Minimal translation surfaces in Lorentz Heisenberg $3-$space with flat metric

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Abstract. In the present paper we study some types of minimal translation surfaces parametrized by $X(x,y) = \alpha(x) * \beta(y)$ or $X(x,y) = \beta(y) * \alpha(x)$, obtained as a product of two curves α and β lying in planes, which are not-orthogonal, where $*$ denotes the group operation in the $3-$ dimensional Lorentz Heisenberg space $\mathbb{H}_3$, endowed with the flat metric $g_3 = dx^2 + (xdy + dz)^2 - [(1-x)dy - dz]^2$.

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1 Introduction

Recently, the study of minimal translation surfaces in $3-$ dimensional homogeneous spaces is the main objects of many researches, R.López and M.I.Munteanu in their paper [7] have classified minimal translation surfaces in the model space Sol_3 of solv-geometry in the sense of Thurston.

In [5] J. Inoguchi, R.López and M.I.Munteanu defined six types of translation surfaces in the 3-dimensional Heisenberg group Nil_3 obtained as a product of two planar curves lying in planes which are not orthogonal and they studied the condition of minimality for each type.

D.W.Yoon, C.W.Lee and M.K.Karacan studied some minimal translation surfaces in the 3- dimensional Heisenberg group $\mathbb{H}_3$ see [15].

In [8] and [9] the authors showed that modulo an automorphism of the Lie algebra the three dimensional Heisenberg group has the following classes of left-invariant Lorentz metrics:

$$g_1 = -dx^2 + dy^2 + (xdy + dz)^2,$$

$$g_2 = dx^2 + dy^2 - (xdy + dz)^2,$$

$$g_3 = dx^2 + (xdy + dz)^2 - [(1-x)dy - dz]^2,$$

they proved that the metrics g_1, g_2, g_3 are non-isometrics and g_3 is flat.

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M. Bekkar and Z. Hanifi showed in their paper [1] that the plane, helicoid, hyperbolic paraboloid and some translation surfaces are defined by elliptic integrals verifying the equation of minimal surfaces in Lorentz Heisenberg $3-$space $\mathbb{H}_3$ endowed with a left invariant Lorentzian metric g_ξ given by:

$$g_3 = dx^2 + dy^2 - (dz + \xi(ydx - xdy))^2.$$

In our previous work [3] we have classified three types of minimal translation surfaces in the $3-$ dimensional Lorentz Heisenberg space $\mathbb{H}_3$ endowed with the left invariant Lorentz metric g_1 given above. Our aim in this work is to classify some types of minimal translation surfaces of $\mathbb{H}_3$ endowed with the left invariant Lorentz flat metric g_3.

2 Preliminaries

The Lorentz-Heisenberg group $\mathbb{H}_3$ is represented as the cartesian $3-$space respected to the product

$$(\bar{x}, \bar{y}, \bar{z}) * (x, y, z) = (\bar{x} + x, \bar{y} + y, \bar{z} + z - \bar{x}y + x\bar{y}).$$

Where $*$ denotes the group operation in the Lorentz Heisenber group $\mathbb{H}_3$.
$\mathbb{H}_3$ is endowed with a left invariant Lorentzian metric g_3 which is given by:

$$\begin{aligned} g_3 &= dx^2 + (xdy + dz)^2 - [(1-x)dy - dz]^2 \\ &= (dx, dy, dz)(g_{ij}) \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}. \end{aligned}$$

where (dx, dy, dz) is a vector field and $(g_{ij})_{1 \leq i,j \leq 3}$ is the matrix given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2x-1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The Lie-algebra of $\mathbb{H}_3$ has a pseudo-orthonormal basis $B = \{e_1, e_2, e_3\}$ such as

$$e_1 = \tfrac{\partial}{\partial x}, \quad e_2 = \tfrac{\partial}{\partial y} + (1-x)\tfrac{\partial}{\partial z}, \quad e_3 = \tfrac{\partial}{\partial y} - x\tfrac{\partial}{\partial z}.$$

which verify

$$g_3(e_1, e_1) = g_3(e_2, e_2) = 1, \quad g_3(e_3, e_3) = -1.$$

We can easily calculate the Lie products:

$$[e_2, e_3] = 0, \quad [e_3, e_1] = e_2 - e_3 \quad \text{and} \quad [e_2, e_1] = e_2 - e_3.$$

