Ricci solitons on trans-Sasakian manifolds

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Abstract. This paper deals with the study of a special class of almost contact metric manifold, called trans-Sasakian manifold. We also study the properties of the Ricci solitons in generalized recurrent, Weyl semisymmetric, Einstein semisymmetric, Weyl pseudo symmetric and partially Ricci pseudo symmetric trans-Sasakian manifolds. Example of trans-Sasakian manifold is given in the last section to validate our results.

M.S.C. 2010: 53D15, 53C25, 53B30.
Key words: trans-Sasakian manifolds; generalized recurrent; Weyl conformal curvature tensor; pseudo-symmetric manifold; Ricci solitons.

1 Introduction

A class of almost contact metric manifold known as trans-Sasakian manifold was introduced by Oubino [24] in 1985. In [17], Gray-Hervella classification of the almost Hermite manifolds appeared as a class W_4 of the Hermitian manifolds which are closely related to the locally conformally Kähler manifolds. An almost contact metric structure on an almost contact metric manifold M is called a trans-Sasakian structure if the product manifold $M \times \Re$ belongs to the class W_4. The trans-Sasakian structures also provide a large class of generalized quasi-Sasakian structures. The local structures of trans-Sasakian manifolds of dimension $n \geq 5$ have been completely characterized by Marrero [21]. He proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or $\alpha-$Sasakian or $\beta-$Kenmotsu manifold. Many authors studied the properties of trans-Sasakian manifolds (for instance, [13], [25], [31]).

In 1982, Hamilton [18] made the fundamental observation that the Ricci flow is an excellent tool for simplifying the structure of a manifold. It is a process which deforms the metric of a Riemannian manifold by smoothing out the irregularities. It is given by

$$\frac{\partial g}{\partial t} = -2Ric\, g. \tag{1.1}$$

where $'g'$ is a Riemannian metric, $'Ric'$ is the Ricci curvature tensor and $'t'$ is time. Let $\varphi_t : M \to M, t \in R$ be a family of diffeomorphisms which is one parameter group of transformations, then it gives rise to a vector field, called infinitesimal generator and integral curves. Ricci soliton moves under the Ricci flow simply by diffeomorphisms

Differential Geometry - Dynamical Systems, Vol.20, 2018, pp. 138-158.

of the initial metric, that is they are stationary points of the Ricci flow in space of metrics of $\varphi_t : M \to M$. Here the metric $g(t)$ is the pull back of the initial metric $g(0)$ of φ_t. Ricci soliton in a Riemannian manifold (M, g) is a special solution to the Ricci flow based on a natural generalization of an Einstein metric, which is defined via the triplet (g, V, λ), where g is a Riemannian metric, V a vector field and λ a real scalar such that

$$L_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where S is a Ricci tensor and L_V is the Lie-derivative along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding when λ is negative, zero and positive respectively. In [28], Sharma initiated the study of Ricci soliton in contact Riemannian geometry. Later Tripathi [30], Nagaraja et al. [23] and others extensively studied Ricci soliton in contact metric manifolds. But the study was extended by Calin et al. [4], Bagewadi et al. [1], Debnath et al. [13], for $f-$Kenmotsu, Lorentzian $\alpha-$Sasakian and Trans-Sasakian manifolds respectively using L. P. Eisenhart problem [16] and also by many others ([10], [19], [20]). It is well known that, if the potential vector field is zero or Killing then the Ricci soliton is an Einstein metric. In ([5], [11], [22]), authors proved that there are no Einstein real hypersurfaces of non-flat complex space forms.

Motivated by the above studied, authors start the study of trans-Sasakian manifolds. We organize our present work as: after introduction, we brief the basic known results of trans-Sasakian manifolds and definitions in section 2. Sections 3 and 4 deal with study of Ricci soliton and generalized recurrent trans-Sasakian manifolds. We study the properties of Ricci solitons in Weyl semisymmetric, Einstein semisymmetric, Weyl pseudo symmetric, partially Ricci pseudo symmetric and Weyl Ricci pseudo symmetric trans-Sasakian manifolds in sections 5, 6, 7, 8 and 9 respectively. Next section concerns with the example of trans-Sasakian manifold and hence we validate our results.

