On Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold

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Abstract. The aim of this paper is to study Bochner Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold and a holomorphically projective Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold.

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1 Introduction

An even dimensional differentiable manifold M^n is said to be a Hermitian manifold if a complex structure J of type $(1,1)$ and a pseudo-Riemannian metric g of the manifold satisfy

$$J^2 = -I,$$

and

$$g(JX, JY) = g(X, Y), \tag{1.1}$$

where $X, Y \in \chi(M)$ and $\chi(M)$ is the Lie algebra of vector fields on the manifold. The notion of an Einstein manifold was introduced by Albert Einstein in differential geometry and mathematical physics. An Einstein manifold is a Riemannian or pseudo-Riemannian manifold (M^n, g) $(n \geq 2)$ in which the Ricci tensor is a scalar multiple of the Riemannian metric, i.e.,

$$S(X, Y) = \alpha g(X, Y), \tag{1.2}$$

where S denotes the Ricci tensor of the manifold (M^n, g) $(n \geq 2)$ and α is a non-zero scalar. An Einstein manifold plays an important role in the study of Riemannian geometry and in the general theory of relativity. From the equation (1.2), we get

$$r = n\alpha.$$

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In 2000, M. C. Chaki and R.K. Maity [4] introduced a new type of a non-flat Riemannian manifold, whose non-zero Ricci tensor satisfies

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y), \tag{1.3}$$

and they called it a quasi-Einstein manfold, where α and β are scalars such that $\beta \neq 0$ and A is a non-zero 1-form associated with unit vector field ρ defined by $g(X,\rho) = A(X)$, for every vector field X. ρ is also called generator of the manifold. An n-dimensional quasi-Einstein manifold is denoted by $(QE)_n$. The transvection of the equation (1.3) gives $r = \alpha\, n + \beta$. From the equations (1.1) and (1.3), we can easily write

$$S(X,\rho) = (\alpha+\beta)A(X), \quad S(\rho,\, \rho) = (\alpha+\beta),$$
$$g(J\rho,\, \rho) = 0, \qquad S(J\rho,\, \rho) = 0.$$

A quasi-Einstein manifold came in existence during the study of exact solutions of Einstein fields equations as well as considerations on the quasi-umbilical hypesurfaces of the semi-Euclidean space. The Walker-space time is an example of a quasi-Einstein manifold. Also, a quasi-Einstein manifold can be taken as a model of the perfect fluid space time in general theory of relativity [7].

In 2001, M.C. Chaki [5] introduced the notion of generalized quasi-Einstein manifold. A Riemannian manifold (M^n, g) $(n \geq 2)$ is said to be a generalized quasi-Einstein manifold if the non-zero Ricci tensor of type (0,2) satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma C(X)C(Y), \tag{1.4}$$

where α, β and γ are scalars such that $\beta \neq 0$, $\gamma \neq 0$ and A, C are non-vanishing 1-forms associated with two orthogonal unit vectors ρ and μ by

$$\begin{aligned} &g(X,\rho) = A(X),\, g(X,\mu) = C(X),\\ &g(\rho,\, \rho) = g(\mu,\, \mu) = 1. \end{aligned} \tag{1.5}$$

An n-dimensional generalized quasi-Einstein manifold is denoted by $G(QE)_n$. After the transvection of (1.4), we get

$$r = \alpha\, n + \beta + \gamma.$$

Then (1.1), (1.4) and (1.5), immediately infer

$$S(X,\, \rho) = (\alpha+\beta)A(X),\, S(X,\, \mu) = (\alpha+\gamma)C(X), S(\mu,\, \mu) = \alpha+\gamma,$$
$$S(\rho,\, \rho) = \alpha+\beta,\;\; g(J\rho,\, \rho) = g(J\mu,\mu) = 0, \;\; \textit{and} \quad S(J\mu,\, \mu) = S(J\rho,\, \rho) = 0.$$

