Existence of generalized mixed super quasi-constant curvature

B. B. Chaturvedi and Pankaj Pandey

Abstract. Aim of the present paper is to study the generalization of mixed super quasi-constant curvature. In this paper, we have defined a new type of quasi-constant curvature, called a generalized mixed super quasi-constant curvature. The existence of generalized mixed super quasi-constant curvature has been verified by theoretical as well as by an example. It is proved that a mixed super quasi-umbilical hypersurface of a manifold of constant curvature is a manifold of generalized mixed super quasi-constant curvature. Further, it is proved that a manifold of generalized mixed super quasi-constant curvature is a mixed super quasi-Einsten manifold. We have also introduced a sufficient condition to be a mixed super quasi-Einstein manifold. Finally, we construct an example of a mixed super quasi-Einstein manifold in a general form.

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1 Introduction

An n-dimensional Riemannian manifold M is said to be an Einstein manifold if

$$S(Y,Z) = \frac{r}{n} g(Y,Z), \tag{1.1}$$

where S, r and g denote the Ricci tensor, scalar curvature and the Riemannian metric respectively. In 2000, M. C. Chaki and R. K. Maity [4] introduced the concept of quasi-Einstein manifold. According to them [4], an n-dimensional Riemannian manifold is said to be a quasi-Einstein manifold if the Ricci tensor S has the form

$$S(Y,Z) = ag(Y,Z) + b\omega(Y)\omega(Z), \tag{1.2}$$

where a, b are non-zero scalars and ω is a 1-form associated with a vector field ρ by $\omega(Y) = g(Y, \rho)$.

The quasi-Einstein manifold is generalized by several authors under the different conditions on Ricci tensor S. These generalizations are the generalized quasi-Einstein

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manifold [6, 7, 5], mixed generalized quasi-Einstein manifold [1], super quasi-Einstein manifold [3, 9], mixed super quasi-Einstein manifold [2], nearly quasi-Einstein manifold [8, 10], pseudo quasi-Einstein manifold [13] and the mixed quasi-Einstein manifold [11].

The idea of generalized quasi-Einstein manifold is given by U. C. De and G. C. Ghosh [6]. In generalized quasi-Einstein manifold the Ricci tensor has the form

$$S(Y,Z) = ag(Y,Z) + b\omega(Y)\omega(Z) + cA(Y)YA(Z), \tag{1.3}$$

where ω and A are the 1-forms defined by $\omega(Y) = g(Y,\rho)$ and $A(Y) = g(Y,\mu)$ such that $g(\rho,\mu) = 0$, ρ and μ are unit vector fields. In 2011, U. C. De and S. Mallick [7] proved the existence of generalized quasi-Einstein manifold with some examples.

In [1] Bhattacharya, De and Debnath extended the condition of quasi-Einstein manifold and defined a mixed generalized quasi-Einstein manifold. In such a development U. C. De and A. K. Gazi [8] introduced a nearly quasi-Einstein manifold satisfying the condition

$$S(Y,Z) = ag(Y,Z) + bD(Y,Z), \tag{1.4}$$

for a tensor field D of type (0,2). The same authors [10] also discussed the existence of such manifold with quasi-constant curvature.

The super quasi-Einstein manifold is introduced by M. C. Chaki [3] and studied by P. Debnath and A. Konar [9]. Debnath and Konar [9] considered the conformally flat and conformally conservative property and construct some examples on super quasi-Einstein manifold.

In 2008, A. Bhattacharya, M. Tarafdar and D. Debnath [2] defined a mixed super quasi-Einstein manifold. If the Ricci tensor S of a Riemannian manifold M satisfies the condition

$$\begin{aligned} S(Y,Z) = {} & ag(Y,Z) + b\omega(Y)\omega(Z) + cA(Y)A(Z) \\ & + d[\omega(Y)A(Z) + \omega(Z)A(Y)] + eD(Y,Z), \end{aligned} \tag{1.5}$$

then M is called a mixed super quasi-Einstein manifold. Further, we know that if the Riemannian curvature tensor R of type (0,4) has the form

$$R(X,Y,Z,U) = k[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)], \tag{1.6}$$

then manifold is said to be of constant curvature k. The generalization of this manifold is the manifold of quasi-constant curvature and in this case the curvature tensor has the following form

$$\begin{aligned} R(X,Y,Z,U) = {} & f_1[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] \\ & + f_2[g(Y,Z)\omega(X)\omega(U) - g(X,Z)\omega(Y)\omega(U) \\ & + g(X,U)\omega(Y)\omega(Z) - g(Y,U)\omega(X)\omega(Z)]. \end{aligned} \tag{1.7}$$