The Levi-Civita connection ∇ of g_3 satisfies

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{3} \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

where $\left(\Gamma_{ij}^{k}\right)_{1\leq k\leq 3}$ are the Christoffel symbols of g_3. We have

$$\begin{cases} \nabla_{e_1}e_1 = 0, \nabla_{e_1}e_2 = 0, \nabla_{e_1}e_3 = 0, \\ \nabla_{e_2}e_1 = e_2 - e_3, \nabla_{e_2}e_2 = -e_1, \nabla_{e_2}e_3 = -e_1, \\ \nabla_{e_3}e_1 = e_2 - e_3, \nabla_{e_3}e_2 = -e_1, \nabla_{e_3}e_3 = -e_1. \end{cases}$$

The Ricci tensor is defined by

$$Ricc\,(X,Y) = \sum_{i=1}^{3} \epsilon_i g_3\left(R\left(X,e_i\right)Y,e_i\right),$$

where X,Y are two vectors fields on $\mathcal{H}_3$, $\epsilon_1 = \epsilon_2 = 1$ and $\epsilon_3 = -1$. Thus its components are

$$R_{11} = R_{33} = \frac{1}{2},\ R_{22} = -\frac{1}{2},\ R_{ij} = 0 \text{ for all } i \neq j.$$

3 Minimal surface equations

Let Σ be a surface in the Lorentz-Heisenberg $3-$space $\mathbb{H}_3$ which represents the graph of the function $z = f(x,y)$, parameterized by

$$\begin{array}{cccc} X: & U \subset \mathbb{R}^2 & \to & \mathbb{H}_3 \\ & (x,y) & \mapsto & (x,y,f(x,y)) \end{array}$$

where $X(x,y) = (x,y,f(x,y))$ is the position vector, and where the components

$$X_x = \tfrac{\partial}{\partial x} + f_x \tfrac{\partial}{\partial z} \quad \textit{and} \quad X_y = \tfrac{\partial}{\partial y} + f_y \tfrac{\partial}{\partial z}$$

of the tangent vector are given by

$$\begin{cases} X_x(x,y) = e_1 + f_x e_2 - f_x e_3, \\ X_y(x,y) = (f_y + x)e_2 + (1 - f_y - x)e_3. \end{cases}$$

The first fundamental form I of the surface Σ is

$$I = Edx^2 + 2Fdxdy + Gdy^2,$$

with

$$E = g_3(X_x, X_x) = 1, \quad F = g_3(X_x, X_y) = f_x, \quad G = g_3(X_y, X_y) = 2(x + f_y) - 1$$

and the second fundamental form II of the surface Σ is

$$II = ldx^2 + 2mdxdy + ndy^2,$$

where

$$l = g_3(\nabla_{X_x}X_x, N), \quad m = g_3(\nabla_{X_x}X_y, N), \quad n = g_3(\nabla_{X_y}X_y, N),$$

$$\begin{cases} \nabla_{X_x} X_x &= f_{xx}(x,y)e_2 - f_{xx}(x,y)e_3 \\ \nabla_{X_x} X_y &= [f_{xy}(x,y)+1]e_2 - [f_{xy}(x,y)+1]e_3 \\ \nabla_{X_y} X_y &= -e_1 + f_{yy}(x,y)e_2 - f_{yy}(x,y)e_3 \end{cases}$$

and N is a unit vector field normal to Σ, so it satisfies the following system

$$\begin{cases} g_3(X_x, N) = 0, \\ g_3(X_y, N) = 0, \\ g_3(N, N) = 1. \end{cases}$$

Therefore

$$\begin{aligned} N &= \frac{-f_x e_1 + (1 - f_y - x)e_2 + (f_y + x)e_3}{\sqrt{f_x^2 + 1 - 2(f_y + x)}} \\ &= \frac{1}{W}[-f_x e_1 + (1 - f_y - x)e_2 + (f_y + x)e_3], \end{aligned}$$

where $W = \sqrt{f_x^2 + 1 - 2(f_y + x)}$. We have

$$l = \tfrac{1}{W} f_{xx}, \quad m = \tfrac{1}{W}(1 + f_{xy}), \quad n = \tfrac{1}{W}(f_x + f_{yy}).$$

Definition 3.1. In a $3-$dimensional Lorentz Heisenberg space $\mathbb{H}_3$, the surfaces that locally minimize the areas are called *minimal surfaces*.