Also, in 2004, M.C. Chaki [6] introduced the notion of super quasi-Einstein manifolds. A Riemannian manifold $(M^n, g)(n \geq 2)$ is said to be a super quasi-Einstein manifold if the non-zero Ricci tensor of type $(0,2)$ satisfies

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \gamma[A(X)C(Y) + C(X)A(Y)] + \delta D(X,Y), \tag{1.6}$$

where α, β, γ and δ are non-zero scalars, A, C are non-vanishing 1-forms defined as (1.5) and ρ, μ are orthogonal unit vector fields, D is a symmetric tensor of order $(0,2)$ with zero trace, which satisfies the condition

$$D(X,\rho) = 0, \tag{1.7}$$

for any vector field X. An n-dimensional super quasi-Einstein manifold is denoted by $S(QE)_n$. From the equations (1.1), (1.5), (1.6) and (1.7), we can easily write

$$S(X,\rho)=(\alpha+\beta)A(X)+\gamma C(X),\ S(X,\mu)=\alpha C(X)+\gamma A(X),$$
$$S(\mu,\mu)=\alpha+\delta D(\mu,\mu),\ S(\rho,\rho)=\alpha+\beta+\delta D(\rho,\rho),\ g(J\rho,\rho)=g(J\mu,\mu)=0,$$
$$S(J\mu,\mu)=\gamma A(J\mu)+\delta D(J\mu,\mu),\ S(J\rho,\rho)=\gamma C(J\rho)+\delta D(J\rho,\rho).$$

In 2009, A. A. Shaikh [17] introduced the notion of pseudo quasi-Einstein manifold. A semi-Riemannian manifold (M^n,g) $(n\geq 2)$ is said to be a pseudo quasi-Einstein manifold if the non-zero Ricci tensor of type $(0,2)$ satisfies the condition

$$S(X,Y)=\alpha g(X,Y)+\beta A(X)A(Y)+\delta D(X,Y), \tag{1.8}$$

where $\alpha,\ \beta$, and δ are non-zero scalars and A is a non-zero 1-form defined by $g(X,\rho)=A(X)$. ρ denotes the unit vector called the generator of the manifold and D is a symmetric tensor of type (0, 2) with zero trace defined as in (1.7). An n-dimensional pseudo quasi-Einstein manifold is denoted by $P(QE)_n$. From the equations (1.1), (1.5), (1.7) and (1.8), we get

$$S(X,\rho)=(\alpha+\beta)A(X),\ S(\rho,\rho)=\alpha+\beta,$$
$$g(J\rho,\rho)=0\ \textit{and}\ S(J\rho,\rho)=\delta D(J\rho,\rho).$$

In 2008, A. Bhattacharyya, M. Tarafdar and D. Debnath [2] introduced the notion of mixed super quasi-Einstein manifolds. A non flat Riemannian manifold (M^n,g) $(n\geq 2)$ is said to be a mixed super quasi-Einstein manifold if the non-zero Ricci tensor of type (0, 2) satisfies

$$\begin{aligned}S(X,Y)=&\ \alpha g(X,Y)+\beta A(X)A(Y)+\gamma C(X)C(Y)\\&+\delta[A(X)C(Y)+C(X)A(Y)]+\eta D(X,Y),\end{aligned} \tag{1.9}$$

where β, γ, δ and η are non-zero scalars, A, C are non-vanishing 1-forms defined as in (1.5), ρ and μ are orthogonal unit vector fields and D is a symmetric tensor of type (0, 2) with zero trace, which satisfies $D(X,\rho)=0$. An n-dimensional mixed super quasi-Einstein manifold is denoted by $MS(QE)_n$. From (1.1), (1.5), (1.7) and (1.8), we get

(1.10)
$$S(X,\rho)=(\alpha+\beta)A(X)+\delta C(X),\ S(X,\mu)=(\alpha+\gamma)C(X)+\delta A(X),$$
$$S(\mu,\mu)=\alpha+\gamma+\eta D(\mu,\mu),\ S(\rho,\rho)=\alpha+\beta+\eta D(\rho,\rho),\ g(J\rho,\rho)=g(J\mu,\mu)=0,$$
$$S(J\mu,\mu)=\delta A(J\mu)+\eta D(J\mu,\mu),\ S(J\rho,\rho)=\delta C(J\rho)+\eta D(J\rho,\rho).$$