In 2004, U. C. De and G. C. Ghosh [6] introduced the concept of generalized quasi-constant curvature. A Riemannian manifold is said to be a manifold of generalized quasi-constant curvature if the curvature tensor satisfies

$$
\begin{aligned}
R(X,Y,Z,U) &= f_1[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] \\
&+ f_2[g(Y,Z)\omega(X)\omega(U) - g(X,Z)\omega(Y)\omega(U) \\
&+ g(X,U)\omega(Y)\omega(Z) - g(Y,U)\omega(X)\omega(Z)] \\
&+ f_3[g(Y,Z)A(X)A(U) - g(X,Z)A(Y)A(U) \\
&+ g(X,U)A(Y)A(Z) - g(Y,U)A(X)A(Z)].
\end{aligned} \tag{1.8}
$$

In 2008 Bhattacharya, Tarafdar and Debnath [2] defined a new type of quasi-constant curvature called the mixed super quasi-constant curvature as follows:

$$
\begin{aligned}
R(X,Y,Z,U) &= f_1[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] \\
&+ f_2[g(Y,Z)\omega(X)\omega(U) - g(X,Z)\omega(Y)\omega(U) \\
&+ g(X,U)\omega(Y)\omega(Z) - g(Y,U)\omega(X)\omega(Z)] \\
&+ f_3[g(Y,Z)A(X)A(U) - g(X,Z)A(Y)A(U) \\
&+ g(X,U)A(Y)A(Z) - g(Y,U)A(X)A(Z)] \\
&+ f_4[g(Y,Z)\{\omega(X)A(U) + \omega(U)A(X)\} \\
&- g(X,Z)\{\omega(Y)A(U) + \omega(U)A(Y)\} \\
&+ g(X,U)\{\omega(Y)A(Z) + \omega(Z)A(Y)\} \\
&- g(Y,U)\{\omega(X)A(Z) + \omega(Z)A(X)\}] \\
&+ f_5[D(Y,Z)g(X,U) - D(X,Z)g(Y,U) \\
&+ g(Y,Z)D(X,U) - g(X,Z)D(Y,U)].
\end{aligned} \tag{1.9}
$$

By such generalization of quasi-constant curvature, we are motivated to study the generalized mixed super quasi-constant curvature. In this paper, we have introduced the concept of generalized mixed super quasi-constant curvature and the existence of such a curvature has been verified.

Definition 1.1. A non-flat Riemannian manifold is said to be a manifold of generalized mixed super quasi-constant curvature if the curvature tensor R of type (0,4) has the following form:

$$
\begin{aligned}
R(X,Y,Z,U) = {} & f_1[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] \\
& + f_2[g(Y,Z)\omega(X)\omega(U) - g(X,Z)\omega(Y)\omega(U) \\
& + g(X,U)\omega(Y)\omega(Z) - g(Y,U)\omega(X)\omega(Z)] \\
& + f_3[g(Y,Z)A(X)A(U) - g(X,Z)A(Y)A(U) \\
& + g(X,U)A(Y)A(Z) - g(Y,U)A(X)A(Z)] \\
& + f_4[g(Y,Z)\{\omega(X)A(U) + \omega(U)A(X)\} \\
& - g(X,Z)\{\omega(Y)A(U) + \omega(U)A(Y)\} \\
& + g(X,U)\{\omega(Y)A(Z) + \omega(Z)A(Y)\} \\
& - g(Y,U)\{\omega(X)A(Z) + \omega(Z)A(X)\}] \\
& + f_5[D(Y,Z)g(X,U) - D(X,Z)g(Y,U) \\
& + g(Y,Z)D(X,U) - g(X,Z)D(Y,U)] \\
& + f_6[D(Y,Z)\omega(X)\omega(U) - D(X,Z)\omega(Y)\omega(U) \\
& + D(X,U)\omega(Y)\omega(Z) - D(Y,U)\omega(X)\omega(Z)] \\
& + f_7[D(Y,Z)A(X)A(U) - D(X,Z)A(Y)A(U) \\
& + D(X,U)A(Y)A(Z) - D(Y,U)A(X)A(Z)] \\
& + f_8[D(Y,Z)\{\omega(X)A(U) + \omega(U)A(X)\} \\
& - D(X,Z)\{\omega(Y)A(U) + \omega(U)A(Y)\} \\
& + D(X,U)\{\omega(Y)A(Z) + \omega(Z)A(Y)\} \\
& - D(Y,U)\{\omega(X)A(Z) + \omega(Z)A(X)\}] \\
& + f_9[A(Y)A(Z)\omega(X)\omega(U) - A(X)A(Z)\omega(Y)\omega(U) \\
& + A(X)A(U)\omega(Y)\omega(Z) - A(Y)A(U)\omega(X)\omega(Z)] \\
& + f_{10}[D(Y,Z)D(X,U) - D(X,Z)D(Y,U)],
\end{aligned} \tag{1.10}
$$

