Almost pseudo product structure

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Abstract. The main objective of the paper is to study a new type of structure named as almost pseudo product structure in an n-dimensional Riemannian manifold. Some results involving this structure have been established. The existence of this type of structure is shown with examples. It has also been shown that, every Einstein manifold is an almost product manifold and if the sum of the associated scalars of a quasi Einstein manifold is zero, the manifold is an almost paracontact manifold.

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1 Introduction

Almost product structure on a differentiable manifold were investigated by A.G. Walker [8], Willmore [9], Yano [10], [11] and others [6]. An almost paracontact structure on a differentiable manifold was introduced by Sato [7]. The structure is closely related to almost contact structure [1] and almost product structure. Einstein manifolds have an important role in Riemannian and semi-Riemannian Geometry. Many of the authors have investigated on Einstein manifold equipped with almost product and almost complex structure.

The notion of Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces. For instance, the Robertson-Walker spacetimes are quasi-Einstein manifolds. Many authors investigated different properties of Quasi Einstein manifold [2], [3], [5]. In [4], U. C. De and G. C. Ghosh introduced generalized quasi-Einstein manifold and showed that a 2-quasi umbilical hypersurface of an Euclidean space is a generalized quasi-Einstein manifold. At the time of investigation on structures on manifolds, the author found that every Einstein manifold admits an almost product structure and similarly every quasi Einstein manifold admits an almost paracontact structure provided the sum of the associated scalars is zero. Inspired by these results, at the time of investigation on generalized quasi Einstein manifold, the author felt the importance to introduce the new structure, named as almost pseudo product structure. This paper is divided in four sections. After introduction in section one and definitions in section two, in section three, it has been shown that every Einstein manifold always

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admits an almost product structure and thus it is an almost product manifold. It has been also proved that, if the sum of the associated scalars of a quasi Einstein manifold is zero, the manifold admits an almost paracontact structure. In the last section, the author has defined and discussed about the properties of almost pseudo product structure with examples.

2 Definitions

Let $(M^n, g), n \geq 2$ be a Riemannian manifold. Let us denote the Ricci Tensor of type (0,2) of (M, g) by Ric. We call M an Einstein manifold if, for every vector field X, Y on M, there exists a real constant λ such that

$$Ric(X, Y) = \lambda g(X, Y). \tag{2.1}$$

Let L be the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor Ric of type (0,2) defined by

$$g(LX, Y) = Ric(X, Y), \tag{2.2}$$

for all vector fields X, Y.
Let M be an n-dimensional differentiate manifold and M_x be the tangent space at each point x of the manifold M. If there exist an endomorphism F on each tangent space M_x such that

$$F^2 = I,$$

we say that the mixed tensor of type $(1, 1)$, F gives an almost product structure to the manifold and we call the manifold an almost product manifold.

A non-flat n-dimensional Riemannian manifold (M_n, g), $n \geq 3$ is said to be a quasi Einstein manifold if its Ricci tensor of type (0, 2) is not identically zero and satisfies the condition

$$Ric(X, Y) = ag(X, Y) + bw(X)w(Y), \tag{2.3}$$

where a, b as scalars and w is a non-zero 1-form, metrically equivalent to the unit vector field U i.e., for all vector fields X

$$g(X, U) = w(X), \qquad g(U, U) = 1. \tag{2.4}$$

I. Sato [7], introduced the concept of a structure similar to the almost contact structure, which is known as almost paracontact structure. A differentiable manifold with structure tensors (ϕ, ξ, η) where ϕ is a (1, 1) type tensor, ξ is a vector field and η is a 1-form on the manifold satisfying

$$\phi^2(X) = X - \eta(X)\xi, \qquad \phi(\xi) = 0,$$

for any vector field X, is said to be an almost paracontact manifold.

