On a gradient Ricci soliton in the plane

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Abstract. The object of the present paper is to introduce a special Riemannian metric and study the gradient Ricci solitons for this special metric. We also characterize the potential function for such gradient Ricci soliton.

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1 Introduction

In 1982, Hamilton [6] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman ([10], [11]) used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold (M, g) defined as follows:

$$\frac{\partial}{\partial t} g(t) = -2Ric,$$

where Ric is the Ricci tensor.

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (g, V, ρ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that ([4], [7])

$$\frac{1}{2}\pounds_V g + Ric = \rho g, \tag{1.1}$$

where $\pounds_V$ is the Lie derivative operator along the vector field V on M and ρ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according as ρ is negative, zero and positive respectively.

Moreover, if the vector field V is the gradient of some smooth function f (called potential function) on M then the Riemannian manifold (M, g) is said to be gradient Ricci solition. A Riemannian manifold (M, g) is called a gradient Ricci soliton [2] if there exists a smooth function $f : M \to \mathbb{R}$, sometimes called potential function, satisfying

$$R_{ij} + f_{,ij} = \rho g_{ij}, \tag{1.2}$$

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where ρ is a real number and R_{ij} are the components of the Ricci tensor. The gradient Ricci solitons have been studied by several authors such as [3], [5], [8] and many others.

In the present paper, we introduce a special Riemannian metric and study the gradient Ricci solitons for this special metric. The paper is organized with some preliminaries. Section 2 is concerned as follows. Section 3 is devoted to the study of gradient Ricci solitons of a special Riemannian metric. We also characterize the potential function for such gradient Ricci soliton.

In 1988, Nesterov and Nemirovskii developed a general, polynomial time framework for convex programming problems, presented in their extensive monograph [9]. This framework for interior point methods relies on the notion of self-concordant barrier functions. These functions are special, convex penalty functions which intricately regulate their own behaviour and growth. Then Udrişte [12] study self-concordant barrier functions of the Riemannian context of optimization methods. Thus the notion of self-concordant function has been introduced on Riemannian manifolds due to the necessity to develop a large class of optimization methods. In this section we obtain the condition of potential function of such constructed gradient Ricci soliton to be c-concordant barrier.

2 Preliminaries

This section deals with some preliminaries, which will be required in the sequel.

Let (M, g) be an n-dimensional Riemannian manifold and $(U, x^1, x^2, \cdots, x^n)$ be a coordinate chart on M. The Christoffel symbols of the Levi-Civita connection is denoted by Γ^k_{ij}, is defined by [2]

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left(\frac{\partial g_{li}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right).$$

Using the Christoffel symbols, the components of the Riemann curvature tensor R can be expressed in the following form [2]

$$R^l_{ijk} = \frac{\partial \Gamma^l_{ki}}{\partial x^j} - \frac{\partial \Gamma^l_{ji}}{\partial x^k} + \Gamma^r_{ki}\Gamma^l_{jr} - \Gamma^r_{ji}\Gamma^l_{kr},$$

and the Ricci tensor (Ric) is defined by $R_{ij} = R^l_{ilk}$.
If $f : M \to \mathbb{R}$ is a smooth function, then we consider [2]

$$f_{,i} = \frac{\partial f}{\partial x^i}, \quad f_{,ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^m_{ij} f_{,m}, \quad f_{,ijk} = \frac{\partial f_{,ij}}{\partial x^k} - \Gamma^l_{ki} f_{,lj} - \Gamma^l_{kj} f_{,li}. \tag{2.1}$$

Also we recall the following:

Definition 2.1. [2] The second covariant derivative of the function $f : M \to \mathbb{R}$ is defined by

$$\nabla^2_g f = \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} f_{,k} \right) dx^i \otimes dx^j, \tag{2.2}$$

is called the Hessian of f.