where $f_i, (1 \le i \le 10)$ are the scalars, ω and A are the 1-forms, D is a tensor field of type (0,2) satisfying the following

$$
\begin{aligned}
& \omega(Y) = g(Y,\rho), \quad A(Y) = g(Y,\mu), \quad g(\rho,\mu) = 0, \quad g(\rho,\rho) = g(\mu,\mu) = 1, \\
& D(X,\rho) = 0, \quad D(X,\mu) = 0, \quad trace(D) = D(e_i,e_i) = 0,
\end{aligned} \tag{1.11}
$$

for X, Y, Z, U being arbitrary vector fields.

2 Existence of generalized mixed super quasi-constant curvature

In this section, we have discussed the hypersurface of a Riemannian manifold and the existence of the generalized mixed super quasi-constant curvature defined by (1.10) has been proved in two different ways. Let $(M_n, \tilde{g})$ be a hypersurface of (M_{n+1}, g), for $\tilde{g}$ being induced metric on M_{n+1}. If H, A and ξ denote the second fundamental tensor, a tensor of type (1,1) and a unit normal vector field respectively then we have

$$
\tilde{g}(A_\xi(Y), Z) = g(H(Y,Z), \xi). \tag{2.1}
$$

Let H_ξ be a symmetric tensor of type (0,2) associated to A_ξ such that

$$\tilde{g}(A_\xi(Y), Z) = H_\xi(Y, Z). \tag{2.2}$$

Now, we define a new type of hypersurface called a mixed super quasi-umbilical hypersurface.

Definition 2.1. A hypersurface is said to be a mixed super quasi-umbilical hypersurface if the second fundamental tensor H_ξ has the form

$$\begin{aligned} H_\xi(Y, Z) &= \alpha_1 g(Y, Z) + \alpha_2 \omega(Y)\omega(Z) + \alpha_3 A(Y)A(Z) \\ &\quad + \alpha_4[\omega(Y)A(Z) + \omega(Z)A(Y)] + \alpha_5 D(Y, Z), \end{aligned} \tag{2.3}$$

where, $\alpha_i, 1 \leq i \leq 5$ are the scalars, ω and A are the 1-forms and D be a tensor field of type (0,2) satisfying (1.11).

Remark 2.2. Some conditions on α_i, $1 \leq i \leq 5$ give the following categories of the hypersurface.
(*i*) If $\alpha_i = 0$ for $1 \leq i \leq 5$ then hypersurface is called geodesics.
(*ii*) If $\alpha_i = 0$ for $2 \leq i \leq 5$ then hypersurface is said to be umbilical.
(*iii*) If $\alpha_i = 0$ for $i \neq 2$ or $i \neq 3$ then hypersurface is known as cylindrical.
(*iv*) If $\alpha_i = 0$ for $3 \leq i \leq 5$ then hypersurface is called quasi-umbilical.
(*v*) A hypersurface is said to be generalized quasi-umbilical if $\alpha_i = 0$ for $i = 4, i = 5$.
(*vi*) A hypersurface is known as mixed generalized quasi-umbilical if $\alpha_5 = 0$.
(*vii*) If $\alpha_4 = 0$ then hypersurface is called super quasi-umbilical.
(*viii*) If $\alpha_i = 0$ for $2 \leq i \leq 4$ then hypersurface is called nearly quasi-umbilical.