Such surfaces Σ satisfy the condition $H = 0$, where H is the mean curvature vector field given by the formula:

$$H = \frac{En + Gl - 2Fm}{2(EG - F^2)}. \tag{3.1}$$

Proposition 3.1. *The surface Σ defined above is a minimal surface in $3-$dimensional Lorentz Heisenberg space $\mathbb{H}_3$ if and only if it's mean curvature H satisfies the following condition*

$$H = \frac{1}{2W^3}\left[f_{yy} + [-1 + 2(f_y + x)]f_{xx} - 2f_x f_{xy} - f_x\right] = 0. \tag{3.2}$$

Example 3.2. It's clear that if $f(x,y) = ay + b$ or $f(x,y) = -\frac{1}{2}xy + cx + dy + e$ where a, b, c, d and e are real constants, then the condition (3.2) is satisfied. Consequently the surfaces parameterized by $X(x,y) = (x, y, ay + b)$ and $X(x,y) = (x, y, -\frac{1}{2}xy + cx + dy + e)$ are minimal surfaces in $\mathbb{H}_3$.

4 Minimal translation surfaces

Now we classify some types of translation surfaces in the $3-$ dimensional Lorentz Heisenberg space $\mathbb{H}_3$ obtained as a product of two generating curves which are not orthogonal.

Type 1:

Let first consider a translation surface Σ parameterized by

$$X(x,y) = (x, 0, g(x)) * (0, y, h(y)) = (x, y, g(x) + h(y) - xy),$$

where g and h are two arbitrary surfaces. The condition of minimality given by the equation (3.2) becomes

$$h''(y) + (-1 + 2h'(y))g''(x) + (g'(x) - y) = 0. \tag{4.1}$$

Taking the derivative with respect to x of the equation (4.1), we obtain

$$(-1 + 2h'(y))g'''(x) + g''(x) = 0 \tag{4.2}$$

We remark that if g is affine satisfy the equation (4.1).
To solve the equation (4.1), we distinguish two cases:

- **Case 1** : if $g''(x) = 0$, we have $g(x) = ax + x_0$ where a and x_0 are two real constants.
  Replacing g in the equation (4.1), we obtain $h(y) = \frac{1}{6}y^3 - \frac{a}{2}y^2 + y_1 y + y_0$, where y_0, $y_1 \in \mathbb{R}$.
- **Case 2** : if $g''(x) \neq 0$, we have
  $$\frac{g'''(x)}{g''(x)} = \frac{1}{1 - 2h'(y)} = \xi, \quad \xi \in \mathbb{R}.$$
  and we get
  $$g(x) = \frac{k}{\xi^2} e^{\xi x + \gamma}, \text{ and } h(y) = \frac{\xi - 1}{2\xi} y + \delta,$$
  where $\delta, \gamma \in \mathbb{R}$.

Theorem 4.1. *The minimal translation surface Σ of type 1 in the $3-$ dimensional Lorentz Heisenberg space $\mathbb{H}_3$ are parameterized by $X(x, y) = (x, y, g(x) + h(y) - xy)$ where $g(x)$, and $h(y)$ are given by:*

$$g(x) = ax + x_0$$

and

$$h(y) = \frac{1}{6}y^3 - \frac{a}{2}y^2 + y_1 y + y_0$$

where a, x_0, y_0 and y_1 are real constants.
Or

$$g(x) = \frac{k}{\xi^2} e^{\xi x + \gamma}, \text{ and } h(y) = \frac{\xi - 1}{2\xi} y + \delta,$$

where $\delta, \gamma \in \mathbb{R}$.

