Warped product semi-transversal lightlike submanifolds of indefinite nearly Kaehler manifolds

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Abstract. We prove the non-existence of warped product semi-transversal lightlike submanifolds of the type $N_{\perp} \times_f N_T$ in an indefinite nearly Kaehler manifold. We find a necessary and sufficient condition for a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold to be a semi-transversal lightlike warped product submanifold of the type $N_T \times_f N_{\perp}$. We also derive some characterizations in terms of the canonical structures T and ω on a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold forcing it to be a semi-transversal lightlike warped product submanifold.

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1 Introduction

Bishop and O'Neill, in 1969, to construct a large variety of complete Riemannian manifolds of everywhere negative sectional curvature, introduced the notion of warped product manifolds (see [2]) and since then, it is an active field of research for mathematicians and physicists. The warped product manifolds provide an excellent setting to model space time near black holes or bodies with large gravitational field (see [10]). Moreover, many solutions to the Einstein field equations are expressed in terms of warped products (see [1]), therefore the study of these manifolds assumes significance in general. In view of its physical applications, many research articles have recently appeared exploring the existence (or non-existence) of warped products in known spaces. B.Y. Chen intiated the study of warped products in Kaehler manifolds by proving the non-existence of non-trivial warped product CR-submanifolds of the type $N_{\perp} \times_f N_T$ in a Kaehler manifold (see [3]). K. Sekigawa [13] proved the non-existence of CR-products in S^6 and gave one non-trivial example for the existence of CR-warped products in S^6; this paved interest towards the study of warped products in S^6 and more generally, in nearly Kaehler manifolds. Then, B. Sahin et. al. [12], investigated warped product CR-submanifolds of nearly Kaehler manifolds and proved the non-existence of warped product CR-submanifolds of the type $N_{\perp} \times_f N_T$ in a

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nearly Kaehler manifold and this class of warped products has been further developed by V.A. Khan et. al. (see [8], [9]). Moreover, it is observed that most of the available work on warped products emphasizes on manifolds with positive definite metric and therefore, it may not be applicable to those areas of mathematical physics and relativity, where the metric is not necessarily positive definite. Also, the relativity theory has led to the study of semi-Riemannian manifolds, which turns out to be the most general framework for the study of warped products and it may lead to some salient applications.

With the said motivation, the author we study the geometry of warped product semi-transversal lightlike submanifolds of indefinite nearly Kaehler manifolds. Firstly, we prove the existence of semi-transversal lightlike submanifolds in indefinite nearly Kaehler manifolds of constant holomorphic sectional curvature c and of constant type α (Sahin, introduced this class of lightlike submanifolds in indefinite Kaehler manifolds (for details see [11]). Then, we prove the non-existence of warped product semi-transversal lightlike submanifolds of the type $N_\perp \times_f N_T$ in an indefinite nearly Kaehler manifold. We also find a necessary and sufficient condition for a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold to be a semi-transversal lightlike warped product submanifold of the type $N_T \times_f N_\perp$. Finally, we derive some characterizations in terms of the canonical structures T and ω on a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold forcing it to be a semi-transversal lightlike warped product submanifold.

2 Preliminaries

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g be the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold of $\bar{M}$, (see [4]). For a degenerate metric g on M, $TM^\perp$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus both T_xM and $T_xM^\perp$ are degenerate orthogonal subspaces, but no longer complementary. In this case, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^\perp$, which is known as radical (null) subspace. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$, defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $Rad(TM)$ is called the radical distribution on M.

Screen distribution $S(TM)$ is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is

$$TM = Rad(TM) \perp S(TM) \tag{2.1}$$

and $S(TM^\perp)$ is a complementary vector subbundle to $Rad(TM)$ in $TM^\perp$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $Rad(TM)$ in $S(TM^\perp)^\perp$ respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^\perp). \tag{2.2}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^\perp). \tag{2.3}$$

For a quasi-orthonormal fields of frames on TM, we have

Theorem 2.1. *([4]). Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\mid_u)$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_u$, where u is a coordinate neighborhood of M such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \text{for any} \quad i, j \in \{1, 2, .., r\}, \tag{2.4}$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$, then according to the decomposition (2.3), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{2.5}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, A_U is a linear operator on M and is known as shape operator.

According to (2.2), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ respectively, then Gauss and Weingarten formulas become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.6}$$

where we put $h^l(X, Y) = L(h(X, Y)), h^s(X, Y) = S(h(X, Y)), D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$. As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called the lightlike second fundamental form and the screen second fundamental form on M. In particular,

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \quad \bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.7}$$

where $X \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Using (2.6) and (2.7), we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.8}$$

$$\bar{g}(D^s(X, N), W) = \bar{g}(A_W X, N), \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$, $W \in \Gamma(S(TM^{\perp}))$ and $N \in \Gamma(ltr(TM))$.