As a generalization of quasi Einstein manifold, in [4], U.C. De and G.C. Ghosh introduced the notion of generalized quasi Einstein manifold. A Riemannian manifold

(M_n, g), $n \geq 3$ is said to be generalized quasi-Einstein manifold if its Ricci tensor of type (0, 2) is not identically zero and satisfies:

$$Ric(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y), \tag{2.5}$$

where a, b, c are scalars and A and B are two non-zero 1-forms, metrically equivalent to the unit vector field U and V respectively, i.e., for all vector fields X

$$g(X,U) = A(X), \quad g(U,U) = 1, \quad g(X,V) = B(X), \quad g(V,V) = 1, \quad g(U,V) = 0. \tag{2.6}$$

3 Structures on Einstein and quasi Einstein manifolds

3.1 Every Einstein manifold admits an almost product structure

Theorem 3.1. *Every Einstein manifold always admits an almost product structure and therefore every Einstein manifold is an almost product manifold.*

Proof. Using equations (2.1) and (2.2), we have

$$g(LX,Y) = \lambda g(X,Y). \quad LX = \lambda X.$$

Therefore,

$$L(L)X = L^2 X = \lambda^2 X. \tag{3.1}$$

Now, let us consider an endomorphism F on each tangent space M_x such that

$$F(X) = \frac{1}{\lambda} L(X)$$

So, we have

$$F(F(X)) = F(\frac{1}{\lambda} L(X))$$

i.e.,

$$F(F(X)) = \frac{1}{\lambda^2} L^2(X)$$

Now, using equation (3.1) we get

$$F^2(X) = X,$$

which is an almost product structure. So, we see that an Einstein manifold admits an almost product structure with structure tensors F. □

Corollary 3.2. *The almost product structure on an Einstein manifold is not unique.*

Proof. If we consider the $(1,1)$ tensor field as

$$\Psi(X) = -\frac{1}{\lambda}L(X),$$

we have the same result, i.e.,

$$\Psi^2(X) = X,$$

where $F = -\Psi$ and consequently, the almost product structure on Einstein manifold is not unique. □

Corollary 3.3. *The almost product structure on Einstein manifold defines two complementary distributions on Einstein manifold globally [10].*

Proof. Let us consider

$$P = \frac{1}{2}(I + F), \qquad Q = \frac{1}{2}(I - F).$$

Then we have

$$P + Q = I \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0 \quad F = P - Q.$$

Thus P and Q globally define two complementary distributions. □

3.2 On quasi Einstein manifolds

Theorem 3.4. *Every quasi Einstein manifold with sum of the associated scalars zero, always admits an almost paracontact structure and therefore every quasi Einstein manifold with sum of the associated scalars zero, is an almost paracontact manifold.*

Proof. From equation (2.2) and equation (2.3), we infer

$$g(LX, Y) = ag(X, Y) + bw(X)w(Y).$$

Therefore, we have

$$L(X) = aX + bw(X)U. \tag{3.2}$$

Now, if sum of the associated scalars is zero, i.e. $a + b = 0$, we get

$$L(U) = (a + b)(U) = 0.$$

Further, equation (3.2) leads to

$$\begin{aligned} L(L(X)) &= L(aX + bw(X)U), \\ &= aL(X) + bw(X)LU, \end{aligned}$$

i.e.,

$$L^2(X) = a^2X + abw(X)U.$$

Now, using $a + b = 0$, we get

$$L^2(X) = a^2(X - w(X)U). \tag{3.3}$$

We further consider an endomorphism ϕ at each point of the tangent space M_x, such that

$$\phi(X) = \frac{1}{a} L(X). \tag{3.4}$$

So,

$$\phi^2(X) = \frac{1}{a^2} L^2(X).$$

By using equation (3.3), we get

$$\phi^2(X) = X - w(X)U.$$

Again, using equation (3.2) and equation (3.4), we get

$$\phi(U) = 0.$$

So, we see that a quasi Einstein manifold with sum of the associated scalars zero admits an almost paracontact structure with structure tensors (ϕ, U, w) where ϕ is a (1, 1) type tensor, U is a vector field and w is a 1-form on the manifold satisfying $\phi^2(X) = X - w(X)U$ and $\phi(U) = 0$. □