Let the hypersurface be a mixed super quasi-umbilical hypersurface then equations (2.1), (2.2) and (2.3) together yield

$$\begin{aligned} H(Y, Z) &= \alpha_1 g(Y, Z)\xi + \alpha_2 \omega(Y)\omega(Z)\xi + \alpha_3 A(Y)A(Z)\xi \\ &\quad + \alpha_4[\omega(Y)A(Z) + \omega(Z)A(Y)]\xi + \alpha_5 D(Y, Z)\xi. \end{aligned} \tag{2.4}$$

According to [12] the Gauss equation for the hypersurface is given by

$$\tilde{R}(X, Y, Z, U) = R(X, Y, Z, U) + g(H(X, U), H(Y, Z)) - g(H(Y, U), H(X, Z)), \tag{2.5}$$

where $\tilde{R}$ denotes the curvature tensor of the hypersurface.

Let us suppose the manifold is of constant curvature then by using (1.6) and (2.4) in (2.5), we have

$$
\begin{aligned}
\tilde{R}(X,Y,Z,U) = {} & f_1[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] \\
& + f_2[g(Y,Z)\omega(X)\omega(U) - g(X,Z)\omega(Y)\omega(U) \\
& + g(X,U)\omega(Y)\omega(Z) - g(Y,U)\omega(X)\omega(Z)] \\
& + f_3[g(Y,Z)A(X)A(U) - g(X,Z)A(Y)A(U) \\
& + g(X,U)A(Y)A(Z) - g(Y,U)A(X)A(Z)] \\
& + f_4[g(Y,Z)\{\omega(X)A(U) + \omega(U)A(X)\} \\
& - g(X,Z)\{\omega(Y)A(U) + \omega(U)A(Y)\} \\
& + g(X,U)\{\omega(Y)A(Z) + \omega(Z)A(Y)\} \\
& - g(Y,U)\{\omega(X)A(Z) + \omega(Z)A(X)\}] \\
& + f_5[D(Y,Z)g(X,U) - D(X,Z)g(Y,U) \\
& + g(Y,Z)D(X,U) - g(X,Z)D(Y,U)] \\
& + f_6[D(Y,Z)\omega(X)\omega(U) - D(X,Z)\omega(Y)\omega(U) \\
& + D(X,U)\omega(Y)\omega(Z) - D(Y,U)\omega(X)\omega(Z)] \\
& + f_7[D(Y,Z)A(X)A(U) - D(X,Z)A(Y)A(U) \\
& + D(X,U)A(Y)A(Z) - D(Y,U)A(X)A(Z)] \\
& + f_8[D(Y,Z)\{\omega(X)A(U) + \omega(U)A(X)\} \\
& - D(X,Z)\{\omega(Y)A(U) + \omega(U)A(Y)\} \\
& + D(X,U)\{\omega(Y)A(Z) + \omega(Z)A(Y)\} \\
& - D(Y,U)\{\omega(X)A(Z) + \omega(Z)A(X)\}] \\
& + f_9[A(Y)A(Z)\omega(X)\omega(U) - A(X)A(Z)\omega(Y)\omega(U) \\
& + A(X)A(U)\omega(Y)\omega(Z) - A(Y)A(U)\omega(X)\omega(Z)] \\
& + f_{10}[D(Y,Z)D(X,U) - D(X,Z)D(Y,U)],
\end{aligned} \tag{2.6}
$$

where

$$
\begin{aligned}
& f_1 = (k + \alpha_1^2), \quad f_2 = \alpha_1\alpha_2, \quad f_3 = \alpha_1\alpha_3, \quad f_4 = \alpha_1\alpha_4, \quad f_5 = \alpha_1\alpha_5, \\
& f_6 = \alpha_2\alpha_5, \quad f_7 = \alpha_3\alpha_5, \quad f_8 = \alpha_4\alpha_5, \\
& f_9 = \alpha_4^2 + \alpha_2\alpha_3 \quad f_{10} = \alpha_5^2.
\end{aligned}
$$

Thus, we can state

Theorem 2.1. *A mixed super quasi-umbilical hypersurface of a manifold of constant curvature is a manifold of generalized mixed super quasi-constant curvature.*

Again, we have the following

Theorem 2.2. *A manifold of generalized mixed super quasi-constant curvature is a mixed super quasi-Einstein manifold.*