In 2008, A. A. Shaikh and S. K. Jana [18] introduced the concept of pseudo generalized quasi-Einstein manifold and also verified it by a suitable example. A Riemannian manifold (M^n,g) $(n\geq 2)$ is called a pseudo generalized quasi-Einstein manifold if its Ricci tensor S of type (0, 2) is not identically zero and satisfies the condition

$$S(X,Y)=\alpha g(X,Y)+\beta A(X)A(Y)+\gamma C(X)C(Y)+\delta D(X,Y), \tag{1.11}$$

where β, γ and δ are non-zero scalars, A, C are non-vanishing 1-forms defined as in (1.5), ρ and μ are orthogonal unit vector fields and D is a symmetric tensor of type

(0, 2) with zero trace and satisfies $D(X, \rho) = 0$. Such type of manifold is denoted by $P(GQE)_n$.

Now contracting equation (1.11), we have

$$r = \alpha\, n + \beta + \gamma + \delta D.$$

From (1.5), (1.7) and (1.11), we infer

$$\begin{gathered} S(X,\, \rho) = (\alpha + \beta)A(X), \quad S(X,\, \mu) = (\alpha + \gamma)C(X), \\ S(\rho,\, \rho) = (\alpha + \beta) + \delta D(\rho,\, \rho), \;\; S(\mu,\, \mu) = (\alpha + \gamma) + \delta D(\mu,\, \mu), \\ g(J\rho,\, \rho) = g(J\mu, \mu) = 0, \\ S(J\mu,\, \mu) = \delta D(J\mu,\, \mu), \quad \textit{and} \quad S(J\rho,\, \rho) = \delta D(J\rho,\, \rho). \end{gathered}$$

2 Semi-symmetric and Ricci semi-symmetric manifolds

Let (M^n, g) be a Riemannian manifold and let ∇ be the Levi-Civita connection on (M^n, g). Then, the Riemannian manifold is said to be locally symmetric if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M^n, g). Locally symmetric manifolds have been studied by different geometers through different approaches, and different notions have been developed, e.g., the semi-symmetric manifold (see Szabò, [16]), recurrent manifolds (Walker, [21]), conformally recurrent manifold (Adati and Miyazawa [1]).

According to Z. I. Szabò [16], if the manifold M satisfies the condition

$$(R(X, Y).R)(U, V)W = 0, \quad X, Y, U, V, W \in \chi(M)$$

for all vector fields X and Y, then the manifold is called a semi-symmetric manifold. For a $(0, k)$- tensor field T on M, $k \geq 1$ and a symmetric (0, 2)-tensor field A on M, the $(0, k+2)$-tensor fields $R.T$ and $Q(A, T)$ are defined by

$$\begin{aligned} (R.T)(X_1, \ldots, X_k; X, Y) = \quad & -T(R(X, Y)X_1, X_2, \ldots, X_k) \\ & - \ldots - T(X_1, \ldots, X_{k-1}, R(X, Y)X_k), \\ Q(A, T)(X_1, ...X_k; X, Y) = \quad & -T((X \wedge_A Y)X_1, X_2, \ldots, X_k) \\ & - \ldots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k), \end{aligned}$$

where $X \wedge_A Y$ is the endomorphism given by

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y.$$

Definition 2.1. ([8]) A semi-Riemannian manifold is said to be Ricci semi-symmetric if

$$(R(X,\, Y).S)(Z, W) = -S(R(X, Y)Z, W) - S(Z, R(X, Y)W) = 0,$$

for all $X, Y, Z, W \in \chi(M^n)$.