2 Preliminaries

A differentiable manifold M $(dimM = n = 2m+1)$ is said to be an almost contact metric manifold if it admits a $(1,1)$ tensor field φ, a vector field ζ, a $1-$form η and the Riemannian metric g, which satisfy

$$\varphi^2 = -I + \eta \otimes \zeta, \quad \eta(\zeta) = 1, \quad \varphi\zeta = 0, \quad \eta(\varphi X) = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \zeta), \tag{2.2}$$

for all vector fields X, Y on M. An almost contact metric manifold $M(\varphi, \zeta, \eta, g)$ is said to be trans-Sasakian manifold if $(M \times \Re, J, G)$ belongs to the class W_4 of the Hermitian manifold, where J is the almost complex structure of $M \times \Re$ defined by

$$J(Z, f\frac{d}{dt}) = (\varphi Z + f\zeta, \eta(Z)\frac{d}{dt}),$$

for all vector field Z on M and smooth function f on $M \times \Re$ and G is the product metric on $M \times \Re$. This is expressed by the following restriction

$$(\nabla_X \varphi)Y = \beta\{g(\varphi X, Y)\zeta - \eta(Y)\varphi X\} + \alpha\{g(\varphi X, Y)\zeta - \eta(Y)X\}, \tag{2.3}$$

where α and β are some scalar functions and such a structure is said to be the trans-Sasakian structure of type (α, β). Here ∇ denotes the Levi-civita connection of g. We note that trans-Sasakian manifolds of type $(0,0)$, $(\alpha, 0)$ and $(0, \beta)$ are the cosymplectic, $\alpha-$Sasakian and $\beta-$Kenmotsu manifolds respectively. In particular, if $\alpha = 1, \beta = 0$ and $\alpha = 0, \beta = 1$, then a trans-Sasakian manifold reduces to a Sasakian and Kenmotsu manifolds respectively. From (2.3), it follows that

$$\nabla_X \zeta = \beta\left\{X - \eta(X)\zeta\right\} - \alpha\varphi X, \tag{2.4}$$

$$(\nabla_X \eta)Y = -\alpha g(\varphi X, Y) + \beta g(\varphi X, \varphi Y). \tag{2.5}$$

The trans-Sasakian manifold with structure tensor $(\varphi, \zeta, \eta, g)$ on M satisfies the following relations:

$$\begin{aligned} R(X,Y)\zeta \;&=\; 2\alpha\beta[\eta(Y)\varphi X - \eta(X)\varphi Y] + (Y\alpha)\varphi X - (X\alpha)\varphi Y + (Y\beta)\varphi^2 X \\ &\quad -(X\beta)\varphi^2 Y + \left(\alpha^2 - \beta^2\right)[\eta(Y)X - \eta(X)Y], \end{aligned} \tag{2.6}$$

$$\begin{aligned} R(\zeta, X)Y \;&=\; (\alpha^2 - \beta^2)\{g(X,Y)\zeta - \eta(Y)X\} + 2\alpha\beta\{g(\varphi Y, X)\xi \\ &\quad -\eta(Y)\varphi X\} + (Y\alpha)\varphi X + g(\varphi Y, X)(grad\alpha) \\ &\quad +(Y\beta)(X - \eta(X)\xi) - g(\varphi X, \varphi Y)(grad\beta). \end{aligned} \tag{2.7}$$

$$2\alpha\beta + \zeta\alpha = 0, \tag{2.8}$$

$$S(X, \zeta) = ((n-1)(\alpha^2 - \beta^2) - \zeta\beta)\eta(X) - (n-2)(X\beta) - (\varphi X)\alpha, \tag{2.9}$$

$$Q\zeta = ((n-1)(\alpha^2 - \beta^2) - \zeta\beta)\zeta - (n-1)\, grad\,\beta - \varphi(grad\,\alpha), \tag{2.10}$$

where R is curvature tensor, while Q is the Ricci operator given by $S(X,Y) = g(QX, Y)$. Further if a trans-Sasakian manifold of type (α, β) satisfies

$$(n-1)\, grad\,\beta = \varphi(grad\,\alpha), \tag{2.11}$$

then from (2.6), (2.7), (2.9), (2.10) and (2.11), for constants α and β, we have

$$R(X,Y)\zeta = (\alpha^2 - \beta^2)\{\eta(Y)X - \eta(X)Y\}, \tag{2.12}$$

$$R(\zeta, X)Y = (\alpha^2 - \beta^2)\{g(X,Y)\zeta - \eta(Y)X\}, \tag{2.13}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \beta^2)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{2.14}$$

$$S(X, \zeta) = (n-1)(\alpha^2 - \beta^2)\eta(X), \tag{2.15}$$

$$Q\zeta = (n-1)(\alpha^2 - \beta^2)\zeta. \tag{2.16}$$

An important consequence of (2.4) is that ζ is a geodesic vector field, i. e., $\nabla_\zeta \zeta = 0$. Also for arbitrary vector field X, we have $d\eta(\zeta, X) = 0$.

The $\zeta-$sectional curvature K_ζ on (M, g) is the sectional curvature of a plane spanned by ζ and a unitary vector field X. From (2.12), we conclude that

$$K_\zeta = g(R(X,\zeta)\zeta, X) = (\alpha^2 - \beta^2), \tag{2.17}$$

for arbitrary vector field X on M. It follows from (2.17), $\zeta-$sectional curvature does not depend on X.