Definition 2.2. [1] The function f is said to be *c-self-concordant barrier*, $c \geq 0$, with respect to the Levi-Civita connection ∇ defined on M if the following conditions holds:

$$(df(x)(X_x))^2 \leq c\, \nabla^2 f(x)(X_x, X_x), \quad \forall x \in M \text{ and } X_x \in T_x M. \tag{2.3}$$

3 Gradient Ricci solitons

In this section we introduce a special Riemannian metric and study the gradient Ricci solitons for this special metric.

Let us consider $M = \mathbb{R}^2$ be a Riemannian manifold endowed with the metric of diagonal type (warped metric) $\bar{g}$, given by

$$\bar{g}(x^1, x^2) = \begin{pmatrix} \alpha g(x^2) & 0 \\ 0 & \beta g(x^1) \end{pmatrix}, \tag{3.1}$$

where g is a smooth positive function, (x^1, x^2) are the global coordinates on $\mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}^+$.
The Lagrangian of energy associated with the above metric is of the form

$$L = \frac{1}{2}[\alpha g(x^2)(\dot{x}^1)^2 + \beta g(x^1)(\dot{x}^2)^2]. \tag{3.2}$$

Hence we get

$$\begin{cases} \frac{\partial L}{\partial x^1} = \frac{\beta g'(x^1)(\dot{x}^2)^2}{2}, \quad \frac{\partial L}{\partial x^2} = \frac{\alpha g'(x^2)(\dot{x}^1)^2}{2}, \\ \frac{\partial L}{\partial \dot{x}^1} = \alpha g(x^2)\dot{x}^1, \quad \frac{\partial L}{\partial \dot{x}^2} = \beta g(x^1)\dot{x}^2, \end{cases} \tag{3.3}$$

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^1}\right) = \alpha g(x^2)\ddot{x}^1 + \alpha g'(x^2)\dot{x}^1\dot{x}^2, \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^2}\right) = \beta g(x^1)\ddot{x}^2 + \beta g'(x^1)\dot{x}^1\dot{x}^2. \end{cases} \tag{3.4}$$

Using the Euler-Lagrange equations

$$\begin{cases} \frac{\partial L}{\partial x^1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^1}\right) = 0, \\ \frac{\partial L}{\partial x^2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^2}\right) = 0, \end{cases}$$

the geodesic equations for the metric $\bar{g}$ are given by

$$\begin{cases} \ddot{x}^1 + \frac{g'(x^2)}{g(x^2)}\dot{x}^1\dot{x}^2 - \frac{\beta}{2\alpha}\frac{g'(x^1)}{g(x^2)}(\dot{x}^2)^2 = 0, \\ \ddot{x}^2 + \frac{g'(x^1)}{g(x^1)}\dot{x}^1\dot{x}^2 - \frac{\alpha}{2\beta}\frac{g'(x^2)}{g(x^1)}(\dot{x}^1)^2 = 0. \end{cases} \tag{3.5}$$

Comparing (3.5) with the usual geodesic equations $\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0$, we obtain the Christoffel symbols for our metric as

$$\begin{cases} \Gamma^1_{11} = 0; \quad \Gamma^1_{12} = \Gamma^1_{21} = \frac{g'(x^2)}{2g(x^2)}; \quad \Gamma^1_{22} = -\frac{\beta g'(x^1)}{2\alpha g(x^2)}; \\ \Gamma^2_{22} = 0; \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{g'(x^1)}{2g(x^1)}; \quad \Gamma^2_{11} = -\frac{\alpha g'(x^2)}{2\beta g(x^1)}. \end{cases} \tag{3.6}$$