Let P be the projection morphism of TM on $S(TM)$, then using (2.1), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$\nabla_X PY = \nabla^*_X PY + h^*(X, Y), \quad \nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X PY, A^*_{\xi} X\}$ and $\{h^*(X, Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively. Using (2.6) and (2.10), we obtain

$$\bar{g}(h^l(X, PY), \xi) = g(A^*_{\xi} X, PY), \quad \bar{g}(h^*(X, PY), N) = g(A_N X, PY), \tag{2.11}$$

for any $X, Y \in \Gamma(TM), \xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.

Definition 2.1. ([6]). Let $(\bar{M}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then $\bar{M}$ is called an indefinite nearly Kaehler manifold if

$$(\bar{\nabla}_X \bar{J})Y + (\bar{\nabla}_Y \bar{J})X = 0, \quad \forall \quad X, Y \in \Gamma(T\bar{M}). \tag{2.12}$$

Nearly Kaehler manifold of constant holomorphic curvature c is denoted by $\bar{M}(c)$ and its sectional curvature is given by, (see [14])

(2.13)

$$\bar{R}(X,Y,X,Y) = \frac{c}{4}\{\bar{g}(X,Y)^2 - \bar{g}(X,X)\bar{g}(Y,Y) - 3\bar{g}(X,\bar{J}Y)^2\} - \frac{3}{4}\|(\bar{\nabla}_X \bar{J})(Y)\|^2.$$

A nearly Kaehler manifold is said to be of constant type α (see [6]), if there exists a real valued $C^{\infty}-$ function α on $\bar{M}$ such that

$$\|(\bar{\nabla}_X \bar{J})(Y)\|^2 = \alpha\{\|X\|^2\|Y\|^2 - g(X,Y)^2 - g(X,\bar{J}Y)^2\}. \tag{2.14}$$

3 semi-transversal lightlike submanifolds

Definition 3.1. ([11]). Let M be a lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$, then M is called a semi-transversal lightlike submanifold of $\bar{M}$, if the following conditions are satisfied

(A) $Rad(TM)$ is transversal with respect to $\bar{J}$, that is, $\bar{J}Rad(TM) = ltr(TM)$.

(B) There exists a real non-null distribution $D \subset S(TM)$ such that

$$S(TM) = D \oplus D^{\perp}, \quad \bar{J}D^{\perp} \subset S(TM^{\perp}), \quad \bar{J}(D) = D,$$

where $D^{\perp}$ is orthogonal complementary to D in $S(TM)$.

Thus we obtain that the tangent bundle TM of a semi-transversal lightlike submanifold is decomposed as $TM = D \perp D'$, where $D' = D^{\perp} \perp Rad(TM)$.

Before proceeding further, firstly we prove the existence of semi-transversal lightlike submanifolds in indefinite nearly Kaehler manifolds.

Theorem 3.1. *(Existence Theorem). A lightlike submanifold M of an indefinite nearly Kaehler manifold $\bar{M}(c)$ of constant type α and of constant holomorphic sectional curvature c such that $c = -3\alpha$, where $\alpha \neq 0$ is a semi-transversal lightlike submanifold with $D \neq 0$, if and only if*

(i) The maximal complex subspace of $T_pM, p \in M$ defines a non-degenerate complex distribution D.

(ii) There exist a radical distribution $Rad(TM)$ and a lightlike transversal vector bundle $ltr(TM)$ such that $\bar{g}(\bar{R}(\xi, N)\xi, N) = 0$, for any $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.

(iii) There exists a vector subbundle $D^{\perp}$ on M such that $\bar{g}(\bar{R}(W, W')W, W') = 0$, for any $W, W' \in \Gamma(D^{\perp})$.

(iv) $\bar{g}(\bar{R}(\xi, W)\xi, W) = 0$, *for any* $\xi \in \Gamma(Rad(TM))$ *and* $W \in \Gamma(D^{\perp})$, *where* $D^{\perp}$ *is orthogonal to* D *and* $\bar{R}$ *be curvature tensor of* $\bar{M}(c)$.

Proof. Assume that M is a semi-transversal lightlike submanifold of $\bar{M}(c)$ such that $c = -3\alpha$ and $c \neq 0$. Then by the definition of semi-transversal lightlike submanifolds, D is a maximal subspace. Next for $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$, using (2.13) and (2.14), we have $\bar{g}(\bar{R}(\xi, N)\xi, N) = 3\alpha\bar{g}(\xi, \bar{J}N)^2$. Then from definition of semi-transversal lightlike submanifolds, $\bar{g}(\xi, \bar{J}N) = 0$, therefore we have $\bar{g}(\bar{R}(\xi, N)\xi, N) = 0$. Similarly, using (2.13) and (2.14), we obtain $\bar{g}(\bar{R}(W, W')W, W') = 0$ and $\bar{g}(\bar{R}(\xi, W)\xi, W) = 0$, for $W, W' \in \Gamma(D^{\perp})$ and $\xi \in \Gamma(Rad(TM))$.