The above developments allowed several authors to generalize of the notion of quasi-Einstein manifolds. In this process, generalized quasi-Einstein manifolds were studied by Prakasha and Venkatesha [15] and $N(k)$-quasi Einstein manifolds were studied in [14, 20]. In 2012, S.K. Hui and R.S. Lemence [13] discussed generalized quasi-Einstein manifolds addmitting a W_2- curvature tensor, and proved that if the W_2- curvature tensor satisfies $W_2.S = 0$, then either the associated scalars β and γ are equal, or the curvature tensor R satisfies a definite condition. D.G. Prakasha and H. Venkatesha [15] studied some results on generalized quasi-Einstein manifolds, and proved that in generalized quasi-Einstein manifolds, if the conharmonic curvature tensor satisfies $L.S = 0$, then either M is a nearly quasi-Einstein manifold $N(QE)_n$ or the curvature tensor R satisfies a certain condition. We have studied the above developments in quasi-Einstein manifolds $(QE)_n$, generalized quasi-Einstein manifolds $G(QE)_n$, and super quasi-Einstein manifolds, and decided to study Bochner Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifolds and holomorphically projective Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifolds.

3 Bochner Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold

The notion of Bochner curvature tensor was introduced by S. Bochner [3]. The Bochner curvature tensor B is defined by

(3.1)
$$\begin{aligned} B(Y,Z,U,V) =& R(Y,Z,U,V) - \frac{1}{2(n+2)}\Big\{S(Y,V)g(Z,U) - S(Y,U)g(Z,V) \\ &+ g(Y,V)S(Z,U) - g(Y,U)S(Z,V) + S(JY,V)g(JZ,U) \\ &- S(JY,U)g(JZ,V) + S(JZ,U)g(JY,V) - g(JY,U)S(JZ,V) \\ &- 2S(JY,Z)g(JU,V) - 2g(JY,Z)S(JU,V)\Big\} \\ &+ \frac{r}{(2n+2)(2n+4)}\Big\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V) \\ &+ g(JZ,U)g(JY,V) - g(JY,U)g(JZ,V) - 2g(JY,Z)g(JU,V)\Big\}, \end{aligned}$$

where r is the scalar curvature of the manifold.

In a Hermitian manifold the Bochner curvature tensor satisfies the condition

(3.2)
$$B(X,Y,U,V) = -B(X,Y,V,U).$$

We introduce the following:

Definition 3.1. A Hermitian manifold is said to be a mixed super quasi-Einstein Hermitian manifold if it satisfies (1.9). Throughout this paper, we denote the mixed super quasi-Einstein Hermitian manifold by $MS(QEH)_n$.

Definition 3.2. An even dimensional Hermitian manifold (M^n, g) is said to be a Bochner Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold if the Bochner curvature tensor of the manifold satisfies $B.S = 0$, i.e.,

(3.3)
$$(B(X,\,Y).S)(Z,W) = -S(B(X,Y)Z,W) - S(Z,(B(X,Y)W) = 0,$$

for all $X, Y, Z, W \in \chi(M^n)$.

If we take a Bochner Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold $MS(QEH)_n$, then from the (1.9) and (3.3), we get

$$\begin{aligned}
&\alpha[B(X,Y,Z,U)+B(X,Y,U,Z)]\\
&+\beta[A(B(X,Y)Z)A(U)+A(Z)A(B(X,Y)U)]\\
&+\gamma[C(B(X,Y)Z)C(U)+C(Z)C(B(X,Y)U)]\\
&+\delta[A(B(X,Y)Z)C(U)+C(B(X,Y)Z)A(U)\\
&+A(Z)C(B(X,Y)U)+C(Z)A(B(X,Y)U)]\\
&+\eta[D(B(X,Y)Z,U)+D(Z,B(X,Y)U)]=0,
\end{aligned}\tag{3.4}$$

where $g(B(X,Y)U,Z)=B(X,Y,U,Z)$. Now from the (3.2) and (3.4), we infer

$$\begin{aligned}
&\beta[A(B(X,Y)Z)A(U)+A(Z)A(B(X,Y)U)]\\
&+\gamma[C(B(X,Y)Z)C(U)+C(Z)C(B(X,Y)U)]\\
&+\delta[A(B(X,Y)Z)C(U)+C(B(X,Y)Z)A(U)\\
&+A(Z)C(B(X,Y)U)+C(Z)A(B(X,Y)U)]\\
&+\eta[D(B(X,Y)Z,U)+D(Z,B(X,Y)U)]=0;
\end{aligned}\tag{3.5}$$