We know that, if the associated scalars a, b of a quasi Einstein hypersurface of an Euclidean space has the property $a+b=0$, or its generator vector field U is a parallel vector field, then the manifold is odd dimensional[5]. Thus we have

Corollary 3.5. *Considr a quasi Einstein hypersurface of Euclidean space with $a+b=0$, or with the generator vector field U parallel vector field. Then the quasi Einstein hypersurface of the Euclidean space is an odd dimensional almost paracontact manifold.*

4 On almost pseudo product structure

Let M_n be an $n(\geq 4)$ dimensional manifold and ψ be a tensor field of type $(1,1)$, U and V be two linearly independent vector fields and A, B be two non-zero 1-forms respectively. If (ψ, A, B, U, V) satisfy the conditions

$$\psi^2(X) = X - A(X)U - B(X)V, \tag{4.1}$$

$$\psi(U) = 0, \quad \psi(V) = 0, \tag{4.2}$$

for any vector field X. Then M_n is said to admit an *almost pseudo product structure*, (ψ, A, B, U, V) and such a manifold M_n is called an *almost pseudo product manifold*.

If the sum of the associated vector fields is zero, i.e., $U + V = 0$, then it becomes an almost paracontact structure of dimension $n-1$ with one form $\eta = (A - B)$.

4.1 Preliminaries

For any vector field X in an almost pseudo product manifold M_n, we have

$$\psi^2(X) = X - A(X)U - B(X)V.$$

Now, operating ψ from left and using equation (4.2) we get

$$\psi^3(X) = \psi(X). \tag{4.3}$$

Again replacing X by $\psi(X)$, in equation (4.1), we get

$$\psi^3(X) = \psi(X) - A(\psi(X))U - B(\psi(X))V. \tag{4.4}$$

By comparing the equations (4.3) and (4.4), we have $A(\psi(X))U + B(\psi(X))V = 0$, but U and V are two linearly independent vectors. Thus we have

$$A(\psi(X)) = 0 \ \ \textit{and} \ B(\psi(X)) = 0 \ \ \textit{i.e.} \ \ A \circ \psi = 0 \ \ \textit{and} \ \ B \circ \psi = 0.$$

Moreover, $\psi(U) = 0 \Rightarrow \psi^2(U) = 0$, and hence, putting $X = U$ in equation (4.1), we get

$$\psi^2(U) = U - A(U)U - B(U)V = 0.$$

Since U and V are two linearly independent non zero vector fields, we get

$$A(U) = 1, \quad B(U) = 0.$$

Similarly, putting $X = V$, we get

$$A(V) = 0, \quad B(V) = 1.$$

We also have $\psi(U) = 0$ and $\psi(V) = 0$ in any almost pseudo product manifold, and U and V are linearly independent. Thus $rank\ \psi \leq n - 2$.

Let W be any other vector field with $\psi(W) = 0$. Therefore by equation (4.1), we have

$$W = A(W)U + B(W)V.$$

So, we see that W, U and V are linearly dependent. Hence $ker\ \psi$ is generated by U and V only, and therefore $rank\ \psi = n - 2$. This leads to the following theorem:

Theorem 4.1. *In an almost pseudo product manifold we have*
a) $A \circ \psi = 0$ *and* $B \circ \psi = 0$
b) $A(U)=1$, $B(U)=0$, $A(V)=0$, $B(V)=1$
c) Rank $\psi = n - 2$.

We will now show that the almost pseudo product structure is not unique. Let f be a non singular vector valued linear function on M_n.

