector associated to λ. Since $(xy)x - x(yx) = (yx)x - y(xx)$, we know that $(xy)x - \lambda xy = \lambda^2 y$. Hence

$$\lambda^2 f(y, y) = f((xy)x - \lambda xy, y) = f((xy)x, y) = f(xy, yx) = \lambda f(xy, y) = 0.$$

Therefore $\lambda = 0$ and then $R(x) = 0$. □

Lemma 3.3 ([5]). $AA = [A, A]$.

Proof. In fact, $x \in (AA)^{\perp}$ if and only if $f(x, yz) = 0, \ \ \forall y, z \in A$, if and only if $f(yx, z) = 0, \ \forall y, z \in A$, if and only if $R(x)y = 0, \ \forall y \in A$, if and only if $R(x) = 0$. Then by Lemma 3.2, we know that $AA = [A, A]$. □

Lemma 3.4 ([5]). *Let H be a trivial subalgebra of A and let V be a subspace of A such that $L(x)V \subseteq V$ for any $x \in H$. Then $\{L(x)|_V\}_{x \in H}$ is a family of commutative linear transformations on V.*

Proof. For any $x, y \in H, z \in V$,

$$L(x)(L(y)z) = x(yz) = x(yz) - (xy)z = y(xz) - (yx)z = y(xz) = L(y)(L(x)z)$$

since H is trivial. Therefore $L(x)|_V L(y)|_V = L(y)|_V L(x)|_V$. □

For any subalgebra V in A, we let $C_R(V) = \{x \in V \mid R(x)|_V = 0\}$. In particular, $C_R(A) = [A, A]^\perp$ due to Lemma 3.2.

Theorem 3.5. *As left-symmetric algebras, $[A, A]$ is a proper ideal of A with trivial products.*

Proof. Obviously, AA is an ideal of the left-symmetric algebra A. Then $[A, A]$ is an ideal of A by Lemma 3.3. Moreover, A splits as an orthogonal direct sum

$$A = [A, A]^\perp \oplus [A, A] = C_R(A) \oplus [A, A].$$

By Proposition 2.3, we know that $[A, A] \neq A$. Hence $C_R(A) \neq 0$.

In order to prove that $[A, A]$ is trivial, it is enough to show that $C = [[A, A], [A, A]]$ is not trivial if $[A, A]$ is not trivial. In fact, if $[A, A]$ is not trivial, then both $B = C_R([A, A])$ and C are not zero and there is an orthogonal direct sum $[A, A] = B \oplus C$. Moreover, $C = [[A, A], [A, A]] = (AA)(AA)$ is an ideal of A since C is an ideal of $[A, A]$ and for any $x \in C_R(A), y, z \in AA$, we have

$$x(yz) = (xy)z - (yx)z + y(xz) = (xy)z + y(xz) \in C.$$

If C is still not trivial, then by induction, there would be an infinite series of non-zero ideals $\{A_n\}_{n\in\mathbb{Z}^+}$ of A such that

$$\dim A_1(= [A, A]) > \dim A_2(= C) > \dim A_3 > \cdots > \dim A_n > \dim A_{n+1} > \cdots,$$

which is obviously impossible since A is a finite-dimensional vector space.

Now assume that $[A, A]$ is not trivial and C is trivial. For any $x \in C_R(A), y \in B, z \in C$,

$$f(xy, z) = -f(y, xz) = 0.$$

Hence B is an invariant space of $L(x)$ for any $x \in C_R(A)$. By Lemma 3.4, we know that $L(x)|_B L(y)|_B = L(y)|_B L(x)|_B$ for any $x, y \in C_R(A)$. Obviously, there does not exist a non-zero $y \in B$ such that $L(x)y = 0$ for any $x \in C_R(A)$. Then there exists a basis $\{u_1, \cdots, u_{2n}\}$ of B (hence $\dim B$ is even) such that $f(u_i, u_j) = \delta_{ij}$ and for any $x \in C_R(A)$,

$$L(x)u_{2i-1} = -\alpha_i(x)u_{2i}, L(x)u_{2i} = \alpha_i(x)u_{2i-1}, \quad i = 1, \cdots, n,$$

and for any i $(1 \leq i \leq n)$, there exists an element $x \in C_R(A)$ such that $\alpha_i(x) \neq 0$.