Type 2: Now the translation surface Σ is parameterized by

$$X(x, y) = (0, y, h(y)) * (x, 0, g(x)) = (x, y, g(x) + h(y) + xy),$$

where g and h are two arbitrary surfaces. The condition of minimality given by the equation (3.2) becomes

$$h''(y) + [2(h'(y) + 2x) - 1]g''(x) - 3(g'(x) + y) = 0. \tag{4.3}$$

We take the derivative of the equation (4.3) with respect to x (to y) respectively, we find

$$2h''(y)g'''(x) = 0, \tag{4.4}$$

which implies $h''(y) = 0$ or $g'''(x) = 0$.

- If $h''(y) = 0$ then $h(y) = ay + y_0$ where a and y_0 are two real constants. Replacing in (4.3) we obtain

$$[-1 + 2(a + 2x)]g''(x) - 3g'(x) = 3y. \tag{4.5}$$

The right hand side of the equality (4.5) depends only on x and the left hand side depends only on y which is a contradiction, then the equation (4.4) is satisfied if and only if $g'''(x) = 0$.

- If $g'''(x) = 0$ then $g(x) = ax^2 + bx + c$ with a, b and c are three real constants. Replacing this result in (4.3), we get

$$h''(y) + 4ah'(y) - 3y = -2ax + 2a + 3b. \tag{4.6}$$

This last equation is verified if and only if $a = 0$, which gives:

$$h(y) = \frac{1}{2}y^3 + \frac{3}{2}by^2 + y_1 y + y_0, \tag{4.7}$$

where y_1, y_0 are real constants.

Therefore we establish the following result

Theorem 4.2. *The minimal translation surfaces Σ of type 2 in the $3-$ dimensional Lorentz Heisenberg space $\mathbb{H}_3$ are parameterized by*

$$X(x, y) = (0, y, h(y))(x, 0, g(x)) = (x, y, g(x) + h(y) + xy)\,,$$

where $g(x)$, and $h(y)$ are given by:

$$g(x) = bx + c$$

and

$$h(y) = \frac{1}{2}y^3 + \frac{3}{2}by^2 + y_1 y + y_0,$$

with b, c, y_1, and y_0 real constants.

Type 3:
We suppose now that the translation surface Σ is given by the product

$$X(x, y) = (x, 0, g(x)) * (h(y), y, 0) = (x + h(y), y, g(x) - xy)\,,$$

so it's parameterized by

$$\begin{array}{rccc} X: & U \subset \mathbb{R}^2 & \rightarrow & \mathbb{H}_3 \\ & (x, y) & \mapsto & (x + h(y), y, g(x) - xy) \end{array}$$

where $X(x, y) = (x + h(y), y, g(x) - xy))$ is the position vector and the components of the tangent vector are given by:

$$\begin{cases} X_x(x, y) = e_1 + (g'(x) - y)e_2 - (g'(x) - y)e_3, \\ X_y(x, y) = h'(y)e_1 + e_3. \end{cases}$$

The coefficients of the first fundamental form are:

$$E = g_3(X_x, X_x) = 1, \quad F = g_3(X_x, X_y) = h'(y) + g'(x) - y, \quad G = g_3(X_y, X_y) = (h'(y))^2 - 1.$$

Since

$$\begin{cases} \nabla_{X_x} X_x &= & g''(x)e_2 - g''(x)e_3, \\ \nabla_{X_x} X_y &= & 0, \\ \nabla_{X_y} X_y &= & [h''(y) - 1]e_1 + h'(y)e_2 - h'(y)e_3, \end{cases}$$

and N the unit vector field normal to Σ is given by:

$$N = \frac{1}{W}[(g'(x) - y)e_1 - (1 + h'(y)(g'(x) - y))e_2 + (h'(y)(g'(x) - y))e_3]$$

with

$$W = \sqrt{[g'(x) - y]^2 + 2h'(y)(g'(x) - y) + 1},$$

then the coefficients of the second fundamental form of Σ are:

$$l = -\tfrac{1}{W} g''(x), \quad m = 0, \quad n = \tfrac{1}{W}[(h''(y) - 1)(g'(x) - y) - h'(y)].$$

We follow the same steps as the previous types to calculate the main curvature of the translation surface Σ, we yield:

$$H = \frac{1}{2W^3}[(g'(x) - y)[(h''(y) - 1] - h'(y) - g''(x)(h'^2(y) - 1)]. \tag{4.8}$$