Let us define the $(1,1)$ tensor field ψ^*, the 1-forms A^*, B^* and the unit vector fields U^*, V^* as

$$f \circ \psi^* = \psi \circ f, \tag{4.5}$$

$$A^* = A \circ f, \quad B^* = B \circ f, \tag{4.6}$$

$$(4.7) \qquad fU^* = U, \quad fV^* = V.$$

Now, post multiplying equation (4.5) by ψ^* and using equation (4.1) and equation (4.6), we get

$$\begin{aligned} f \circ \psi^{*2} &= \psi \circ f \circ \psi^* = \psi \circ (f \circ \psi^*) \\ &= \psi^2 \circ f \\ &= (I_n - A \otimes U - B \otimes V) \circ f \\ &= f - A^* \otimes U - B^* \otimes V \ . \end{aligned}$$

Applying equation (4.7), we infer

$$f \circ \psi^{*2} = f \circ (I_n - A^* \otimes U^* - B^* \otimes V^*) \ .$$

Since f is non singular, we have

$$\psi^{*2} = I_n - A^* \otimes U^* - B^* \otimes V^*.$$

Now, $f \circ \psi^* U^* = \psi \circ fU^* = \psi(U) = 0$, by equation (4.5) and (4.7)

$$\psi^* U^* = 0.$$

Similarly,

$$\psi^* V^* = 0.$$

Therefore, with the help of these, we can state the following theorem:

Theorem 4.2. *The almost pseudo product structure in an almost pseudo product manifold is not unique.*

4.2 Necessary and sufficient condition for being an almost pseudo product manifold

To find the necessary and sufficient condition for M_n to be an almost pseudo product manifold, we need the following results:

Theorem 4.3. *The eigenvalues of the structure tensor ψ are the roots of the equation $\alpha(\alpha^2 - 1) = 0$.*

Proof. Let α be the eigen value of ψ and ζ be the corresponding eigenvector. Then

$$\psi(\zeta) = \alpha\zeta \text{ and } \psi^2(\zeta) = \alpha^2\zeta \ .$$

Now, using equation (4.1), we get

$$(\alpha^2 - 1)\zeta + A(\zeta)U + B(\zeta)V = 0 \ .$$

Operating by ψ on this equation and using $A \circ \psi = B \circ \psi = 0$, we get $\alpha(\alpha^2 - 1) = 0$. □

Corollary 4.4. *The eigenvalues of ψ are 0, 1 and -1.*

Theorem 4.5. *The necessary and sufficient condition that a manifold M_n will be an almost pseudo product manifold is that at each point of the manifold M_n, it contains a tangent bundle Π_p of dimension p, a tangent bundle Π_q of dimension q and a tangent bundle Π_2 of dimension 2 such that $\Pi_p \cap \Pi_q = \{\Phi\}, \Pi_p \cap \Pi_2 = \{\Phi\}, \Pi_q \cap \Pi_2 = \{\Phi\}$ (where $\{\Phi\}$ is the null set) and $\Pi_p \cup \Pi_q \cup \Pi_2 =$ a tangent bundle of dimension n, projection L, M, N on Π_p, Π_q and Π_2 respectively being given by*

$$a)\ 2L = \psi^2 + \psi \qquad b)\ 2M = \psi^2 - \psi, \qquad c)\ N = -\psi^2 + I_n.$$

Proof. Let P_i be p linearly independent eigenvectors corresponding to the eigenvalue 1 of ψ, Q_j be q linearly independent eigenvectors corresponding to -1 and R_k be 2 linearly independent eigenvectors corresponding to the eigenvalue 0 respectively where $i = 1, 2, ...p$, $j = 1, 2, ...q$ and $k = 1, 2$ and $p + q + 2 = n$. Using Einstein's summation , we have $a^i P_i = 0 \Rightarrow a^i = 0$, $b^j Q_j = 0 \Rightarrow b^j = 0$ and $c^k R_k = 0 \Rightarrow c^k = 0$, for scalars a^i, b^j and c^k and for all i, j and k.