putting $Z=U=\rho$ in (3.5), we get

$$\delta B(X,Y,\rho,\mu)=0,$$

which implies either $\delta=0$ or

$$B(X,Y,\rho,\mu)=0.\tag{3.6}$$

Using (3.6), if $\delta=0$, from (1.9), we yield

$$S(X,Y)=\alpha g(X,Y)+\beta A(X)A(Y)+\gamma C(X)C(Y)+\eta D(X,Y),$$

which is the condition of having a pseudo-generalized quasi-Einstein manifold. Thus we come to the following conclusion:

Theorem 3.1. *A Bochner Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold is either a Bochner Ricci semi-symmetric pseudo generalised quasi-Einstein Hermitian manifold or*

$$B(X,Y,\rho,\mu)=0.$$

If we take a Bochner flat curvature tensor then from equation (3.1), we have

$$\begin{aligned}
R(Y,Z,U,V)=&\frac{1}{2(n+2)}\Big\{S(Y,V)g(Z,U)-S(Y,U)g(Z,V)\\
&+g(Y,V)S(Z,U)-g(Y,U)S(Z,V)+S(JY,V)g(JZ,U)\\
&-S(JY,U)g(JZ,V)+S(JZ,U)g(JY,V)-g(JY,U)S(JZ,V)\\
&-2S(JY,Z)g(JU,V)-2g(JY,Z)S(JU,V)\Big\}\\
&-\frac{r}{(2n+2)(2n+4)}\Big\{g(Z,U)g(Y,V)-g(Y,U)g(Z,V)\\
&+g(JZ,U)g(JY,V)-g(JY,U)g(JZ,V)-2g(JY,Z)g(JU,V)\Big\},
\end{aligned}\tag{3.7}$$

Since the manifold satisfies $R(X,Y).S = 0$, we get

$$S(R(Y,Z)U,V) + S(U,R(Y,Z)V) = 0. \tag{3.8}$$

Using (3.7) in (3.8), we infer

$$\begin{aligned}
&\Big\{S(QY,V)g(Z,U) - g(Y,U)S(QZ,V) + S(QJY,V)g(JZ,U) \\
&- g(JY,U)S(JQZ,V) - 2g(JY,Z)S(JQU,V) \\
&+ S(QY,U)g(Z,V) - g(Y,V)S(QZ,U) + S(QJY,U)g(JZ,V) \\
&- g(JY,V)S(JQZ,U) - 2g(JY,Z)S(JQV,U)\Big\} \\
&- \frac{r}{(2n+2)}\Big\{g(Z,U)S(Y,V) - g(Y,U)S(Z,V) + g(JZ,U)S(JY,V) \\
&- g(JY,U)S(JZ,V) + g(Z,V)S(Y,U) - g(Y,V)S(Z,U) \\
&+ g(JZ,V)S(JY,U) - g(JY,V)S(JZ,U)\Big\} = 0,
\end{aligned} \tag{3.9}$$

If we take λ be an eigenvalue of Q and JQ corresponding to the eigenvectors X and JX, then $QX = \lambda X$ and $QJX = \lambda JX$, i.e., $S(X,U) = \lambda g(X,U)$ (where the manifold is not Einstein), and hence

$$S(QX,U) = \lambda S(X,U) \quad \textit{and} \quad S(QJX,U) = \lambda S(JX,U). \tag{3.10}$$

Using (3.10) in (3.9), we have

$$\begin{aligned}
&\left(\lambda - \frac{r}{(2n+2)}\right)\Big\{S(Y,V)g(Z,U) - g(Y,U)S(Z,V) \\
&+ S(Y,U)g(Z,V) - g(Y,V)S(Z,U) + S(JY,V)g(JZ,U) \\
&- S(JZ,V)g(JY,U) + S(JY,U)g(JZ,V) - g(JY,V)S(JZ,U)\Big\} = 0,
\end{aligned} \tag{3.11}$$

