Characterizations of real hypersurfaces with structure Lie operator in a nonflat complex space form

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Abstract. In this paper, we prove that a real hypersurface M in a complex space form $M_n(c)$, whose structure Lie operator and structure tensor symmetric and skew-symmetric, is a Hopf hypersurface. We characterize such Hopf hypersurface of $M_n(c)$.

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1 Introduction

A complex n-dimensional Kaeherian manifold of constant holomorphic sectional curvature c is called a *complex space form*, which is denoted by $M_n(c)$. As is well-known, a complete and simply connected complex space form is complex analytically isometric to a complex projective space $P_n(\mathbb{C})$, a complex Euclidean space $\mathbb{C}^n$ or a complex hyperbolic space $H_n(\mathbb{C})$, according to $c > 0$, $c = 0$ or $c < 0$.

We consider a real hypersurface M in a complex space form $M_n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, g, ξ, η) induced from the Kaehler metric and complex structure J on $M_n(c)$. The structure vector field ξ is said to be *principal* if $A\xi = \alpha\xi$ is satisfied, where A is the shape operator of M and $\alpha = \eta(A\xi)$. In this case, it is known that α is locally constant ([3]) and that M is called a *Hopf hypersurface*.

Typical examples of Hopf hypersurfaces in $P_n(\mathbb{C})$ are homogeneous ones, namely those real hypersurfaces are given as orbits under subgroup of the projective unitary group $PU(n+1)$. Takagi [11] completely classified such hypersurfaces as six model spaces which are said to be A_1, A_2, B, C, D and E. On the other hand, real hypersurfaces in $H_n(\mathbb{C})$ have been investigated by Berndt [1], Montiel and Romero [6] and so on. Berndt [1] classified all homogeneous Hopf hyersurfaces in $H_n(\mathbb{C})$ as four model spaces which are said to be A_0, A_1, A_2 and B. If M is a real hypersurface of type A_1 or A_2 in $P_n(\mathbb{C})$ or type A_0, A_1 or A_2 in $H_n(\mathbb{C})$, then M is said to be *of type A* for simplicity.

As a typical characterization of real hypersurfaces of type A, the following is due to Okumura [8] for $c > 0$ and Montiel and Romero [6] for $c < 0$.

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Theorem 1.1. *([6],[8]) Let M be a real hypersurface of $M_n(c)$, $c \neq 0$, $n \geq 2$. It satisfies $A\phi - \phi A = 0$ on M if and only if M is locally congruent to one of the model spaces of type A.*

The induced operator L_ξ on real hypersurface M from the 2-form $\mathcal{L}_\xi g$ is defined by $(\mathcal{L}_\xi g)(X, Y) = g(L_\xi X, Y)$ for any vector field X and Y on M, where $\mathcal{L}_\xi$ denotes the operator of the Lie derivative with respect to the structure vector field ξ. This operator L_ξ is given

$$L_\xi = \phi A - A\phi$$

on M, and call it structure Lie operator of M. One of the most interesting problems in the study of real hypersurfaces M with commuting operators and anti-commuting operators in $M_n(c)$ is to investigate a geometric characterization of these model spaces. With respect to the above conditions, some characterizations of real hypersurfaces in $M_n(c)$ are determined by the above conditions of real hypersurfaces and many important results on them have been obtained by many differential geometers ([2], [3], [5] and [9] etc).

As for the commuting Ricci operator, Kimura ([9]) for $c > 0$ and Ki and Suh ([3]) showed the following.

Theorem 1.2. *([9]) Let M be a real hypersurface of $P_n(\mathbb{C})$, $n \geq 3$. Then M satisfies $\phi S = S\phi$ and ξ is principal if and only if M lies on a tube of radius r over one of the following kähler submanifolds:*

(a) totally geodesic $P^k(c)$ $(1 \leq k \leq n-1)$, where $0 < r < \frac{\pi}{2}$,

(b) complex quadric $Q^n - 1(c)$, where $0 < r < \frac{\pi}{4}$ and $cot^2 2r = n - 2$,

(c) $P^1(c) \times P^{\frac{n-1}{2}}$, where $0 < r < \frac{\pi}{4}$ and $cot^2 2r = \frac{1}{(n-2)}$ and $n(\geq 5)$ is odd,

(d) complex Grassmann $G_{2.5}(c)$, where $0 < r < \frac{\pi}{4}$, $cot^2 2r = \frac{3}{5}$ and $n = 9$,

(e) Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$, $cot^2 2r = \frac{5}{9}$ and $n = 15$,

(f) k-dimensional Kähler submanifold on which the rank of each shape operator is not greater than 2 and non zero principal curvature$\neq \pm((2k-1)/(2n-2k-1))^{1/2}4$ with $cot^2 r = (2k-1)/(2n-2k-1)$.