The minimality condition $H = 0$ implies the following equation

$$(g'(x) - y)[h''(y) - 1] - h'(y) - g''(x)(h'^2(y) - 1) = 0 \tag{4.9}$$

Taking the derivative with respect to x, we obtain the following differential system:

$$\frac{g'''(x)}{g''(x)} = \frac{h''(y) - 1}{h'^2(y) - 1} \tag{4.10}$$

Since the left hand side of the equality (4.10) depends only on y and the right hand side depends only on x, thus for all λ a real constant we have

$$\frac{g'''(x)}{g''(x)} = \frac{h''(y) - 1}{h'^2(y) - 1} = \lambda, \lambda \in \mathbb{R}$$

which implies the following differential system

$$\begin{cases} g'''(x) = \lambda g''(x) \\ h''(y) - 1 = \lambda[h'^2(y) - 1] \end{cases}$$

Solving this system, we obtain first in the particular case $\lambda = 0$:
$g'''(x) = 0$, then $g(x) = ax^2 + bx + c$ with a, b and c are real constants.
Replacing g in the minimality condition (4.9), we obtain

$$(2ax + b - y)(h''(y) - 1) - h'(y) = 2a[(h'(y))^2 - 1] \tag{4.11}$$

The equation (4.9) is satisfied if $h''(y)-1=0$ or $a=0$.
Moreover, if $h''(y)-1=0$ then we get $h'(y)=y+y_0$ where $y_0\in\mathbb{R}$ and $2a[(h'(y))^2-1]+h'(y)=0$, which implies

$$2a[(y+y_0)^2-1]+(y+y_0)=0 \tag{4.12}$$

This last equation is not true for all $y\in\mathbb{R}$, so we conclude that $h''(y)-1\neq 0$, consequently we have $a=0$ and the equation (4.11) becomes

$$(b-y)h''(y)-h'(y)=b-y,$$

which has the solution

$$h(y)=\frac{1}{4}(b-y)^2-c_1\ln|b-y|+c_0$$

where c_1, $c_0\in\mathbb{R}$.
We suppose now that $\lambda\neq 0$ then

$$g(x)=C(x+\frac{1}{\lambda})+C_1\exp(\lambda x)+C_0$$

where C_1, C_0, $C\in\mathbb{R}$. Replacing g in the equation (4.9), we get

$$(C-y)(h''(y)-1)-h'(y)=C_1\lambda\exp(\lambda x)[1-h''(y)+\lambda(h'(y))^2-1].$$

Since the right hand side side of this equality depends only on y and the left hand side depends on x and y then the minimality condition is satisfied if and only if $C_1=0$ so we get

$$(C-y)h''(y)-h'(y)=C-y,$$

which is already solved above. Finally we announce the following theorem summarizing this result

Theorem 4.3. *The minimal translation surfaces Σ of type 3 in the $3-$ dimensional Lorentz Heisenberg space $\mathbb{H}_3$ are parameterized by $X(x,y)=(x+h(y),y,g(x)-xy)$ where g and h are given by:*

$$g(x)=bx+c$$

and

$$h(y)=\frac{1}{4}(b-y)^2-c_1\ln|b-y|+c_0,$$

where b, c, c_1 and c_0 are real constants.

Type 4:
Now we suppose now that the translation surface Σ of type 4 is given by the product

$$X(x,y)=(x,0,g(x))*(h(y),y,0)=(x+h(y),y,g(x)+xy)$$

so it's parameterized by

$$\begin{array}{cccc} X: & U\subset\mathbb{R}^2 & \to & \mathbb{H}_3 \\ & (x,y) & \mapsto & (x+h(y),y,g(x)+xy), \end{array}$$

where $X(x, y) = (x + h(y), y, g(x) - xy))$ is the position vector and the components of the tangent vector are given by:

$$\begin{cases} X_x(x, y) = e_1 + (g'(x) + y)e_2 - (g'(x) + y)e_3 \\ X_y(x, y) = h'(y)e_1 + 2xe_2 + (1 - 2x)e_3. \end{cases}$$