If we take $\lambda \neq \frac{r}{(2n+2)}$, then from (3.11), we obtain

$$\begin{gathered}
S(Y,V)g(Z,U) - g(Y,U)S(Z,V) \\
+S(Y,U)g(Z,V) - g(Y,V)S(Z,U) + S(JY,V)g(JZ,U) \\
-S(JZ,V)g(JY,U) + S(JY,U)g(JZ,V) - g(JY,V)S(JZ,U) = 0.
\end{gathered}$$

Now putting $V = \rho$ and $U = \rho$, we get

$$2[S(Y,\rho)g(Z,\rho) - g(Y,\rho)S(Z,\rho) + S(JY,\rho)g(JZ,\rho) - S(JZ,\rho)g(JY,\rho)] = 0. \tag{3.12}$$

Now using (1.10) in (3.12), we get

$$\delta[A(Z)C(Y) - A(Y)C(Z) + A(JZ)C(JY) - A(JY)C(JZ)] = 0.$$

This implies either $\delta = 0$ or

$$A(Z)C(Y) - A(Y)C(Z) = A(JY)C(JZ) - A(JZ)C(JY). \tag{3.13}$$

If we take $\lambda \neq \frac{r}{(2n+2)}$ and $\delta \neq 0$, then from (1.1) and (1.5), equation (3.13) implies

$$g(Z,\rho)\, g(Y,\mu) - g(Y,\rho)\, g(Z,\mu) = g(Y,J\rho)\, g(Z,J\mu) - g(Z,J\rho)\, g(Y,J\mu),$$

i.e., $g(Z,\rho)\, g(Y,\mu) = g(Y,\rho)\, g(Z,\mu)$ if and only if $g(Y,J\rho)\, g(Z,J\mu) = g(Z,J\rho)\, g(Y,J\mu)$. Therefore we can say that the vector fields ρ and μ corresponding to the 1-forms A and C respectively are codirectional if and only if the vector fields $\overline{\rho}$ and $\overline{\mu}$ corresponding to the 1-forms A and C respectively are codirectional.

Thus we conclude:

Theorem 3.2. *In a Bochner flat Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold, if* $\frac{r}{(2n+2)}$ *is not an eigenvalue of the Ricci operators* Q *and* JQ*, and* $\delta \neq 0$*, then the vector fields* ρ *and* μ *corresponding to the 1-forms* A *and* C *respectively are codirectional if and only if the vector fields* $\bar{\rho}$ *and* $\bar{\mu}$ *corresponding to the 1-forms* A *and* C *respectively are codirectional.*

4 Holomorphically projective Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold

The holomorphically projective curvature tensor is defined by [22]

$$\begin{aligned} P(X,Y,Z,W) = R(X,Y,Z,W) - \frac{1}{n-2}[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ + S(JX,Z)g(JY,W) - S(JY,Z)g(JX,W)]. \end{aligned} \tag{4.1}$$

This tensor has the following properties

$$P(X,Y,Z,W) = -P(Y,X,Z,W), \quad P(JX,JY,Z,W) = P(X,Y,Z,W).$$

Now we introduce the following:

Definition 4.1. An even dimensional Hermitian manifold (M^n, g) is said to be a holomorphically projective Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold if the holomorphically projective curvature tensor of the manifold satisfies $P.S = 0$, i.e.,

$$(P(X,\, Y).S)(Z,W) = -S(P(X,Y)Z,W) - S(Z,(P(X,Y)W) = 0, \tag{4.2}$$

for all $X, Y, Z, W \in \chi(M^n)$.