Theorem 1.3. *([3]) Let M be a connected real hypersurface of $H_n(\mathbb{C})$, $n \geq 3$. Then the Ricci tensor of M commutes with the almost contact structure of M induced from $H_n(\mathbb{C})$ if and only if M is of type A_1, A_2.*

On the other hand, Cho and Ki ([2]) and Sohn and Lim ([4]) have proved the followings.

Theorem 1.4. *([2]) Let M be a real hypersurface of $P_n(\mathbb{C})$, $c \neq 0$. It satisfies $R_\xi \phi A = A\phi R_\xi$ on M, Then M is locally congruent to one of the model space of type A.*

Theorem 1.5. *([4]) Let M be a real hypersurface of $P_n(\mathbb{C})$, $c \neq 0$. Then M satisfies $R_\xi \phi A + A\phi R_\xi = 0$ if and only if M is locally congruent to one of the model space of type A.*

With respect to the structure Lie operator, the present author and W.H. Sohn ([5]) have proved the following.

Theorem 1.6. *([5]) Let M be a real hypersurface of $M_n(c)$, $c \neq 0$. If M has $L_\xi \phi A = A\phi L_\xi$, Then M is locally congruent to a real hypersurface of type A.*

In this paper, we shall study a real hypersurface in a nonflat complex space form $M_n(c)$ with symmetric or skew-symmetric operators of L_ξ and ϕ and give some characterizations of such a real hypersurface in $M_n(c)$. Namely, we shall prove the following theorems

Theorem A *Let M be a real hypersurface satisfying $L_\xi \phi = \phi L_\xi$ in a complex space form $M_n(c)$, $c \neq 0$. Then M is a Hopf hypersurface in $M_n(c)$.*

Theorem B *Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. Then it satisfies $L_\xi \phi = \phi L_\xi$ on M if and only if M is locally congruent to one of the model spaces of type A.*

Theorem C *Let M be a real hypersurface satisfying $L_\xi \phi + \phi L_\xi = 0$ in a complex space form $M_n(c)$, $c \neq 0$. Then M is a Hopf hypersurface in $M_n(c)$.*

All manifolds in the present paper are assumed to be connected and of class C^∞ and the real hypersurfaces supposed to be orientable.

2 Preliminaries

Let M be a real hypersurface immersed in a complex space form $M_n(c)$, and N be a unit normal vector field of M. By $\widetilde{\nabla}$ we denote the Levi-Civita connection with respect to the Fubini-Study metric tensor $\widetilde{g}$ of $M_n(c)$. Then the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \qquad \widetilde{\nabla}_X N = -AX,$$

for any vector fields X and Y tangent to M, where g denotes the Riemannian metric tensor of M induced from $\widetilde{g}$, and A is the shape operator of M in $M_n(c)$. For any vector field X on M we put

$$JX = \phi X + \eta(X)N, \qquad JN = -\xi,$$

where J is the almost complex structure of $M_n(c)$. Then we see that M induces an almost contact metric structure (ϕ, g, ξ, η), that is,

$$\begin{aligned} &\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \end{aligned} \tag{2.1}$$

for any vector fields X and Y on M. Since the almost complex structure J is parallel, we can verify from the Gauss and Weingarten formulas the followings :

$$\nabla_X \xi = \phi AX, \tag{2.2}$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \tag{2.3}$$

Since the ambient manifold is of constant holomorphic sectional curvature c, we have the following Gauss and Codazzi equations respectively :

$$\begin{aligned} R(X,Y)Z = \ & \tfrac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & -2g(\phi X, Y)\phi Z\} + g(AY,Z)AX - g(AX,Z)AY, \end{aligned} \tag{2.4}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{2.5}$$

for any vector fields X, Y and Z on M, where R denotes the Riemannian curvature tensor of M. By the use of (2), we have $(\mathcal{L}_\xi g)(X,Y) = g((\phi A - A\phi)X, Y)$, for any vector fields X and Y on M, and hence the induced operator L_ξ from $\mathcal{L}_\xi g$ is given

$$L_\xi X = (\phi A - A\phi)X. \tag{2.6}$$

Let W be a unit vector field on M with the same direction of the vector field $-\phi\nabla_\xi\xi$, and let μ be the length of the vector field $-\phi\nabla_\xi\xi$ if it does not vanish, and zero (constant function) if it vanishes. Then it is easily seen from (2) that

$$A\xi = \alpha\xi + \mu W, \tag{2.7}$$

where $\alpha = \eta(A\xi)$. We notice here that W is orthogonal to ξ.