Now, let us consider the equation

$$a^i P_i + b^j Q_j + c^k R_k = 0, \tag{4.8}$$

where $i = 1, 2, ..., p$, $j = 1, 2, ..., q$ and $k = 1, 2$. Applying ψ on equation (4.8), we get

$$a^i \psi(P_i) + b^j \psi(Q_j) = 0 \tag{4.9}$$

$$\Rightarrow a^i P_i - b^j Q_j = 0.$$

Operating ψ once again, we get

$$a^i P_i + b^j Q_j = 0\ . \tag{4.10}$$

Thus, from equation (4.8), (4.9) and (4.10), we get

$$a^i P_i = b^j Q_j = c^k R_k = 0.$$

Therefore $a^i = b^j = c^k = 0$, i.e., $\{P_i, Q_j, R_k\}$ is a linearly independent set.

Now, let L, M, N be projection maps on Π_p, Π_q and Π_2 respectively and we see that

$$\begin{array}{lll} LP_i = P_i & LQ_j = 0 & LR_k = 0 \\ MP_i = 0 & MQ_j = Q_j & MR_k = 0 \\ NP_i = 0 & NQ_j = 0 & NR_k = R_k. \end{array}$$

Conversely, suppose that there is a tangent bundle Π_p , Π_q and Π_2 of dimension p, q and 2 respectively at each point of M_n, such that $\Pi_p \cap \Pi_q = \Pi_p \cap \Pi_2 = \Pi_q \cap \Pi_2 = \{\Phi\}$, also $\Pi_p \cup \Pi_q \cup \Pi_2 =$ a tangent bundle of dimension n. Let P_i, Q_j and U, V be p, q and two linearly independent vectors in Π_p, Π_q and Π_2 respectively, where $i = 1, 2, ..., p$ and $j = 1, 2, ..., q$. Let $\{P_i, Q_j, U, V\}$ span a tangent bundle of dimension n. Then $\{P_i, Q_j, U, V\}$ is a linearly independent set.

Let us define the inverse set $\left\{p'^i, q'^j, A, B\right\}$ such that

$$I_n = p'^i \otimes P_i + q'^j \otimes Q_j + A \otimes U + B \otimes V\ . \tag{4.11}$$

We define

$$\psi = p^{'i} \otimes P_i - q^{'j} \otimes Q_j.$$

Therefore

$$\psi^2 = p^{'i} \otimes P_i + q^{'j} \otimes Q_j.$$

Thus, by the help of equation (4.11), we infer

$$\psi^2 = I_n - A \otimes U - B \otimes V.$$

Thus, we see that M_n admits an almost pseudo product structure. Hence the condition is sufficient. □

Now, let us define a vector valued (1,1) tensor field $\psi^{'}$ such that

$$(\psi^{'})^2(X) = \psi^2(X) + A(X)U + B(X)V, \quad \text{and} \quad (\psi^{'})^2(X) = X.$$

So, we see that the almost pseudo product structure induces an almost product structure for all vector fields X in M_n. Thus we have the following theorem

Theorem 4.6. *Every almost pseudo product structure induces an almost product structure in an almost pseudo product manifold.*

Consider the Nijenhuis tensor $N(X,Y)$ of the induced almost product structure $\psi^{'}$, where

$$N(X,Y) = [\psi^{'}X, \psi^{'}Y] - \psi^{'}[\psi^{'}X, Y] - \psi^{'}[X, \psi^{'}Y] - [X,Y].$$

In N vanishes, i.e., the almost product structure becomes integrable, we will call the almost pseudo product structure a pseudo product structure.

Though the almost pseudo product structure always induces an almost product structure, the basic difference is that almost pseudo product structure is singular but almost product structure is non-singular.