For an $n-$dimensional almost contact metric manifold M, the Weyl conformal curvature tensor $\breve{W}$ is given by

$$\begin{aligned} \breve{W}(X,Y)Z &= R(X,Y)Z - \frac{1}{(n-2)}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY\} + \frac{\kappa}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{2.18}$$

where κ represents the scalar curvature of the manifold. In consequence of (2.1), (2.2), (2.13), (2.15), (2.16) and (2.18), we find that

$$\begin{aligned} \breve{W}(\zeta,Y)Z &= \frac{1}{(n-1)(n-2)}[\kappa - (n-1)(\alpha^2-\beta^2)]\{g(Y,Z)\zeta - \eta(Z)Y\} \\ &\quad -\frac{1}{(n-2)}\{S(Y,Z)\xi - \eta(Z)QY\}, \end{aligned} \tag{2.19}$$

$$\begin{aligned} \breve{W}(\zeta,Y)\zeta &= \frac{1}{(n-1)(n-2)}[\kappa - (n-1)(\alpha^2-\beta^2)]\{\eta(Y)\zeta - Y\} \\ &\quad -\frac{1}{(n-2)}\{(n-1)(\alpha^2-\beta^2)\eta(Y)\xi - QY\}, \end{aligned} \tag{2.20}$$

$$\breve{W}(\zeta,\zeta)Z = 0. \tag{2.21}$$

Here we illustrate some definitions that are useful to deduce our results.

Definition 2.1. An $n-$dimensional trans-Sasakian manifold (M, g) is called generalized recurrent [12] if its non-vanishing curvature tensor R satisfies the following restriction

$$(\nabla_X R)(Y,Z)U = \psi(X)R(Y,Z)U + \chi(X)\{g(Z,U)Y - g(Y,U)Z\}, \tag{2.22}$$

for arbitrary vector fields X, Y, Z and U on M, where ψ and χ are $1-$forms such that χ is non-zero and are defined by

$$\psi(X) = g(X,A),\ \ \chi(X) = g(X,B), \tag{2.23}$$

where A and B are vector fields associated with $1-$forms ψ and χ respectively and ∇ is the Riemannian connection of g. In particular if $\chi = 0$, then generalized recurrent manifold reduces to recurrent manifold ([26], [27]).

Definition 2.2. An $n-$dimensional trans-Sasakian manifold (M, g) is called generalized Ricci recurrent [12] if its Ricci tensor S satisfies the following restriction

$$(\nabla_X S)(Y, Z) = \psi(X)S(Y, Z) + (n-1)\chi(X)g(Y, Z), \tag{2.24}$$

where ψ and χ are $1-$forms, χ is non-zero and these are defined by (2.23).

Definition 2.3. An $n-$dimensional trans-Sasakian manifold (M, g) is called generalized concircular recurrent [12] if its concircular curvature tensor $\breve{C}$ [32].

$$\breve{C}(X, Y)Z = R(X, Y)Z - \frac{\kappa}{n(n-1)}\{g(Y, Z)X - g(X, Z)Y\}, \tag{2.25}$$

satisfies the following condition

$$(\nabla_X \breve{C})(Y, Z)U = \psi(X)\breve{C}(Y, Z)U + \chi(X)\{g(Z, U)Y - g(Y, U)Z\}, \tag{2.26}$$

where ψ and χ are defined as in (2.23) and κ is the scalar curvature of (M, g).

Definition 2.4. An $n-$dimensional trans-Sasakian manifold (M, g) is called Weyl-semisymmetric [29] if $R \cdot \breve{W} = 0$.

Definition 2.5. An $n-$dimensional trans-Sasakian manifolds (M, g) is called Einstein-semisymmetric [29] if $R \cdot E = 0$, where E is the Einstein tensor given by

$$E(Y, Z) = S(Y, Z) - \frac{\kappa}{n}g(Y, Z). \tag{2.27}$$

Definition 2.6. An $n-$dimensional trans-Sasakian manifold (M, g) is called Weyl pseudosymmetric [14, 15] if the tensors $R \cdot \breve{W}$ and $Q(g, W)$ are linearly dependent that are defined by

$$R \cdot \breve{W} = L_{\breve{W}} Q(g, \breve{W}), \tag{2.28}$$

holds on the set $\bigcup_{\breve{W}} = \{x \in M : \breve{W} \neq 0 \ \ at \ x\}$, where $L_{\breve{W}}$ is some function on $\bigcup_{\breve{W}}$.