Proof. By substituting $X = U = e_i$, $1 \le i \le n$ and summing over i, equation (2.6) leads to

$$
\begin{aligned}
\tilde{S}(Y,Z) = {} & \alpha g(Y,Z) + \beta\omega(Y)\omega(Z) + \gamma A(Y)A(Z) \\
& + \delta[\omega(Y)A(Z) + \omega(Z)A(Y)] + \epsilon E(Y,Z),
\end{aligned} \tag{2.7}
$$

where

$$\alpha = (n-1)f_1 + f_2 + f_3, \quad \beta = (n-2)f_2 + f_9, \quad \gamma = (n-2)f_3 + f_9,$$
$$\delta = (n-2)f_4, \quad \epsilon E(Y,Z) = [(n-2)f_5 + f_6 + f_7]D(Y,Z) - f_{10}D(TY,Z)$$
$$\textit{such that} \quad D(Y,Z) = g(TY,Z).$$

for a tensor field T of type (1,1). □

Also, from the theorems (2.3) and (2.4), we conclude the following

Corollary 2.3. *A mixed super quasi-umbilical hypersurface of a manifold of constant curvature is a mixed super quasi-Einstein manifold.*

Now, we will construct an example of generalized mixed super quasi-constant curvature to verify the definition given in (1.10).

Let us define a metric g by

$$ds^2 = e^{x^2}(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2. \tag{2.8}$$

The non-zero components of the christoffel symbols of second kind Γ^i_{jk} are given by

$$\Gamma^1_{12} = \frac{1}{2}, \quad \Gamma^2_{11} = -\frac{1}{2}e^{x^2}. \tag{2.9}$$

The non-zero components of the curvature tensor R_{ijkh} are given by

$$R_{1221} = R_{2112} = \frac{1}{4}e^{x^2} \tag{2.10}$$

For the non-zero components of the curvature tensor R_{ijkh}, equation (1.10) reduces to

$$\begin{aligned}
R_{1221} &= f_1[g_{11}g_{22} - g_{12}g_{21}] + f_2[g_{22}\omega_1\omega_1 - g_{12}\omega_1\omega_2 + g_{11}\omega_2\omega_2 - g_{21}\omega_1\omega_2] \\
&+ f_3[g_{22}A_1A_1 - g_{12}A_1A_2 + g_{11}A_2A_2 - g_{21}A_1A_2] \\
&+ f_4[2g_{22}\omega_1A_1 - g_{12}(\omega_1A_2 + \omega_2A_1) + 2g_{11}\omega_2A_2 - g_{21}(\omega_1A_2 + \omega_2A_1)] \\
&+ f_5[D_{22}g_{11} - D_{12}g_{21} + D_{11}g_{22} - D_{21}g_{12}] \\
&+ f_6[D_{22}\omega_1\omega_1 - D_{12}\omega_1\omega_2 + D_{11}\omega_2\omega_2 - D_{21}\omega_1\omega_2] \\
&+ f_7[D_{22}A_1A_1 - D_{12}A_1A_2 + D_{11}A_2A_2 - D_{21}A_1A_2] \\
&+ f_8[2D_{22}\omega_1A_1 - D_{12}(\omega_1A_2 + \omega_2A_1) + 2D_{11}\omega_2A_2 - D_{21}(\omega_1A_2 + \omega_2A_1] \\
&+ f_9[\omega_1\omega_1A_2A_2 - \omega_1\omega_2A_1A_2 + \omega_2\omega_2A_1A_1 - \omega_1\omega_2A_1A_2] \\
&+ f_{10}[D_{11}D_{22} - D_{12}D_{21}].
\end{aligned} \tag{2.11}$$

If we set

(2.12)

$$\omega_1 = A_1 = \sqrt{e^{x^2}}, \quad \omega_2 = 1 = A_2,$$
$$D_{11} = -e^{x^2}, \quad D_{22} = e^{x^2}, \quad D_{12} = \sqrt{e^{x^2}}, \quad D_{21} = -\sqrt{e^{x^2}}$$
$$f_2 = f_3 = -f_4, \quad f_5 = f_{10}, \quad f_6 = -f_7, \quad f_1 = \frac{1}{4}[1 - 8f_8(e^{x^2}+1)], \quad f_9 \ \textit{is arbitrary},$$

and using the values of g_{ij} from (2.8), the right hand side of (2.11) reduces to $\frac{1}{4}e^{x^2}$, which is equal to right hand side of (2.10). Thus, it satifies the condition of generalized mixed super quasi-constant curvature.
Hence, we can state