Conversely, suppose that $(i), (ii), (iii)$ and (iv) are satisfied. From (i), a non-degenerate complex subspace $D_p, p \in M$ implies that $D \subset S(TM)$. Now, consider an orthogonal complement distribution $D^{\perp}$ to D in $S(TM)$, then using (2.13), (2.14) and (iii), we obtain $3\alpha\bar{g}(W, \bar{J}W') = 0$, for $W, W' \in \Gamma(D^{\perp})$. Since $\alpha \neq 0$, therefore we get $\bar{g}(W, \bar{J}W') = 0$, which implies that $\bar{J}D^{\perp} \perp D^{\perp}$. As D is invariant, thus we have $\bar{g}(X, \bar{J}W) = -g(\bar{J}X, W) = 0$, which implies that $\bar{J}D^{\perp} \cap D = \{0\}$. Similarly, on using (2.13), (2.14) and (ii), we have $3\alpha\bar{g}(\bar{J}\xi, N)^2 = 0$ for $\xi \in \Gamma(Rad(TM)), N \in \Gamma(ltr(TM))$. Since $\alpha \neq 0$, we conclude that $\bar{g}(\bar{J}\xi, N) = 0$ that is $\bar{J}Rad(TM) \cap Rad(TM) = \{0\}$. As D is a non-degenerate and complex distribution, therefore $\bar{g}(X, \bar{J}\xi) = -g(\bar{J}X, \xi) = 0$, thus we derive $\bar{J}Rad(TM) \cap D = \{0\}$. Furthermore, we have $0 = g(\xi, W) = \bar{g}(\bar{J}\xi, \bar{J}W)$, which implies that $\bar{J}D^{\perp} \cap \bar{J}Rad(TM) = \{0\}$. Then using (2.13), (2.14) and (iv), we get $3\alpha\bar{g}(\bar{J}\xi, W)^2 = 0$. As $\alpha \neq 0$, thus we have $\bar{g}(\bar{J}\xi, W) = 0$, which implies that $\bar{J}Rad(TM) \cap D^{\perp} = \{0\}$ and $Rad(TM) \cap \bar{J}D^{\perp} = \{0\}$. Summing up, we obtain $\bar{J}Rad(TM) \cap TM = \{0\}$, $\bar{J}Rad(TM) \cap \bar{J}D^{\perp} = \{0\}$ and $\bar{J}D^{\perp} \cap TM = \{0\}$. Thus we conclude that $\bar{J}Rad(TM) = ltr(TM)$ as $dim(Rad(TM)) = dim(ltr(TM))$. Similarly, we have $\bar{J}D^{\perp} \subset S(TM^{\perp})$, this completes the proof. □

Let M be a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. Let Q, P_1, P_2 and P be the projections on D, $Rad(TM)$, $D^{\perp}$ and D' respectively. Then for any $X \in \Gamma(TM)$, we have

$$X = QX + P_1X + P_2X. \tag{3.1}$$

Applying $\bar{J}$ to (3.1), we obtain $\bar{J}X = \bar{J}QX + \bar{J}P_1X + \bar{J}P_2X$, hence we have $\bar{J}X = TQX + wP_1X + wP_2X$. Put $wP_1 = w_1$ and $wP_2 = w_2$, then we have

$$\bar{J}X = TX + w_1X + w_2X, \tag{3.2}$$

where $TX \in \Gamma(D), w_1X \in \Gamma(ltr(TM))$ and $w_2X \in \Gamma(\bar{J}D^{\perp}) \subset S(TM^{\perp})$. Similarly,

$$\bar{J}V = BV + CV, \tag{3.3}$$

for any $V \in \Gamma(tr(TM))$, where BV and CV are the sections of TM and $tr(TM)$, respectively. Differentiating (3.2) and using (2.6), (2.7) and (3.3), we obtain

$$(\nabla_X T)Y + (\nabla_Y T)X = A_{\omega_1 Y}X + A_{\omega_2 Y}X + A_{\omega_1 X}Y + A_{\omega_2 X}Y + 2Bh(X, Y), \tag{3.4}$$

$$(\nabla_X \omega_1)Y + (\nabla_Y \omega_1)X = -h^l(X, TY) - h^l(TX, Y) - D^l(X, \omega_2 Y) - D^l(Y, \omega_2 X), \tag{3.5}$$

$$
\begin{aligned}
(\nabla_X \omega_2)Y + (\nabla_Y \omega_2)X &= 2Ch^s(X,Y) - h^s(X,TY) - h^s(TX,Y) \\
&\quad - D^s(X,\omega_1 Y) - D^s(Y,\omega_1 X). \qquad (3.6)
\end{aligned}
$$

for any $X, Y \in \Gamma(TM)$.
Using nearly Kaehlerian property of $\bar{\nabla}$ with (2.5), we have the following lemma.