Next, for an arbitrary function $f(x^1, x^2)$ on M, we compute

$$\begin{aligned} f_{,11} &= \frac{\partial^2 f}{\partial (x^1)^2} - \Gamma^1_{11}\frac{\partial f}{\partial x^1} - \Gamma^2_{11}\frac{\partial f}{\partial x^2} \\ &= \frac{\partial^2 f}{\partial (x^1)^2} + \frac{\alpha}{2\beta}\frac{g'(x^2)}{g(x^1)}\frac{\partial f}{\partial x^2}; \end{aligned} \tag{3.7}$$

$$\begin{aligned} f_{,12} = f_{,21} &= \frac{\partial^2 f}{\partial x^1 \partial x^2} - \Gamma^1_{12}\frac{\partial f}{\partial x^1} - \Gamma^2_{12}\frac{\partial f}{\partial x^2} \\ &= \frac{\partial^2 f}{\partial x^1 \partial x^2} - \frac{g'(x^2)}{2g(x^2)}\frac{\partial f}{\partial x^1} - \frac{g'(x^1)}{2g(x^1)}\frac{\partial f}{\partial x^2}; \end{aligned} \tag{3.8}$$

$$\begin{aligned} f_{,22} &= \frac{\partial^2 f}{\partial (x^2)^2} - \Gamma^1_{22}\frac{\partial f}{\partial x^1} - \Gamma^2_{22}\frac{\partial f}{\partial x^2} \\ &= \frac{\partial^2 f}{\partial (x^2)^2} + \frac{\beta}{2\alpha}\frac{g'(x^1)}{g(x^2)}\frac{\partial f}{\partial x^1}. \end{aligned} \tag{3.9}$$

Also for 2-dimensional manifolds, the components of Ricci tensor are

$$\begin{cases} R_{11} = R^1_{111} + R^2_{121}, \quad R_{12} = R^1_{112} + R^2_{122}, \\ R_{21} = R^1_{211} + R^2_{221}, \quad R_{22} = R^1_{212} + R^2_{222}. \end{cases} \tag{3.10}$$

In view of (3.6), we compute by direct calculation that

$$R^2_{121} = \frac{\alpha}{4\beta}\left(\frac{\{g'(x^2)\}^2 - 2g''(x^2)g(x^2)}{g(x^1)g(x^2)}\right) + \frac{\{g'(x^1)\}^2 - 2g(x^1)g''(x^1)}{4\{g(x^1)\}^2}, \tag{3.11}$$

$$R^1_{111} = 0,\ R^1_{112} = 0,\ R^2_{122} = 0, \tag{3.12}$$

$$R^1_{212} = \frac{\beta}{4\alpha}\frac{\{g'(x^1)\}^2}{g(x^1)g(x^2)} - \frac{\beta}{2\alpha}\frac{g''(x^1)}{g(x^2)} + \frac{\{(g'(x^2)\}^2 - 2g''(x^2)g(x^2)}{4\{g(x^2)\}^2},\ R^2_{222} = 0. \tag{3.13}$$

By virtue of (3.11)-(3.13), we obtain from (3.10) that

$$\begin{cases} R_{11} = \frac{\alpha}{4\beta}\left(\frac{\{g'(x^2)\}^2 - 2g''(x^2)g(x^2)}{g(x^1)g(x^2)}\right) + \frac{\{g'(x^1)\}^2 - 2g(x^1)g''(x^1)}{4\{g(x^1)\}^2}, \\ R_{12} = 0, \\ R_{22} = \frac{\beta}{4\alpha}\frac{\{g'(x^1)\}^2}{g(x^1)g(x^2)} - \frac{\beta}{2\alpha}\frac{g''(x^1)}{g(x^2)} + \frac{\{(g'(x^2)\}^2 - 2g''(x^2)g(x^2)}{4\{g(x^2)\}^2}. \end{cases} \tag{3.14}$$

Using (3.14), the relation (1.2) reduces to the following equations:

$$\begin{cases} R_{11} + f_{,11} = \rho g_{11}, \\ R_{12} + f_{,12} = \rho g_{12}, \\ R_{22} + f_{,22} = \rho g_{22}. \end{cases} \tag{3.15}$$