If we take a holomorphically projective Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold, then from the (1.9) and (4.2), we have

$$\begin{aligned} &\alpha[P(X,Y,Z,W) + P(X,Y,W,Z)] \\ &+ \beta[A(P(X,Y)Z)A(W) + A(Z)A(P(X,Y)W)] \\ &+ \gamma[C(P(X,Y)Z)C(W) + C(Z)C(P(X,Y)W)] \\ &+ \delta[A(P(X,Y)Z)C(W) + A(W)C(P(X,Y)Z) \\ &+ A(Z)C(P(X,Y)W) + C(Z)A(P(X,Y)W)] \\ &+ \eta[D(P(X,Y)Z,W) + D(Z,P(X,Y)W)] = 0. \end{aligned} \tag{4.3}$$

Now putting $Z = U = \rho$ in (4.3), we have

$$(\alpha + \beta)P(X, Y, \rho, \rho) + \delta P(X, Y, \rho, \mu) = 0. \tag{4.4}$$

Using $Z = W = \rho$ in (4.1), we infer

$$\begin{aligned} P(X, Y, \rho, \rho) = -\frac{\delta}{n-2}[C(Y)A(X) - A(Y)C(X) \\ + C(JX)A(JY) - C(JY)A(JX)]. \end{aligned} \tag{4.5}$$

Similarly, putting $Z = \rho$ and $W = \mu$ in (4.1), we get

$$\begin{aligned} P(X, Y, \rho, \mu) = R(X, Y, \rho, \mu) - \frac{(\alpha + \beta)}{n-2}[A(Y)C(X) - C(Y)A(X) \\ + C(JY)A(JX) - C(JX)A(JY)]. \end{aligned} \tag{4.6}$$

Using equations (4.5) and (4.6) in (4.4), we get

$$\delta[R(X, Y, \rho, \mu) = 0.$$

This implies that either $\delta = 0$, or $R(X, Y, \rho, \mu) = 0$. If $\delta = 0$, then from equation (1.9), we get the condition of pseudo-generalised quasi-Einstein manifolds. Thus we can conclude:

Theorem 4.1. *A holomorphically projective Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold is either a holomorphically projective Ricci semi-symmetric pseudo generalised quasi-Einstein Hermitian manifold or*

$$R(X, Y, \rho, \mu) = 0.$$

Putting $Z = \rho$ and $W = \mu$ in equation (4.3), we get

$$\begin{aligned} \alpha[P(X, Y, \rho, \mu) + P(X, Y, \mu, \rho)] + \beta P(X, Y, \mu, \rho) \\ + \gamma[P(X, Y, \rho, \rho) + P(X, Y, \mu, \mu)] = 0. \end{aligned} \tag{4.7}$$

Again, putting $Z = U = \mu$ in (4.1), we get

$$\begin{aligned} &P(X, Y, \mu, \mu) \\ &= -\frac{\gamma}{n-2}[A(Y)C(X) - C(Y)A(X) + C(JY)A(JX) - C(JX)A(JY)]. \end{aligned} \tag{4.8}$$

Adding equations (4.5) and (4.8), we infer

$$P(X, Y, \mu, \mu) + P(X, Y, \rho, \rho) = 0, \tag{4.9}$$

from (4.7) and (4.9), we have

$$\alpha[P(X, Y, \rho, \mu) + P(X, Y, \mu, \rho)] + \beta P(X, Y, \mu, \rho) + \gamma P(X, Y, \rho, \mu) = 0. \tag{4.10}$$

Now putting $Z = \mu$ and $W = \rho$ in equation (4.1), we yield

$$\begin{aligned} P(X, Y, \mu, \rho) = R(X, Y, \mu, \rho) - \frac{(\alpha + \gamma)}{n-2}[C(Y)A(X) - A(Y)C(X) \\ + C(JX)A(JY) - C(JY)A(JX)]. \end{aligned} \tag{4.11}$$

From equations (4.6), (4.10) and (4.11), we have

$$(\gamma - \beta)[R(X, Y, \mu, \rho) = 0. \tag{4.12}$$

This implies either $\beta = \gamma$, or $R(X, Y, \mu, \rho) = 0$.