Theorem 2.4. *A 4-dimensional Riemannian manifold equipped with a metric* $ds^2 = e^{x^2}(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2$ *is a manifold of generalized mixed super quasi-constant curvature.*

3 Mixed super quasi-Einstein manifold

As the Theorem 2.4 explained that a manifold of generalized mixed super quasi-constant curvature is a mixed super quasi-Einstein manifold. In this sequence, we have introduced a sufficient condition for the existence of mixed super quasi-Einstein manifold which is different from the condition given in [2].

Theorem 3.1. *A sufficient condition for a Riemannian manifold to be a mixed super quasi-Einstein manifold is that the Ricci tensor S is of the form*

$$\begin{aligned} S(Y,Z)S(X,U) - S(X,Z)S(Y,U) = f_1[S(Y,U)g(X,Z) + S(X,Z)g(Y,U)] \\ + f_2[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)] + f_3S(X,U)D(Y,Z), \end{aligned} \tag{3.1}$$

for a tensor field D of type (0,2).

Proof. Let the Riemannian manifold satisfies the condition (3.1).
Putting $X = U = \rho$ in (3.1), we get

$$\begin{aligned} S(Y,Z) = k_1g(Y,Z) + k_2\omega(Y)\omega(Z) + k_3\omega(LY)\omega(LZ) \\ + k_4[\omega(Y)\omega(LZ) + \omega(Z)\omega(LY)] + k_5D(Y,Z), \end{aligned} \tag{3.2}$$

where

$$k_1 = \frac{f_2g(\rho,\rho)}{S(\rho,\rho)}, \quad k_2 = \frac{-f_2}{S(\rho,\rho)}, \quad k_3 = \frac{1}{S(\rho,\rho)},$$
$$k_4 = \frac{f_1}{S(\rho,\rho)}, \quad k_5 = f_3 \ \textit{and} \ \ S(Y,Z) = g(LY,Z).$$

We take $\omega(LY) = A(Y)$ for 1-form A then (3.2) can be written as

$$\begin{aligned} S(Y,Z) = k_1g(Y,Z) + k_2\omega(Y)\omega(Z) + k_3A(Y)A(Z) \\ + k_4[\omega(Y)A(Z) + \omega(Z)A(Y)] + k_5D(Y,Z), \end{aligned} \tag{3.3}$$

which is same as (1.5). Hence, a Riemannian manifold satisfying (3.1) is a mixed super quasi-Einstein manifold.
This completes the proof. □

Further, in the same sequence we will construct an example of mixed super quasi-Einstein manifold in general form as follows:

Example 3.1. Let us define a metric g by

$$ds^2 = f(x^2)(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2, \tag{3.4}$$

for a non-vanishing arbitrary function f.
The non-zero components of the christoffel symbols of second kind Γ^i_{jk} are given by

$$\Gamma^1_{12} = \frac{1}{2}\frac{f'(x^2)}{f(x^2)} \quad and \quad \Gamma^2_{11} = -\frac{1}{2}f'(x^2), \tag{3.5}$$

where f' denotes the derivative of arbirary function f with respect to x^2.
The non-zero components of the curvature tensor R_{ijkh} and the Ricci tensor R_{ij} are given by

$$\begin{aligned} R_{1221} &= R_{2112} = \frac{1}{2}f''(x^2) - \frac{1}{4}\frac{(f'(x^2))^2}{f(x^2)}, \\ R_{11} &= \frac{1}{2}f''(x^2) - \frac{1}{4}\frac{(f'(x^2))^2}{f(x^2)}, \\ R_{22} &= \frac{1}{f(x^2)}\Big[\frac{1}{2}f''(x^2) - \frac{1}{4}\frac{(f'(x^2))^2}{f(x^2)}\Big]. \end{aligned} \tag{3.6}$$

For the non-vanishing components of the Ricci tensor R_{ij}, the condition of mixed super quasi-Einstein manifold given in (1.5) reduces to

$$\begin{aligned} (i)R_{11} &= ag_{11} + b\omega_1\omega_1 + cA_1A_1 + 2d\omega_1A_1 + eD_{11}, \\ (ii)R_{22} &= ag_{22} + b\omega_2\omega_2 + cA_2A_2 + 2d\omega_2A_2 + eD_{22}. \end{aligned} \tag{3.7}$$