Lemma 3.2. *Let M be a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. Then we have*

$$(\nabla_X T)Y + (\nabla_Y T)X = A_{\omega Y}X + A_{\omega X}Y + 2Bh(X,Y), \tag{3.7}$$

$$(\nabla^t_X \omega)Y + (\nabla^t_Y \omega)X = 2Ch(X,Y) - h(X,TY) - h(TX,Y), \tag{3.8}$$

for any $X, Y \in \Gamma(TM)$, where

$$(\nabla_X T)Y = \nabla_X TY - T\nabla_X Y, \quad (\nabla^t_X \omega)Y = \nabla^t_X \omega Y - \omega \nabla_X Y. \tag{3.9}$$

Lemma 3.3. *([14]). If $\bar{M}$ is a nearly Kaehler manifold, then*

$$(\bar{\nabla}_X \bar{J})Y + (\bar{\nabla}_{\bar{J}X}\bar{J})\bar{J}Y = 0, \quad N(X,Y) = -4\bar{J}((\bar{\nabla}_X \bar{J})(Y)), \tag{3.10}$$

for any $X, Y \in \Gamma(T\bar{M})$, where $N(X,Y)$ is the Nijenhuis tensor given by

$$N(X,Y) = [\bar{J}X, \bar{J}Y] - \bar{J}[X, \bar{J}Y] - \bar{J}[\bar{J}X, Y] - [X,Y]. \tag{3.11}$$

Theorem 3.4. *Let M be a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. If D is integrable, then*

$$h(X, \bar{J}Y) = h(\bar{J}X, Y), \forall X, Y \in \Gamma(D).$$

Proof. For any $X, Y \in \Gamma(D)$, using (2.5) and (3.10), we have

$$(\nabla_X \bar{J}Y - \nabla_Y \bar{J}X) + (h(X, \bar{J}Y) - h(Y, \bar{J}X)) = \frac{1}{2}\bar{J}N(X,Y) + \bar{J}[X,Y]. \tag{3.12}$$

Since D is integrable, therefore it follows that $\bar{J}N(X,Y) \in \Gamma(D)$ and $\bar{J}[X,Y] \in \Gamma(D)$. Thus equating the transversal components in (3.12), the result follows. □

Theorem 3.5. *Let M be a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. Then D' is integrable if and only if*

$$2g(\nabla_Z X, V) = g(A_{\bar{J}Z}V + A_{\bar{J}V}Z, \bar{J}X),$$

for any $Z, V \in \Gamma(D')$ and $X \in \Gamma(D)$.

Proof. For any $Z, V \in \Gamma(D')$ and $X \in \Gamma(D)$, using (2.6), (2.7) and (2.12), we have

$$
\begin{aligned}
g([Z,V],X) &= \bar{g}(\bar{\nabla}_Z V, X) - g(\nabla_V Z, X) \\
&= \bar{g}(-(\bar{\nabla}_Z \bar{J})V + \bar{\nabla}_Z \bar{J}V, \bar{J}X) - g(\nabla_V Z, X) \\
&= \bar{g}((\bar{\nabla}_V \bar{J})Z), \bar{J}X) + \bar{g}(\bar{\nabla}_Z \bar{J}V, \bar{J}X) - g(\nabla_V Z, X) \\
&= \bar{g}(\bar{\nabla}_V \bar{J}Z, \bar{J}X) - \bar{g}(\bar{\nabla}_V Z, X) + \bar{g}(\bar{\nabla}_Z \bar{J}V, \bar{J}X) - g(\nabla_V Z, X) \\
&= -g(A_{\bar{J}Z}V + A_{\bar{J}V}Z, \bar{J}X) - 2g(\nabla_V Z, X) \\
&= -g(A_{\bar{J}Z}V + A_{\bar{J}V}Z, \bar{J}X) - 2g([V,Z],X) + 2\bar{g}(V, \nabla_Z X). \qquad (3.13)
\end{aligned}
$$

On simplifying (3.13), we derive $g([Z,V],X) = g(A_{\bar{J}Z}V + A_{\bar{J}V}Z, \bar{J}X) - 2g(\nabla_Z X, V)$, which proves our assertion. □

Theorem 3.6. *Let M be a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. If D defines a totally geodesic foliation in M, then*

$$h(X, \bar{J}Y) = h(\bar{J}X, Y) = \bar{J}h(X, Y), \forall X, Y \in \Gamma(D). \tag{3.14}$$

Proof. Suppose that D defines a totally geodesic foliation in M, then clearly D is integrable. Thus, first part of equality holds in view of Theorem (3.4). Now let $X, Y \in \Gamma(D)$, then using hypothesis alongwith (3.5) and (3.6), we have $h(X, \bar{J}Y) = Ch^s(X, Y)$. Further using $\bar{J}h(X, Y) = Bh(X, Y) + Ch^s(X, Y)$, we obtain $h(\bar{J}X, Y) = \bar{J}h(X, Y) - Bh(X, Y)$. As $X, Y \in \Gamma(D)$, therefore from (3.4), we have $(\nabla_X T)Y + (\nabla_Y T)X = 2Bh(X, Y)$. Since D defines a totally geodesic foliation in M with (2.12), we obtain $Bh(X, Y) = 0$ and hence $h(\bar{J}X, Y) = \bar{J}h(X, Y)$. □