The coefficients of the first fundamental form of the surface Σ are:

$$\begin{array}{lcll} E & = & g_3(X_x, X_x) & = 1, \\ F & = & g_3(X_x, X_y) & = h'(y) + g'(x) + y, \\ G & = & g_3(X_y, X_y) & = (h'(y))^2 + 4x - 1. \end{array}$$

We have also

$$\begin{cases} \nabla_{X_x} X_x & = & g''(x)e_2 - g''(x)e_3 \\ \nabla_{X_x} X_y & = & 2e_2 - 2e_3 \\ \nabla_{X_y} X_y & = & [h''(y) - 1]e_1 + h'(y)e_2 - h'(y)e_3 \end{cases}$$

and N the unit vector field normal to Σ is given by:

$$N = \frac{1}{W}[-(g'(x) + y)e_1 + (1 - 2x + h'(y)(g'(x) + y))e_2 + (2x - h'(y)(g'(x) + y))e_3]$$

with

$$W = \sqrt{[g'(x) + y]^2 + 2h'(y)(g'(x) + y) + 1 - 4x}.$$

The coefficients of the second fundamental form of Σ are:

- $l = \frac{1}{W} g''(x)$,
- $m = \frac{2}{W}$,
- $n = \frac{1}{W}[(1 - h''(y))(g'(x) + y) + h'(y)]$.

Therefore we get the main curvature of the translation surface Σ of type 4:
(4.13)

$$H = \frac{1}{2W^3}[(1-h''(y))(g'(x)+y)+h'(y)+(h'^2(y)+4x-1)g''(x)-4(h'(y)+g'(x)+y)],$$

where

$$W = \sqrt{(g'(x) + y))^2 + 2h'(y)(g'(x) + y)) + 1 - 4x}.$$

The minimality condition $H = 0$ yields the following equation

$$[1 - h''(y)](g'(x) + y) + (h'^2(y) + 4x - 1)g''(x) - 3h'(y) - 4(g'(x) + y) = 0. \tag{4.14}$$

Taking the derivative firstly with respect to x then secondly with respect to y, we obtain

$$\frac{-g'''(x)}{g''(x)} = \frac{h'''(y)}{2h'(y)h''2(y)}. \tag{4.15}$$

Since the left hand side of the equality (4.15) depends only on y and the right hand side depends only on x, then for any real constant λ, we have

$$\frac{-g'''(x)}{g''(x)} = \frac{h'''(y)}{2h''(y)h'(y)} = \lambda, \quad \lambda \in \mathbb{R},$$

which implies the following differential system

$$\begin{cases} g'''(x) = -\lambda g''(x) \\ h'''(y) = 2\lambda h''(y)h'(y). \end{cases}$$

To solve this system we distinguish two cases:

1. $\lambda = 0$: we find
$$g(x) = ax^2 + bx + c,$$
where a, b, and c are real constants. Replacing g in the minimality condition (4.14), we obtain
$$[1 - h''(y)](b + y) + 2a(h'^2(y) - 1) - 3h'(y) - 4(b + y) = -2ax(1 - h''(y)).$$
This equation is verified if $a = 0$ or $(1 - h''(y)) = 0$.
   - If $(1 - h''(y)) = 0$, then the function $h'(y) = y + y_0$, where $y_0 \in \mathbb{R}$ is a solution of the equation $2a(h'(y))^2(y) - 1) - 3h'(y) - 4(b + y) = 0$, which is not true for any $y \in \mathbb{R}$.
   - If $a = 0$ then we get $(b + y)h''(y) + 3h'(y) = -3(b + y)$, which has the following solution,
   $$h(y) = \frac{-3}{8}(b + y)^2 - \frac{c_0}{2}(b + y)^{-2} + c_1,$$
   where c_0 and c_1 are real constants.
2. $\lambda \neq 0$: Solving the equation
$$g'''(x) + \lambda g''(x) = 0,$$
we find
$$g(x) = \frac{1}{\lambda}(b_1 x + b_0) - \frac{b_1}{\lambda^2} + b_2 \exp(-\lambda x),$$
where b_2, b_1 and b_0 are real constants. Replacing this result in the minimality condition (4.14), we obtain
(4.16)
$$[1-h''(y)](\frac{b_1}{\lambda}+y)-3h'(y)-4(\frac{b_1}{\lambda}+y) = \lambda b_2 \exp(-\lambda x)[1-h''(y)-\lambda(h'(y))^2+4x-1)-4],$$
and hence $b_2 = 0$. Consequently $g(x) = \frac{1}{\lambda}(b_1 x + b_0) - \frac{b_1}{\lambda^2} = bx + c$, where b and c are real constants. In addition, the equation (4.16) becomes
$$(b + y)h''(y) + 3h'(y) = -3(b + y),$$
which has the solution
$$h(y) = \frac{-3}{8}(b + y)^2 - \frac{c_0}{2}(b + y)^{-2} + c_1,$$
where c_0 and c_1 are real constants.