Furthermore, by the assumption that C is trivial, there exists an element $u_k \in B$ $(1 \leq k \leq 2n)$ such that $L(u_k)|_C \neq 0$. If $k = 2l$, then there exists an element $x \in C_R(A)$ such that $L(x)u_k = \alpha_l(x)u_{k-1}$ and $\alpha_l(x) \neq 0$. Let $e_1 = -\frac{x}{\alpha_l(x)}, e_2 = u_k, e_3 = u_{k-1}$, then $\{e_1, e_2, e_3\}$ satisfies

$$f(e_2, e_2) = f(e_3, e_3) = 1, f(e_2, e_3) = 0, \quad e_1e_2 = -e_3, e_1e_3 = e_2, \quad L(e_2)|_C \neq 0.$$

If $k = 2l-1$, then there exists an element $x \in C_R(A)$ such that $L(x)u_k = -\alpha_l(x)u_{k+1}$ and $\alpha_l(x) \neq 0$. Let $e_1 = \frac{x}{\alpha_l(x)}, e_2 = u_k, e_3 = u_{k-1}$, then $\{e_1, e_2, e_3\}$ still satisfies

$$f(e_2, e_2) = f(e_3, e_3) = 1, f(e_2, e_3) = 0, \quad e_1e_2 = -e_3, e_1e_3 = e_2, \quad L(e_2)|_C \neq 0.$$

By Lemma 3.4 again, $L(x)|_C L(y)|_C = L(y)|_C L(x)|_C$ for any $x, y \in B$. Similarly, there also exists a basis $\{v_1, \cdots, v_{2m}\}$ of C such that $f(v_i, v_j) = \delta_{ij}$ and for any $x \in B$,

$$L(x)v_{2i-1} = -\beta_i(x)v_{2i}, \;\; L(x)v_{2i} = \beta_i(x)v_{2i-1}, \;\; i = 1, \cdots, m.$$

and for any i $(1 \leq i \leq m)$, there exists an element $x \in B$ such that $\beta_i(x) \neq 0$. Since $L(e_2)|_C \neq 0$, there exists j such that $\beta_j(e_2) \neq 0$. Because C is an ideal of A, we set

$$e_1 v_{2j-1} = \sum_{i=1}^{2m} \lambda_i v_i, \;\; e_1 v_{2j} = \sum_{i=1}^{2m} \mu_i v_i.$$

Since $f(e_1 v_{2j-1}, v_{2j-1}) = f(e_1 v_{2j}, v_{2j}) = 0$ and $f(e_1 v_{2j-1}, v_{2j}) + f(e_1 v_{2j}, v_{2j-1}) = 0$, we have

$$\lambda_{2j-1} = \mu_{2j} = 0, \;\; \lambda_{2j} + \mu_{2j-1} = 0.$$

Set $e_1' = e_1 + \frac{\lambda_{2j}}{\beta_j(e_2)} e_2$, then

$$e_1' v_{2j-1} = \sum_{i \neq 2j-1, 2j} \lambda_i v_i, \;\; e_1' v_{2j} = \sum_{i \neq 2j-1, 2j} \mu_i v_i.$$

Let H be the vector space linearly spanned by $\{v_1, \cdots, v_{2j-2}, v_{2j+1}, \cdots, v_{2m}\}$. Then $e_1' v_{2j-1} \in H, e_1' v_{2j} \in H$. Hence

$$-\beta_j(e_2)v_{2j} = (e_1' e_3)v_{2j-1} = e_1'(e_3 v_{2j-1}) + (e_3 e_1')v_{2j-1} - e_3(e_1' v_{2j-1}) \in H,$$

which is a contradiction. □

It is clear that $\dim[A, A]$ is even from the proof of Theorem 3.5. From the above discussion, we have the following structure theory.