4 Warped product lightlike submanifolds

Definition 4.1. ([2]). Let (B, g_B) and (F, g_F) be two Riemannian manifolds with Riemannian metric g_B and g_F respectively and f a positive differentiable function on B. The warped product of B and F is the Riemannian manifold $B \times_f F = (B \times F, g)$, where

$$g = g_B + f^2 g_F.$$

More explicitly, if U is tangent to $M = B \times_f F$ at (p, q), then

$$\|U\|^2 = \|d\pi_1(U)\|^2 + f^2(p)\|d\pi_1(U)\|^2,$$

where $\pi_i (i = 1, 2)$ are canonical projections of $B \times F$ onto B and F respectively and $d\pi_i$'s are their differentials. Here function f is called the warping function of the warped product. For differentiable function f on M, the gradient ∇f is defined by $g(\nabla f, U) = Uf$, for all $U \in \Gamma(TM)$.

Theorem 4.1. *([2]). Let $M = B \times_f F$ be a warped product manifold. If $X, Y \in T(B)$ and $U, V \in T(F)$, then*

$$\nabla_X Y \in T(B), \tag{4.1}$$

$$\nabla_X V = \nabla_V X = \left(\frac{Xf}{f}\right) V, \tag{4.2}$$

$$\nabla_U V = -\frac{g(U, V)}{f} \nabla f. \tag{4.3}$$

Corollary 4.2. *([2]). On a warped product manifold, $M = B \times_f F$,*

(i) B is totally geodesic in M.

(ii) F is totally umbilical in M.

Theorem 4.3. *There does not exist a proper warped product semi-transversal lightlike submanifold of the type $M = N_\perp \times_f N_T$ in an indefinite nearly Kaehler manifold $\bar{M}$ such that $N_\perp$ is a totally real submanifold and N_T is a holomorphic submanifold of $\bar{M}$.*

Proof. Suppose that M be a warped product semi-transversal lightlike submanifold of the type $M = N_\perp \times_f N_T$ in an indefinite nearly Kaehler manifold $\bar{M}$. Then for any $X \in \Gamma(D)$ and $Z \in \Gamma(D')$, using (4.2), we have

$$\nabla_X Z = \nabla_Z X = (Zlnf)X. \tag{4.4}$$

Thus, we have $g(\nabla_X Z, X) = (Zlnf)\|X\|^2 = g(\nabla_{\bar{J}X} Z, \bar{J}X)$. Further taking into account (2.6) and (2.12), we obtain

$$(Zlnf)\|X\|^2 = \bar{g}(\bar{\nabla}_{\bar{J}X} Z, \bar{J}X) = \bar{g}(\bar{J}Z, \bar{\nabla}_{\bar{J}X} X) = \bar{g}(\bar{J}Z, h^s(\bar{J}X, X)). \tag{4.5}$$

On changing X to $\bar{J}X$ in (4.5), we get

$$(Zlnf)\|X\|^2 = -\bar{g}(\bar{J}Z, h^s(\bar{J}X, X)). \tag{4.6}$$

Adding (4.5) and (4.6), we derive $(Zlnf)\|X\|^2 = 0$, then using non-degeneracy of D, we get $Zlnf = 0$. This implies that f is constant on $N_\perp$, which shows that M is a usual product. Hence, the proof is complete. □

From Theorem (4.3), we observe that there exist no warped product semi-transversal lightlike submanifolds of the type $M = N_\perp \times_f N_T$ in an indefinite nearly Kaehler manifold $\bar{M}$. Therefore, in the proceeding part of the paper, we consider warped product semi-transversal lightlike submanifolds of the type $M = N_T \times_f N_\perp$ in an indefinite nearly Kaehler manifold $\bar{M}$. For simplification, we call a warped product semi-transversal lightlike submanifold of the type $M = N_T \times_f N_\perp$ a semi-transversal lightlike warped product.

Lemma 4.4. *Let M be a semi-transversal lightlike warped product submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. Then, we have*

$$\bar{g}(h^s(X, Z), \bar{J}V) = -\bar{J}X(lnf)g(Z, V),$$

for any $X \in \Gamma(D)$ and $Z, V \in \Gamma(D^\perp) \subset \Gamma(D')$.