We summarize by the following theorem

Theorem 4.4. *The minimal translation surfaces Σ of type 4 in the $3-$ dimensional Lorentz Heisenberg space $\mathbb{H}_3$ are parameterized by $X(x,y) = (x + h(y), y, g(x) + xy)$ with*

$$g(x) = bx + c,$$

and

$$h(y) = \frac{-3}{8}(b+y)^2 - \frac{c_0}{2}(b+y)^{-2} + c_1,$$

where b, c, c_0 and c_1 are real constants.

Type 5:

Let the curves $\beta(y) = (0, y, h(y))$ and $\alpha(x) = (x, g(x), 0)$. The translation surface Σ given by the product $\Sigma = \beta(y) * \alpha(x)$ is parameterized by:

$$X(x,y) = (0, y, h(y)) * (x, g(x), 0) = (x, y + g(x), h(y) + xy).$$

So, we have

$$\begin{gathered} X : U \subset \mathbb{R}^2 \to \mathbb{H}_3 \\ (x,y) \to (x, y + g(x), h(y) + xy), \end{gathered}$$

such that

$$\begin{cases} X_x(x,y) = e_1 + (y + xg'(x))e_2 - (xg'(x) + y - g'(x))e_3 \\ X_y(x,y) = (h'(y) + 2x)e_2 - [h'(y) + 2x - 1]e_3. \end{cases}$$

The coefficients of the first fundamental form of Σ are given by:

$$\begin{gathered} E = 1 + 2(g'(x))^2(2x-1) + 2yg'(x), \\ F = y + g'(x)[h'(y) + 3x - 1], \\ G = 2(h'(y) + 2x) - 1. \end{gathered}$$

We have

$$\begin{cases} \nabla_{X_x} X_x &= & -(g'(x))^2 e_1 + (2g'(x) + xg''(x))e_2 - [2g'(x) + xg''(x) - g''(x)]e_3 \\ \nabla_{X_x} X_y &= & -g'(x)e_1 + 2e_2 - 2e_3 \\ \nabla_{X_y} X_y &= & -e_1 + h''(y)e_2 - h''(y)e_3 \end{cases}$$

and N, the unit vector field normal to Σ, is given by:

$$N = \frac{1}{W}([g'(x)(h'(y) + x) - y]e_1 - [h'(y) + 2x - 1]e_2 + [h'(y) + 2x]e_3)$$

with

$$W = \sqrt{[g'(x)(h'(y) + x) - y]^2 - 2(h'(y) + 2x) + 1}.$$

The coefficients of the second fundamental form of the surface Σ are:

- $l = \frac{1}{W}[-(g'(x))^3(h'(y) + x) + y(g'(x))^2 + 2g'(x) - g''(x)(h'(y) + x)]$,
- $m = \frac{1}{W}[-(g'(x))^2(h'(y) + x) + yg'(x) + 2]$,
- $n = \frac{1}{W}[-g'(x)(h'(y) + x) + h''(y) + y]$.

The minimality condition given in the definition (3.1) implies the following equation

$$\begin{aligned}(2x-1)(g'(x))^2h''(y)+(x+h'(y)+2yh''(y))g'(x)\\ +g''[(x+h'(y))(1-2(2x+h'(y)))]=3y-h''(y).\end{aligned} \tag{4.17}$$

Since one hand side of (4.17) depends only on y, and the other depends on x and y then, for any minimal translation surface of type 5, we have $g''(x)=g'(x)=0$. Consequently, g is a constant function and $h=\frac{1}{2}y^3+y_1y+y_0$, where y_0 and y_1 are integration constants.