We put

$$\Omega = \{p \in M \mid \mu(p) \neq 0\}. \tag{2.8}$$

Then Ω is an open subset of M.

3 Proof of Theorems

In this section, we shall prove Theorems A, B and C . Now Now we need the following lemmas in order to prove the our main results.

Lemma 3.1([3], [7]) *If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

Lemma 3.2([7]) *Assume that ξ is a principal curvature vector and the corresponding principal is α. Then*

$$A\phi A - \frac{\alpha}{2}(A\phi + \phi A) - \frac{c}{4} = 0. \tag{3.1}$$

Proof of Theorem A.

We assume that the Lie structure operator L_ξ and structure tensor field ϕ are commute, that is, $L_\xi \phi = \phi L_\xi$. Hence, we have

$$2(\phi A \phi + A)X = (2\alpha\eta(X) + \mu w(X))\xi + \mu\eta(X)W, \tag{3.2}$$

for any vector field X on Ω, where w is the dual 1-form of the unit vector field W.

If we put $X = \xi$ into (3.2), then we have

$$2A\xi = 2\alpha\xi + \mu W. \tag{3.3}$$

If we substitute (2.7) into (3.3) then we obtains $\mu = 0$ on Ω, and it is a contradiction.Thus the set Ω is empty and hence M is a Hopf hypersurface. $\square$

Proof of Theorem B.

By Theorem A, M is a Hopf hypersurface in $M_n(c)$, that is $A\xi = \alpha\xi$. Therefore the assumption $L_\xi\phi = \phi L_\xi$ is equivalent to

$$(\phi A\phi + A)X = \alpha\eta(X)\xi. \tag{3.4}$$

For any vector field X on M such that $AX = \lambda X$, it follows from (3.1) that

$$\left(\lambda - \frac{\alpha}{2}\right) A\phi X = \frac{1}{2}\left(\alpha\lambda + \frac{c}{2}\right)\phi X. \tag{3.5}$$

We can choose an orthonormal frame field $\{X_0 = \xi, X_1, X_2, \ldots, X_{2(n-1)}\}$ on M such that $AX_i = \lambda_i X_i$ for $1 \le i \le 2(n-1)$. If $\lambda_i \ne \frac{\alpha}{2}$ for $1 \le i \le p \le 2(n-1)$, then we see from (3.5) that ϕX_i is also a principal direction, say $A\phi X_i = \mu_i\phi X_i$. From (3.4), we have $\mu_i = \lambda_i$ and hence $A\phi X_i = \phi A X_i$ for $1 \le i \le p$. If $\lambda_i \ne \frac{\alpha}{2}$ and $\lambda_j = \frac{\alpha}{2}$ for $1 \le i \le p$ and $p+1 \le j \le 2(n-1)$ respectively, then we gets $c = -\alpha^2$ and it follows from (3.4) that

$$\phi A\phi X_j + AX_j = 0. \tag{3.6}$$

Taking inner product of (3.6) with X_i, we obtain $(\lambda_i - \mu_i)g(X_i, X_j) = 0$ for $1 \le i \le p$. If $\lambda_i = \mu_i$ for $1 \le i \le p$, then we have

$$\lambda_i = \mu_i = \frac{\alpha \pm \sqrt{\alpha^2 + c}}{2}. \tag{3.7}$$

Since $c = -\alpha^2$, it follows that $\lambda_i = \frac{\alpha}{2}$, and hence a contradiction. If $\lambda_j = \frac{\alpha}{2}$ for $1 \le j \le 2(n-1)$, then it is easily seen that $A\phi X_j = \phi AX_j$ for all j. Therefore we have $L_\xi = \phi A - A\phi = 0$ on M. Theorem B follows Theorem 1. $\square$

Proof of Theorem C.

Assume that the Lie structure operator L_ξ and structure tensor field ϕ are skew-symmetric, that is, $L_\xi\phi + \phi L_\xi = 0$. Hence, we have

$$\eta(X)A\xi - \eta(AX)\xi = 0. \tag{3.8}$$

If we put $X = \xi$ into (3.8), then we have

$$A\xi = \alpha\xi. \tag{3.9}$$

Comparing (2.7) with (3.9), we get $\mu = 0$ and hence a contradiction. Thus the set Ω is empty and hence M is a Hopf hypersurface. □

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