Proof. For any $X \in \Gamma(D)$ and $Z, V \in \Gamma(D^\perp)$, using (2.6) and (2.12), we have

$$\begin{aligned} g(A_{\bar{J}Z} X, V) &= -\bar{g}(\bar{\nabla}_X \bar{J}Z, V) = -\bar{g}((\bar{\nabla}_X \bar{J})Z + \bar{J}\bar{\nabla}_X Z, V) \\ &= \bar{g}((\bar{\nabla}_Z \bar{J})X, V) + \bar{g}(\bar{\nabla}_X Z, \bar{J}V) \\ &= \bar{g}(\bar{\nabla}_Z \bar{J}X, V) + \bar{g}(\bar{\nabla}_Z X, \bar{J}V) + \bar{g}(\bar{\nabla}_X Z, \bar{J}V) \\ &= g(\nabla_Z \bar{J}X, V) + 2\bar{g}(h^s(Z, X), \bar{J}V). \end{aligned} \tag{4.7}$$

Then using (2.8) and (4.2) in (4.7), we get

$$\bar{g}(h^s(X, V), \bar{J}Z) = \bar{J}X(lnf)g(Z, V) + 2\bar{g}(h^s(Z, X), \bar{J}V). \tag{4.8}$$

Now, interchanging the role of Z and V in (4.8), we obtain

$$\bar{g}(h^s(X, Z), \bar{J}V) = \bar{J}X(lnf)g(V, Z) + 2\bar{g}(h^s(V, X), \bar{J}Z). \tag{4.9}$$

Thus from (4.8) and (4.9), we get $\bar{g}(h^s(X, Z), \bar{J}V) = -\bar{J}X(lnf)g(Z, V)$, which proves the result. □

Definition 4.2. ([5]). A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be totally umbilical in $\bar{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that

$$h^l(X,Y) = H^l g(X,Y), \quad h^s(X,Y) = H^s g(X,Y), \quad D^l(X,W) = 0, \tag{4.10}$$

for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$.

Theorem 4.5. *A proper totally umbilical semi-transversal lightlike submanifold M of an indefinite nearly Kaehler manifold $\bar{M}$ is locally a semi-transversal lightlike warped product if and only if*

$$A_{\bar{J}Z}X = -(\bar{J}X)(\mu)Z, \tag{4.11}$$

for each $X \in \Gamma(D), Z \in \Gamma(D')$ and μ is a $C^{\infty}-$ function on M such that $W\mu = 0$ for each $W \in \Gamma(D')$ and

$$2g(\nabla_Z X, V) = g(A_{\bar{J}Z}V + A_{\bar{J}V}Z, \bar{J}X), \tag{4.12}$$

for any $Z, V \in \Gamma(D')$ and $X \in \Gamma(D)$.

Proof. Assume that M be a proper totally umbilical semi-transversal lightlike warped product submanifold of the type $N_T \times_f N_{\perp}$. As $\bar{M}$ is a nearly Kaehler manifold, therefore for each $X \in \Gamma(D)$ and $Z \in \Gamma(D')$, from (2.12), we have $\bar{\nabla}_X \bar{J}Z + \bar{\nabla}_Z \bar{J}X = \bar{J}\bar{\nabla}_X Z + \bar{J}\bar{\nabla}_Z X$, which on using (2.5), (4.2) and (4.10) gives that $-A_{\bar{J}Z}X + \nabla^t_X \bar{J}Z = \bar{J}X(lnf)Z$. Further equating tangential components on both sides, we derive $A_{\bar{J}Z}X = -\bar{J}X(lnf)Z$. Also, $\mu = lnf$ is a function on N_T, therefore $W(\mu) = W(lnf) = 0$ for all $W \in \Gamma(D')$. As M is a semi-transversal lightlike warped product submanifold, therefore D' is integrable, which proves (4.12) using Theorem (3.5).
Conversely, let M be a proper totally umbilical semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$ satisfying (4.11) and (4.12). For $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, using (4.11), we have $g(A_{\bar{J}Z}X, Y) = -g(((\bar{J}X)\mu)Z, Y) = 0$, then using (2.8), we get $\bar{g}(h^s(X,Y), \bar{J}Z) = 0$, that is, $\bar{g}(h^s(D,D), \bar{J}Z) = 0$. Now for $Z \in \Gamma(Rad(TM))$, we have $\bar{g}(h^s(D,D), \bar{J}Z) = 0$. Also, $\bar{g}(h^l(D,D), \bar{J}Z) = 0$, for each $Z \in \Gamma(D')$. Therefore, we have $\bar{g}(h(D,D), \bar{J}D') = 0$, that is, $h(D,D)$ has no component in $\bar{J}D'$, which implies that D defines a totally geodesic foliation in M and using Theorem (3.6), D is integrable.