Feeding (3.7), (3.8), (3.9) and (3.14) in (3.15) we get respectively the followings:

$$\begin{aligned}\frac{\alpha}{4\beta}\frac{\{g'(x^2)\}^2-2g''(x^2)g(x^2)}{g(x^1)g(x^2)}+\frac{\{g'(x^1)\}^2-2g(x^1)g''(x^1)}{4\{g(x^1)\}^2}+\frac{\partial^2 f}{\partial(x^1)^2}\\+\frac{\alpha}{2\beta}\frac{g'(x^2)}{g(x^1)}\frac{\partial f}{\partial x^2}=\alpha\rho g(x^2),\end{aligned}\tag{3.16}$$

$$\frac{\partial^2 f}{\partial x^1\partial x^2}-\frac{g'(x^2)}{2g(x^2)}\frac{\partial f}{\partial x^1}-\frac{g'(x^1)}{2g(x^1)}\frac{\partial f}{\partial x^2}=0,\tag{3.17}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial(x^2)^2}+\frac{\beta}{2\alpha}\frac{g'(x^1)}{g(x^2)}\frac{\partial f}{\partial x^1}+\frac{\beta}{4\alpha}\frac{\{g'(x^1)\}^2}{g(x^1)g(x^2)}-\frac{\beta}{2\alpha}\frac{g''(x^1)}{g(x^2)}\\+\frac{\{g'(x^2)\}^2-2g''(x^2)g(x^2)}{4\{g(x^2)\}^2}=\beta\rho g(x^1).\end{aligned}\tag{3.18}$$

The Hessian of the function f with respect to metric (3.1) is

$$H_f=\begin{pmatrix}\frac{\partial^2 f}{\partial(x^1)^2}+\frac{\alpha}{2\beta}\frac{g'(x^2)}{g(x^1)}\frac{\partial f}{\partial x^2} & \frac{\partial^2 f}{\partial x^1\partial x^2}-\frac{g'(x^2)}{2g(x^2)}\frac{\partial f}{\partial x^1}-\frac{g'(x^1)}{2g(x^1)}\frac{\partial f}{\partial x^2}\\ \frac{\partial^2 f}{\partial x^1\partial x^2}-\frac{g'(x^2)}{2g(x^2)}\frac{\partial f}{\partial x^1}-\frac{g'(x^1)}{2g(x^1)}\frac{\partial f}{\partial x^2} & \frac{\partial^2 f}{\partial(x^2)^2}+\frac{\beta}{2\alpha}\frac{g'(x^1)}{g(x^2)}\frac{\partial f}{\partial x^1}\end{pmatrix}.\tag{3.19}$$

Imposing the condition $det\,(H_f)\geq 0$, we get

$$\begin{aligned}&\frac{\partial^2 f}{\partial(x^1)^2}\frac{\partial^2 f}{\partial(x^2)^2}+\frac{\alpha}{2\beta}\frac{g'(x^2)}{g(x^1)}\frac{\partial f}{\partial x^2}\frac{\partial^2 f}{\partial(x^2)^2}+\frac{\beta}{2\alpha}\frac{g'(x^1)}{g(x^2)}\frac{\partial f}{\partial x^1}\frac{\partial^2 f}{\partial(x^1)^2}\\&+\frac{\partial^2 f}{\partial x^1\partial x^2}\left\{\frac{g'(x^2)}{g(x^2)}\frac{\partial f}{\partial x^1}+\frac{g'(x^1)}{g(x^1)}\frac{\partial f}{\partial x^2}\right\}\\&\geq\left(\frac{\partial^2 f}{\partial x^1\partial x^2}\right)^2+\frac{\{g'(x^2)\}^2}{4\{g(x^2)\}^2}\left(\frac{\partial f}{\partial x^1}\right)^2+\frac{1}{4}\frac{g'(x^1)g'(x^2)}{g(x^1)g(x^2)}\frac{\partial f}{\partial x^1}\frac{\partial f}{\partial x^2}.\end{aligned}\tag{3.20}$$

Thus we can state the following:

Theorem 3.1. *Let $M=\mathbb{R}^2$ be a Riemannian manifold endowed with the metric $\bar{g}$ given in* (3.1)*. If the following conditions are satisfied*

$$\begin{cases}\frac{\alpha}{4\beta}\frac{\{g'(x^2)\}^2-2g''(x^2)g(x^2)}{g(x^1)g(x^2)}+\frac{\{g'(x^1)\}^2-2g(x^1)g''(x^1)}{4\{g(x^1)\}^2}+\frac{\partial^2 f}{\partial(x^1)^2}+\frac{\alpha}{2\beta}\frac{g'(x^2)}{g(x^1)}\frac{\partial f}{\partial x^2}=\alpha\rho g(x^2);\\ \frac{\partial^2 f}{\partial x^1\partial x^2}-\frac{g'(x^2)}{2g(x^2)}\frac{\partial f}{\partial x^1}-\frac{g'(x^1)}{2g(x^1)}\frac{\partial f}{\partial x^2}=0;\\ \frac{\partial^2 f}{\partial(x^2)^2}+\frac{\beta}{2\alpha}\frac{g'(x^1)}{g(x^2)}\frac{\partial f}{\partial x^1}+\frac{\beta}{4\alpha}\frac{\{g'(x^1)\}^2}{g(x^1)g(x^2)}-\frac{\beta}{2\alpha}\frac{g''(x^1)}{g(x^2)}+\frac{\{g'(x^2)\}^2-2g''(x^2)g(x^2)}{4\{g(x^2)\}^2}=\beta\rho g(x^1)\end{cases}$$

then $(\mathbb{R}^2,\bar{g})$ is a gradient Ricci soliton having $f:M\to\mathbb{R}$ as potential function, where the Hessian of f is given in (3.19) *and the condition for the potential function f of the metric* (3.1) *to define a Riemannian metric of Hessian type is given in* (3.20)*.*

If we take $g(x^1)=\frac{1}{\beta}=$ constant and $\alpha=1$ in (3.1) then we can state the following:

Corollary 3.2. *Let $M = \mathbb{R}^2$ be a Riemannian manifold endowed with the metric of diagonal type $\bar{g}$, where $\bar{g}(x^1, x^2) = diag(g(x^2), 1)$, with g positive function, of C^∞ class. If the following conditions are satisfied*

$$\begin{cases} \frac{\partial^2 f}{\partial(x^1)^2} + \frac{g'(x^2)}{2}\frac{\partial f}{\partial x^2} + \frac{(g'(x^2))^2 - 2g''(x^2)g(x^2)}{4g(x^2)} = \rho g(x^2); \\ \frac{\partial^2 f}{\partial x^1 \partial x^2} - \frac{g'(x^2)}{2g(x^2)}\frac{\partial f}{\partial x^1} = 0; \\ \frac{\partial^2 f}{\partial(x^2)^2} + \frac{(g'(x^2))^2 - 2g''(x^2)g(x^2)}{4g(x^2)} = \rho \end{cases}$$

then $(\mathbb{R}^2, \bar{g})$ is a gradient Ricci soliton having $f : M \to \mathbb{R}$ as potential function [2], where the Hessian of f is given in

$$H_f = \begin{pmatrix} \frac{\partial^2 f}{\partial(x^1)^2} + \frac{g'(x^2)}{2}\frac{\partial f}{\partial x^2} & \frac{\partial^2 f}{\partial x^1 \partial x^2} - \frac{g'(x^2)}{2g(x^2)}\frac{\partial f}{\partial x^1} \\ \frac{\partial^2 f}{\partial x^1 \partial x^2} - \frac{g'(x^2)}{2g(x^2)}\frac{\partial f}{\partial x^1} & \frac{\partial^2 f}{\partial(x^2)^2} \end{pmatrix}$$

and the condition for the potential function f of the metric $\bar{g}(x^1, x^2) = diag(g(x^2), 1)$ to define a Riemannian metric of Hessian type is given by

$$\frac{\partial^2 f}{\partial(x^1)^2}\frac{\partial^2 f}{\partial(x^2)^2} + \frac{g'(x^2)}{2}\frac{\partial f}{\partial x^2}\frac{\partial^2 f}{\partial(x^2)^2} + \frac{\partial^2 f}{\partial x^1 \partial x^2}\frac{g'(x^2)}{g(x^2)}\frac{\partial f}{\partial x^1}$$
$$\geq \left(\frac{\partial^2 f}{\partial x^1 \partial x^2}\right)^2 + \frac{\{g'(x^2)\}^2}{4\{g(x^2)\}^2}\left(\frac{\partial f}{\partial x^1}\right)^2.$$