Theorem 3.6 ([5]). *A left-symmetric algebra A has a positive definitive symmetric left invariant bilinear form if and only if A splits as an orthogonal direct sum $A = [A, A] \oplus C_R(A)$, where $C_R(A)$ is a non-zero trivial subalgebra, $[A, A]$ is a trivial ideal with even dimension, and where the linear transformations $R(x) = 0$ and $L(x)$ is skew-adjoint for any $x \in C_R(A)$.*

Let $\mathrm{Ann}(A) = \{x \in A | xy = yx = 0, \; \forall y \in A\}$ be the annihilator of A. Obviously, $\mathrm{Ann}(A)$ is an ideal of A and $\mathrm{Ann}(A) \subseteq C_R(A)$. Set $\mathfrak{b} = \{x \in C_R(A) | f(x, y) = 0, \;\; \forall y \in \mathrm{Ann}(A)\}$. Therefore $C_R(A)$ splits as an orthogonal direct sum $C_R(A) = \mathfrak{b} \oplus \mathrm{Ann}(A)$.

Consider the corresponding structure of the sub-adjacent Lie algebra $\mathfrak{g}$ of the left-symmetric algebra A, and set $\mathfrak{u} = \mathrm{Ann}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$. Then the Milnor's classification is obtained immediately.

Theorem 3.7 ([9], Theorem 1.5). *A Lie group G with left invariant metric is flat if and only if the associated Lie algebra $\mathfrak{g}$ splits as an orthogonal direct sum $\mathfrak{b} \oplus \mathfrak{u}$, where $\mathfrak{b}$ is a commutative subalgebra, $\mathfrak{u}$ is a commutative ideal, and where the linear transformation* ad$b$ *is skew-adjoint for any $b \in \mathfrak{b}$. Furthermore, if these conditions are satisfied, then*

$$\nabla_u = 0, \nabla_b = \mathrm{ad}(b), \forall u \in \mathfrak{u}, b \in \mathfrak{b}. \tag{3.3}$$

Acknowledgements. This work was supported by the National Natural Science Foundation of China (11001133). We would like to express our thanks to Prof. Chengming Bai for the helpful suggestions and discussion.

References

[1] A. Aubert, A. Medina, *Pseudo-Riemannian Lie groups* (in French), Tohoku Math. J. 55 (2003), 487-506.

[2] L. Auslander, *Simply transitive groups of affine motions*, Amer. J. Math. 99 (1977), 809-826.

[3] O. Baues, *Prehomogeneous affine representations and flat pseudo-Riemannian manifolds*, in: V. Cortes (Ed.), Handbook of pseudo-Riemannian Geometry and Supersymmetry, in: IRMA Lect. Math. Theor. Phys., vol. 16, Euro. Math. Soc., Zurich, 2010, 731-820.

[4] D. Burde, *Left-symmetric algebras, or pre-Lie algebras in geometry and physics*, Cent. Eur. J. Math. 4 (2006), 323-357.

[5] Z. Chen, D. Hou, C. Bai, *A left-symmetric algebraic approach to left invariant flat (pseudo-)metrics on Lie groups*, J. Geom. Phys. 62 (2012), 1600-1610.

[6] H. Kim, *Complete left-invariant affine structures on nilpotent Lie groups*, J. Diff. Geom. 24 (1986), 373-394.

[7] J.-L. Koszul, *Homogeneous bounded domains and orbits of affine transformation groups* (in French), Bull. Soc. Math. France 89 (1961), 515-533.

[8] A. Medina, *Flat left-invariant connections adapted to the automorphism structure of a Lie group*, J. Diff. Geom. 16 (1981), 445-474.

[9] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Adv. Math. 21 (1976), 293-329.

[10] A.A. Sagle, *Nonassociative algebras and Lagrangian mechanics on homogeneous spaces*, Algebras, Groups Geom. 2 (1985), 478-494.

Authors' address:

Zhiqi Chen *and* Mengying Liu
School of Mathematical Sciences and LPMC, Nankai University,
Tianjin 300071, P.R. China.
E-mail: chenzhiqi@nankai.edu.cn