On taking inner product of (4.11) with $U \in \Gamma(D')$ and using hypothesis alongwith (2.6), (2.12), (4.2) and (4.10), we have

$$\begin{aligned} g(((\bar{J}X)\mu)Z, U) &= -g(A_{\bar{J}Z}X, U) = -g(\bar{J}Z, \nabla_X U) = -g(\bar{J}Z, \nabla_U X) \\ &= \bar{g}(\bar{\nabla}_U \bar{J}Z, X) = \bar{g}(-(\bar{\nabla}_Z \bar{J})U + \bar{J}\bar{\nabla}_U Z, X) \\ &= -\bar{g}(\bar{\nabla}_Z \bar{J}U, X) + \bar{g}(\bar{J}\bar{\nabla}_Z U, X) - \bar{g}(\bar{\nabla}_U Z, \bar{J}X) \\ &= -g(\nabla_Z U, \bar{J}X) - g(\nabla_U Z, \bar{J}X), \end{aligned} \tag{4.13}$$

where $X \in \Gamma(D)$ and $Z \in \Gamma(D')$. Then using the definition of gradient $g(\nabla\phi, X) = X\phi$ in (4.13), we get

$$g(\nabla_Z U, \bar{J}X) + g(\nabla_U Z, \bar{J}X) = -g(\nabla\mu, \bar{J}X)g(Z, U). \tag{4.14}$$

Let h' be the second fundamental form of D' in M and let ∇' be the metric connection of D' in M, then from (4.14), we derive

$$g(h'(Z,U), \bar{J}X) = -\frac{1}{2}g(\nabla\mu, \bar{J}X)g(Z,U), \tag{4.15}$$

thus using non-degeneracy of D, from (4.15), we get

$$h'(Z,U) = -\frac{1}{2}\nabla\mu g(Z,U), \tag{4.16}$$

which implies that the distribution D' is totally umbilical in M. From (4.12) and using Theorem 3.5, the totally real distribution D' is integrable and further, using (4.16) and the condition $W\mu = 0$ for each $W \in \Gamma(D')$ implies that each leaf of D' is an intrinsic sphere in M. Thus by virtue of the result of [7], which states that "*If the tangent bundle of a Riemannian manifold M splits into an orthogonal sum $TM = E_0 \oplus E_1$ of non-trivial vector sub-bundles such that E_1 is spherical and its orthogonal complement E_0 is auto parallel, then the manifold M is locally isometric to a warped product $M_0 \times_f M_1$*", thus we conclude that M is locally a semi-transversal lightlike warped product of the type $N_T \times_f N_\perp$ in $\bar{M}$, where $f = e^\mu$. Hence the proof is complete. □

Lemma 4.6. *Let $M = N_T \times_f N_\perp$ be a semi-transversal lightlike warped product submanifold of an indefinite nearly Kaehler manifold $\bar{M}$, then*

$$(\nabla_Z T)X = TX(lnf)Z, \quad (\nabla_U T)Z = T(\nabla lnf)g(U,Z),$$

for any $U \in \Gamma(TM), X \in \Gamma(D)$ and $Z \in \Gamma(D')$, where $\nabla(lnf)$ denotes the gradient of lnf.

Proof. For $X \in \Gamma(D)$ and $Z \in \Gamma(D')$, from (3.9) and (4.2), we have $(\nabla_Z T)X = \nabla_Z TX = TX(lnf)Z$. Again using (3.9) for $U \in \Gamma(TM)$ and $Z \in \Gamma(D')$, we get $(\nabla_U T)Z = -T\nabla_U Z$, which implies that $(\nabla_U T)Z \in \Gamma(D)$. Then for any $X \in \Gamma(D)$, we have $g((\nabla_U T)Z, X) = -g(T\nabla_U Z, X) = g(\nabla_U Z, TX) = \bar{g}(\bar{\nabla}_U Z, TX) = -\bar{g}(Z, \nabla_U TX) = -TX(lnf)g(Z,U)$, then using definition of gradient of f and non-degeneracy of D, the result follows. □

Theorem 4.7. *Let M be a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$ with totally real distribution D' being integrable. Then M is locally a semi-transversal lightlike warped product submanifold if and only if*

$$(\nabla_U T)U = ((TU)\mu)PU + \|PU\|^2\bar{J}\nabla\mu, \tag{4.17}$$

for each $U \in \Gamma(TM)$, where μ is a $C^\infty-$ function on M satisfying $Z\mu = 0$ for each $Z \in \Gamma(D')$.

Proof. Assume that M be a semi-transversal lightlike warped product submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. Then, for any $U \in \Gamma(TM)$, we have

$$(\nabla_U T)U = (\nabla_{QU} T)QU + (\nabla_{PU} T)QU + (\nabla_U T)PU. \tag{4.18}$$

Since D defines a totally geodesic foliation in M, therefore using (3.4), we have

$$(\nabla_{QU}T)QU = 0. \tag{4.19}$$

Further, using Lemma (4.6), we obtain

$$(\nabla_{PU}T)QU = T(QU)(lnf)PU, \tag{4.20}$$

$$(\nabla_U T)PU = g(U, PU)T(\nabla lnf) = \|PU\|^2 T(\nabla lnf). \tag{4.21}$$

Then from (4.18) - (4.21), we derive (4.17).
Conversely, let M be a semi-transversal lightlike submanifold of an indefinite nearly Kaehler manifold $\bar{M}$ satisfying (4.17). The relation (4.17) is equivalent to

$$(\nabla_U T)V + (\nabla_V T)U = ((TU)\mu)PV + ((TV)\mu)PU + 2g(PU, PV)\bar{J}\nabla\mu. \tag{4.22}$$