Theorem 4.2. *In a holomorphically projective Ricci semi-symmetric mixed super quasi-Einstein Hermitian manifold, if* $\beta \neq \gamma$ *then* $R(X, Y, \mu, \rho) = 0$.

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References

[1] T. Adati and T. Miyazawa, *On a Riemannian space with recurrent conformal curvature*, Tensor N. S. 18(1967), 348-354.

[2] A. Bhattacharyya, M. Tarafdar and D. Debnath, *On mixed super quasi-Einstein manifolds*, Differential Geometry-Dynamical Systems, 10(2008), 44-57.

[3] S. Bochner, *Curvature and Betti numbers II*, Ann. of Math., 50(1949), 77-93.

[4] M.C. Chaki and R.K. Maity, *On quasi-Einstein manifolds*, Publ. Math. Debrecen, 57(2000), 297-306.

[5] M.C. Chaki, *On generalized quasi-Einstein manifolds*, Publ. Math. Debrecen, 58(2001), 683-691.

[6] M.C. Chaki, *On super quasi-Einstein manifold*, Publ. Math. Debrecen, 64(2004), 481-488.

[7] U.C. De and G.C. Ghosh, *On quasi-Einstein and special quasi-Einstein manifolds*, Proc. of the Conf. of Mathematics and Its Applications, Kuwait University, April 5-7 (2004), 178-191.

[8] F. Defever, *Ricci-semisymmetric hypersurfaces*, Balkan Journal of Geometry and Its Appl. 5(2000), 81-91.

[9] P. Debnath and A. Konar, *On quasi-Einstein manifolds and quasi-Einstein spacetimes*, Differ. Geom. Dyn. Syst. 12(2010), 73-82.

[10] R. Deszcz, *On pseudo-symmetric spaces*, Bull. Soc. Math. Belg. Ser. A 44(1992)1-34.

[11] S. Guha, *On quasi-Einstein and generalized quasi-Einstein manifolds*, Facta Universitatis, Series Mechanics, Automatic Control and Robotics, 3(2003), 821-842.

[12] A. Hosseinzadeh and A. Taleshian, *On conformal and quasi-conformal curvature tensors of an N(k)-quasi-Einstein manifold*, Commun. Korean Math. Soc., 27(2012), 317-326.

[13] S.K. Hui and R.S. Lemence, *On generalized quasi Einstein manifold admitting* W_2*-curvature tensor*, Int. Journal of Math. Analysis, 6, 23(2012), 1115 - 1121.

[14] C. Ozg ur and S. Sular, *On N(k)-quasi-Einstein manifolds satisfying certain conditions*, Balkan J. Geom. Appl., 13(2008), 74-79.

[15] D.G. Prakasha and H. Venkatesha, *Some results on generalized quasi-Einstein manifolds*, Chinese Journal of Mathematics (Hindawi Publishing Corporation), Volume 2014, Article ID 563803 (2014), 5 pages.

[16] Z.I. Szabo, *Structure theorems on Riemannian spaces satisfying R(X,Y).R=0. The local version*, J. Diff. Geom. 17(1982), 531-582.
[17] A.A. Shaikh, *On pseudo quasi Einstein manifold*, Period. Math. Hungar., 59(2009), 119-146.
[18] A.A. Shaikh and S.K.Jana, *On pseudo-generalised quasi Einstein manifold*, Tamkang Journal of Mathematics, 39(2008), 9-24.
[19] A. Taleshian and A.A. Hosseinzadeh, *Investigation of some conditions on N(k)-quasi Einstein manifolds*, Bull. Malays. Math. Sci. Soc., 34(2011), 455-464.
[20] M.M. Tripathi and J.S Kim, *On N(k)-quasi-Einstein manifolds*, Commun. Korean Math. Soc., 22(3) (2007), 411-417.
[21] A.G. Walker, *On Ruse′s spaces of recurrent curvature*, Proc. London Math. Soc. 52(1950), 36-64.
[22] K. Yano., *Differential Geometry of Complex and Almost Complex Spaces*, Pergamon Press, New York, 1965.

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