Following the same treatment of [2], if we take

$$\begin{cases} h = \frac{\partial f}{\partial x^1} \Rightarrow \frac{\partial h}{\partial x^2} = \frac{\partial^2 f}{\partial x^1 \partial x^2}, \\ k = \frac{\partial f}{\partial x^2} \Rightarrow \frac{\partial k}{\partial x^2} = \frac{\partial^2 k}{\partial x^1 \partial x^2}. \end{cases}$$

then from (3.17) we obtain

$$\begin{cases} \frac{\partial h}{\partial x^2} = \frac{g'(x^2)}{2g(x^2)}h + \frac{g'(x^1)}{2g(x^1)}k, \\ \frac{\partial k}{\partial x^1} = \frac{g'(x^2)}{2g(x^2)}h + \frac{g'(x^1)}{2g(x^1)}k. \end{cases} \tag{3.21}$$

Taking an integration of first equation of (3.21) with respect to x^2 and second equation of (3.21) with x^1 respectively, we get

Theorem 3.3. *The potential function f of the gradient Ricci soliton in Theorem 3.1 must satisfies the following conditions:*

$$\begin{cases} \frac{\partial f}{\partial x^1} = a(x^1)\sqrt{g(x^2)} + \frac{g'(x^1)}{2g(x^1)}f\left(x^1, x^2\right), \\ \frac{\partial f}{\partial x^2} = b(x^2)\sqrt{g(x^1)} + \frac{g'(x^2)}{2g(x^2)}f\left(x^1, x^2\right). \end{cases}$$

We now prove the following:

Theorem 3.4. *The potential function f of the gradient Ricci soliton in Theorem 3.1* *is c-concordant barrier, if*

$$
\begin{aligned}
(3.22)\quad \left(\frac{\partial f}{\partial x^1}u+\frac{\partial f}{\partial x^2}v\right)^2 \quad &\leq \quad c\left[\left(\frac{\partial^2 f}{\partial (x^1)^2}+\frac{\alpha}{2\beta}\frac{g'(x^2)}{g(x^1)}\frac{\partial f}{\partial x^2}\right)u^2\right]\\
&+\quad c\left[\left(\frac{\partial^2 f}{\partial x^1\partial x^2}-\frac{g'(x^2)}{2g(x^2)}\frac{\partial f}{\partial x^1}-\frac{g'(x^1)}{2g(x^1)}\frac{\partial f}{\partial x^2}\right)uv\right]\\
&+\quad c\left[\left(\frac{\partial^2 f}{\partial (x^2)^2}+\frac{\beta}{2\alpha}\frac{g'(x^1)}{g(x^2)}\frac{\partial f}{\partial x^1}\right)v^2\right].
\end{aligned}
$$

Proof. The proof can be done immediatly if we replace $X_x=(u,v)$ in the Definition 2.2. □

If we take $g(x^1)=\frac{1}{\beta}=$ constant and $\alpha=1$ in (3.1) then we can state the following:

Corollary 3.5. *The potential function f of the gradient Ricci soliton in Corollary 3.1 is c-concordant barrier, if*

$$\left(\frac{k^2c+1}{k^2(x^1)^2}-\frac{cg'(x^2)}{2x^2}\right)u^2+\left(\frac{2}{kx^1x^2}+\frac{cg'(x^2)}{2x^1}g(x^2)\right)uv+\frac{k^2c+1}{k^2(x^2)^2}v^2\leq 0.$$

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