Let $U, V \in \Gamma(D)$, then (4.22) implies that $(\nabla_U T)V + (\nabla_V T)U = 0$; further using (3.4), we obtain $Bh(U, V) = 0$, which shows that $h(U, V)$ has no component in $\bar{J}D'$, for each $U, V \in \Gamma(D)$. Thus D defines a totally geodesic foliation in M and consequently D is integrable using Theorem (3.6).
For $U, V \in \Gamma(D')$, from (4.22), we have

$$(\nabla_U T)V + (\nabla_V T)U = 2g(PU, PV)\bar{J}\nabla\mu. \tag{4.23}$$

Now taking inner product of (4.23) with $X \in \Gamma(D)$, we obtain

$$g((\nabla_U T)V + (\nabla_V T)U, X) = 2g(PU, PV)g(\bar{J}\nabla\mu, X). \tag{4.24}$$

Also using (3.4) for $X \in \Gamma(D)$, we get

$$\begin{aligned} g((\nabla_U T)V + (\nabla_V T)U, X) &= g(A_{\omega V}U, X) + g(A_{\omega U}V, X) \\ &= -\bar{g}(\bar{\nabla}_U \bar{J}V, X) - \bar{g}(\bar{\nabla}_V \bar{J}U, X) \\ &= g(\nabla_U V, \bar{J}X) + g(\nabla_V U, \bar{J}X). \end{aligned} \tag{4.25}$$

From (4.24) and (4.25), we have

$$g(\nabla_U V, \bar{J}X) + g(\nabla_V U, \bar{J}X) = 2g(PU, PV)g(\bar{J}\nabla\mu, X). \tag{4.26}$$

Let h' be the second fundamental form of D' in M and let ∇' be the metric connection of D' in M, then from (4.26), we have $g(h'(U, V), \bar{J}X) = -g(PU, PV)g(\nabla\mu, \bar{J}X)$, then the non-degeneracy of D implies that $h'(U, V) = -\nabla\mu g(PU, PV)$, this shows that the distribution D' is totally umbilical in M. Moreover by hypothesis, the totally real distribution D' is integrable and in view of condition that $Z\mu = 0$, for each $Z \in \Gamma(D')$, each leaf of D' is an intrinsic sphere. Thus, by similar argument as in Theorem (4.5), M is locally a semi-transversal lightlike warped product of the type $N_T \times_f N_\perp$ in $\bar{M}$ with a warping function $f = e^\mu$, which completes the proof. □

Theorem 4.8. *Let M be a semi-transversal lightlike warped product submanifold of an indefinite nearly Kaehler manifold $\bar{M}$, then*

$$\bar{g}((\nabla^t_U\omega)V + (\nabla^t_V\omega)U, \bar{J}W) = -QU(\mu)g(V, W) - QV(\mu)g(U, W), \tag{4.27}$$

for any $U, V \in \Gamma(TM)$, where μ is a $C^\infty-$ function on M satisfying $W\mu = 0$ for each $W \in \Gamma(D')$.

Proof. Let M be semi-tranversal lightlike warped product submanifold of an indefinite nearly Kaehler manifold $\bar{M}$. Therefore, the distribution D defines a totally geodesic foliation in M, thus using (3.9) for $U, V \in \Gamma(D)$, we have

$$\bar{g}((\nabla^t_U \omega)V + (\nabla^t_V \omega)U, \bar{J}W) = -g(\nabla_U V, W) - g(\nabla_V U, W) = 0. \tag{4.28}$$

For $U, V \in \Gamma(D')$, using (3.8), we derive

$$\bar{g}((\nabla^t_U \omega)V + (\nabla^t_V \omega)U, \bar{J}W) = 2\bar{g}(Ch(U, V), \bar{J}W) = 0. \tag{4.29}$$

Now for $U \in \Gamma(D)$ and $V \in \Gamma(D')$, using (3.8) and Lemma (4.4), we get

$$\bar{g}((\nabla^t_U \omega)V + (\nabla^t_V \omega)U, \bar{J}W) = -\bar{g}(h^s(V, TU), \bar{J}W) = -QU(lnf)g(V, W). \tag{4.30}$$

Similarly, for $U \in \Gamma(D')$ and $V \in \Gamma(D)$, using (3.8) and Lemma (4.4), we obtain

$$\bar{g}((\nabla^t_U \omega)V + (\nabla^t_V \omega)U, \bar{J}W) = -QV(lnf)g(U, W). \tag{4.31}$$

Hence, (4.27) follows from (4.28)-(4.31), which completes the proof. □

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