If we take

$$a = -\frac{f'(x^2)}{f(x^2)\sqrt{f(x^2)}}, \quad b = \frac{1}{2}, \quad c = -\frac{1}{4}, \quad d = \frac{1}{\sqrt{f''(x^2)}}, \quad e = \frac{1}{\sqrt{f(x^2)}}, \tag{3.8}$$

$$\begin{aligned} &\omega_1 = \sqrt{f''(x^2)}, \quad \omega_2 = \sqrt{\frac{f''(x^2)}{f(x^2)}}, \quad A_1 = \frac{f'(x^2)}{\sqrt{f(x^2)}}, \\ &A_2 = \frac{4}{\sqrt{f(x^2)}} + \sqrt{\frac{16}{f(x^2)} + \frac{4f'(x^2)}{\sqrt{f(x^2)}} - \frac{4f'(x^2)}{\sqrt{f(x^2)}f(x^2)} + \frac{[f'(x^2)]^2}{[f(x^2)]^2}}, \end{aligned} \tag{3.9}$$

$$D_{11} = -f'(x^2), \quad D_{22} = f'(x^2). \tag{3.10}$$

then we can easily verify that the equation (3.7) holds good for both the conditions (i) and (ii).
In particular, If we take $f(x^2) = e^{x^2}$ then (3.5), (3.6), (3.8), (3.9) and (3.10) reduces to

$$\Gamma^1_{12} = \frac{1}{2}, \quad \Gamma^2_{11} = -\frac{1}{2}e^{x^2}, \tag{3.11}$$

$$R_{1221} = R_{2112} = \frac{1}{4}e^{x^2}, \quad R_{11} = \frac{1}{4}e^{x^2}, \quad R_{22} = \frac{1}{4}, \tag{3.12}$$

$$a = -\frac{1}{\sqrt{e^{x^2}}}, \quad b = \frac{1}{2}, \quad c = -\frac{1}{4}, \quad d = e = \frac{1}{\sqrt{e^{x^2}}}, \tag{3.13}$$

$$\omega_1 = A_1 = \sqrt{e^{x^2}}, \quad \omega_2 = 1, \quad A_2 = \frac{4}{\sqrt{e^{x^2}}} + \sqrt{\frac{16}{e^{x^2}} + 4\sqrt{e^{x^2}} - \frac{4}{\sqrt{e^{x^2}}} + 1}, \tag{3.14}$$

$$D_{11} = -e^{x^2}, \quad D_{22} = e^{x^2}. \tag{3.15}$$

By the help of (3.12), (3.13), (3.14) and (3.15), it can be again verify that the equation (3.7) holds for both the conditions (i) and (ii).
Hence, this is an example of mixed super quasi-Einstein manifold.
Thus, we have

Theorem 3.2. *Let M be a 4-dimensional Riemannian manifold equipped with a metric $ds^2 = f(x^2)(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2$. Then it is a mixed super quasi-Einstein manifold.*

4 Discussion and conclusions

In this paper, we have generalized the work of Bhattacharya, Tarafdar and Debnath [2] and tried to extend the concept of quasi-constant curvature on a Riemannian manifold. We have proved the existence of generalized mixed super quasi-constant curvature by theoretical as well as by an example. In 2008 Bhattacharya, Tarafdar and Debnath [2] introduced a sufficient condition to be a mixed super quasi-Einstein manifold. We have also introduced a sufficient condition to be a mixed super quasi-Einstein manifold, which is different from [2]. Finally, we have constructed an example of mixed super quasi-Einstein manifold in a general form in terms of a non-vanishing arbitrary function. These investigated manifolds are very important in the study of general relativity; in particular, they are relevant in the study of rotating universe, space-time and the study of perfect fluids.

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Authors' addresses:

B.B. Chaturvedi
Department of Pure and Applied Mathematics,
Guru Ghasidas Vishwavidyalaya Bilaspur (C.G.)-495009, India.
Email: brajbhushan25@gmail.com

Pankaj Pandey
Department of Mathematics,
Invertis University, Bareilly (U.P.)-243006, India.
Email: pankaj.anvarat@gmail.com