4.3 Metric on almost pseudo product-manifold

Let us now try to find a metric on almost pseudo product manifold. We first prove the following Lemma:

Lemma 4.7. *Every almost pseudo product manifold M_n admits a Riemannian metric tensor field h, such that $h(X,U) = A(X)$ and $h(X,V) = B(X)$ for every vector field X on M_n.*

Proof. Since M_n admits a metric tensor field f (which exists, provided that M_n is paracompact), we obtain h by setting

$$h(X,Y) = f(-X+A(X)U+B(X)V,\ -Y+A(Y)U+B(Y)V)+A(X)A(Y)+B(X)B(Y).$$

Now, putting $Y = U$ and using theorem (4.1), we get

$$h(X,U) = A(X).$$

Similarly, by putting $Y = V$, we get

$$h(X,V) = B(X).$$

□

Theorem 4.8. *Every almost pseudo product manifold M_n admits a Riemannian metric tensor field g such that*

$$g(X,U) = A(X),\ \ g(X,V) = B(X) \tag{4.12}$$

and

$$g(\psi X, \psi Y) = g(X,Y) - A(X)A(Y) - B(X)B(Y)\ . \tag{4.13}$$

Proof. Let us put

$$g(X,Y) = \frac{1}{2}[h(X,Y) + h(\psi X, \psi Y) + A(X)A(Y) + B(X)B(Y)] \tag{4.14}$$

Now, using Lemma (4.7), it can be easily verified that $g(X,U) = A(X))$ and $g(X,V) = B(X)$. Again, from equation (4.14) and Theorem 4.1,

$$g(\psi X, \psi Y) = \frac{1}{2}[h(\psi X, \psi Y) + h(\psi^2 X, \psi^2 Y)],$$

whence we infer $g(\psi X, \psi Y) = g(X,Y) - A(X)A(Y) - B(X)B(Y)$. □

This leads to the following result:

Corollary 4.9. *The structure tensor ψ of the almost pseudo contact structure is symmetric with respect to the metric tensor field g.*

Proof. Putting ψY for Y in the equation (4.13)and (4.14) and using $A \circ \psi = B \circ \psi = 0$, we get $g(\psi X, Y) = g(X, \psi Y)$. □

4.4 Example of almost pseudo product structure in 4-dimensional Euclidean space

Example 4.1. Let R_4 be any 4-dimensional Euclidean space and let us define

$$\psi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\ .$$

So, $\psi^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ Now, let us choose $U = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $V = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ and correspondingly $A = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}$.

Thus $A \otimes U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $B \otimes V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

Therefore $\psi^2 = I_4 - A \otimes U - B \otimes V$. Again,

$$\psi(U) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\psi(V) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Thus, we conclude that the structure is an almost pseudo product structure and R_4 is an almost pseudo product manifold.

Example 4.2. Let us consider a generalized quasi Einstein manifold (M_n, g), $n \geq 4$ with Ricci tensor of type (0, 2) is not identically zero and satisfying

$$Ric(X,Y) = ag(X,Y) + bA(X)A(Y) + cB(X)B(Y),$$

where a, b, c are scalars with the property $a + b = 0$ and $a + c = 0$. A and B are two non-zero 1-forms, metrically equivalent to the unit vector field U and V respectively, i.e. for all vector fields X

$$g(X,U) = A(X), \quad g(U,U) = 1, \quad g(X,V) = B(X), \quad g(V,V) = 1, \quad g(U,V) = 0.$$

Using (2.2) and (2.5), we get

$$LX = aX + bA(X)U + cB(X)V.$$

Therefore

$$L^2X = a^2X + 2abA(X)U + 2acB(X)V + b^2A(X)U + b^2B(X)V.$$

Then, using $a + b = 0$ and $a + c = 0$, we infer

$$L^2X = a^2(X - A(X)U - B(X)V).$$

Now, let us consider an endomorphism ψ on each tangent space M_n such that

$$\psi(X) = \frac{1}{a}L(X).$$

Therefore, we see that

$$\psi^2(X) = X - A(X)U - B(X)V,$$

and that $\psi(U) = 0$ and $\psi(V) = 0$. Hence the mixed tensor of type $(1, 1)$, ψ gives an almost pseudo product structure on the manifold. So, we conclude that every $G(QE)_n$ is an almost pseudo product manifold.

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