Theorem 4.5. *The minimal translation surfaces Σ of type 5 in the $3-$ dimensional Lorentz Heisenberg space $\mathbb{H}_3$ are parameterized by $X(x,y)=(x,y+g(x),h(y)+xy)$, where $g(x)$ is constant and $h(y)=\frac{1}{2}y^3+y_1y+y_0$, with $y_0,y_1\in\mathbb{R}$.*

Type 6:
Consider two curves $\alpha(x)=(x,g(x),0)$ and $\beta(y)=(0,y,h(y))$. The translation surface Σ given by the product $\Sigma=\alpha(x)*\beta(y)$ is parameterized by:

$$X(x,y)=((x,g(x),0)*(0,y,h(y))=(x,y+g(x),h(y)-xy).$$

So, we have

$$\begin{array}{rccc} X: & U\subset\mathbb{R}^2 & \to & \mathbb{H}_3\\ & (x,y) & \mapsto & (x,y+g(x),h(y)-xy)\end{array}$$

such that

$$\begin{cases} X_x(x,y)=e_1+(xg'(x)-y)e_2-(xg'(x)-y-g'(x))e_3\\ X_y(x,y)=h'(y)e_2+[1-h'(y)]e_3\end{cases}$$

The coefficients of the first fundamental form of Σ are given by:

$$\begin{gathered}E=1+(g'(x))^2(2x-1)-2yg'(x),\\ F=-y+g'(x)[h'(y)+x-1],\\ G=2h'(y)-1.\end{gathered}$$

We have

$$\begin{cases}\nabla_{X_x}X_x &=& -(g'(x))^2e_1+(2g'(x)+xg''(x))e_2-[2g'(x)+xg''(x)-g''(x)]e_3,\\ \nabla_{X_x}X_y &=& -g'(x)e_1,\\ \nabla_{X_y}X_y &=& -e_1+h''(y)e_2-h''(y)e_3,\end{cases}$$

and N the unit vector field normal to Σ is given by:

$$N=\frac{1}{W}([g'(x)(h'(y)-x)+y]e_1+[1-h'(y)]e_2+h'(y)e_3)$$

with

$$W=\sqrt{[g'(x)(h'(y)-x)+y]^2-2h'(y)+1}.$$

The coefficients of the second fundamental form of the surface Σ are:

- $l=\frac{1}{W}[-(g'(x))^3(h'(y)-x)-y(g'(x))^2+2g'(x)+g''(x)(x-h'(y))]$,

- $m = \frac{1}{W}[-(g'(x))^2(h'(y) - x)]$,
- $n = \frac{1}{W}[-g'(x)(h'(y) - x) + h''(y) - y]$.

Since the main curvature of the translation surface Σ of type 6 is given by
(4.18)

$$\frac{1}{2W^3}[(g'(x))^2((2x-1)h''(y) - 2y + (x + h'(y) - 1))+ \\ g'(x)(-2yh''(y) + 3h'(y) + 2y^2 + x - 2) + g''(x - h'(y))(2h'(y) - 1) - y + h''(y)]$$

then the minimality condition yields
(4.19)

$$(g'(x))^2((2x-1)h''(y) - 2y + (x + h'(y) - 1))+ \\ g'(x)(-2yh''(y) + 3h'(y) + 2y^2 + x - 2) + g''(x - h'(y))(2h'(y) - 1) = y - h''(y).$$

It's clear that the right hand side of (4.19) depends on x and y and the left one depends only on y. Therefore, for any minimal translation surface of type 6, we have $g''(x) = g'(x) = 0$.
Consequently, g is a constant function and $h = \frac{1}{6}y^3 + y_1 y + y_0$, where y_0 and y_1 are integration constants.

Theorem 4.6. *The minimal translation surfaces Σ of type 6 in the 3− dimensional Lorentz Heisenberg space $\mathbb{H}_3$ are parameterized by $X(x, y) = (x, y + g(x), h(y) - xy)$, where $g(x)$ is constant and $h(y) = \frac{1}{6}y^3 + y_1 y + y_0$, with $y_0, y_1 \in \mathbb{R}$.*

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