Definition 2.7. An $n-$dimensional trans-Sasakian manifold $(M, g)$is called partially Ricci-pseudosymmetric [3] if and only if the relation defined by

$$R \cdot S = f(q) \ Q(g, S), \tag{2.29}$$

holds on the set $\bigcup = \{ \ x \in M : Q(g, S) \neq 0 \ \ at \ x \ \}$, where $f \in C^{\infty}(M)$ for $q \in \bigcup$. $R \cdot S$, $Q(g, S)$ and $(X \wedge_g Y)$ are respectively defined as

$$(R(X, Y) \cdot S)(U, V) = -S(R(X, Y)U, V) - S(U, R(X, Y)V). \tag{2.30}$$

$$Q(g, S) = ((X \wedge_g Y) \cdot S)(U, V). \tag{2.31}$$

$$(X \wedge_g Y)Z = g(Y, Z)X - g(X, Z)Y, \tag{2.32}$$

for all X, Y, U and V on M.

Definition 2.8. An $n-$dimensional trans-Sasakian manifold (M, g) is called Weyl Ricci pseudosymmetric [14, 15] if the tensors $\breve{W} \cdot S$ and $Q(g, S)$ are linearly dependent, that is defined by

$$\breve{W} \cdot S = L_S Q(g, S), \tag{2.33}$$

and holds on the set $\bigcup_S = \{x \in M : \breve{W} \neq 0 \ \ at \ x\}$, where L_S is some function on $\bigcup_S$.

The different classes of symmetric spaces in different extents have been studied by many authors in ([26], [27], [29], [14], [15], [3], [12], [6], [7], [8], [9]).

3 Ricci solitons on $(M, \varphi, \zeta, \eta, g)$

We call the notion of Ricci soliton from [11]. Thus from equation (1.2) we have

$$(L_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{3.1}$$

where L_V is the Lie-derivative operator along the vector field V and λ is a real constant. We have two natural situations regarding the vector field V : V $\in$ Spanζ and $V \perp \zeta$. We investigate only the case $V = \xi$. A straightforward calculation from (2.4) gives

$$(L_\zeta g)(X, Y) = 2\beta\{g(X, Y) - \eta(X)\eta(Y)\}. \tag{3.2}$$

With reference to (3.2), equation (3.1) reduces to

$$\mathrm{S(X, Y)} = -(\beta + \lambda)\mathrm{g(X, Y)} + \beta\eta(\mathrm{X})\eta(\mathrm{Y}), \tag{3.3}$$

$$\mathrm{S(X,}\zeta) = \mathrm{S}(\zeta, \mathrm{X}) = -\lambda\eta(\mathrm{X}), \tag{3.4}$$

$$\mathrm{S}(\zeta,\zeta) = -\lambda, \tag{3.5}$$

$$\mathrm{QX} = -(\beta + \lambda)\mathrm{X} + \beta\eta(\mathrm{X})\xi, \tag{3.6}$$

$$\kappa = -\lambda\mathrm{n} - (\mathrm{n} - 1)\beta, \tag{3.7}$$

$$\mathrm{Q}\zeta = -\lambda\xi, \tag{3.8}$$

$$\lambda = -(n - 1)(\alpha^2 - \beta^2) = -(n - 1)K_\zeta. \tag{3.9}$$

In a 3$-$dimensional trans-Sasakian manifold [31], we have

$$S(X, Y) = \left\{\frac{\kappa}{2} - (\alpha^2 - \beta^2)\right\} g(X, Y) - \left\{\frac{\kappa}{2} - 3(\alpha^2 - \beta^2)\right\} \eta(X)\eta(Y). \tag{3.10}$$

Let us consider

$$h(X,Y) = (L_\zeta g)(X,Y) + 2S(X,Y). \tag{3.11}$$

In view of (3.2) and (3.10), equation (3.11) reduces to

$$h(X,Y) = \left\{ \kappa - 2(\alpha^2 - \beta^2) + 2\beta \right\} g(X,Y) - \left\{ \kappa - 6(\alpha^2 - \beta^2) + 2\beta \right\} \eta(X)\eta(Y).$$

Replacing X = Y = ζ in (3.11), we yield

$$h(\zeta,\zeta) = 4(\alpha^2 - \beta^2). \tag{3.12}$$

Before going to state our result, we will recall the

Theorem 3.1. *Let (M,φ,ζ,η,g) be a trans-Sasakian manifold with non-vanishing $\xi-$sectional curvature and endowed with a tensor field $h \in \Gamma(T_2^0(M))$ which is symmetric and $\varphi-$skew-symmetric. If h is parallel with respect to ∇ then it is a constant multiple of the metric tensor g (p. 221, [31]).*

In view of theorem 3.1 and equation (3.12), we are in position to state the result as follow:

Theorem 3.2. *Let $(M^3,\varphi,\zeta,\eta,g)$ be a $3-$dimensional trans-Sasakian manifold, then (g,ζ,λ) yields a Ricci soliton on $(M^3,\varphi,\zeta,\eta,g)$.*

As a consequence of theorem 3.2, we have

Corollary 3.3. *A Ricci soliton generated by (g,ζ,λ) in a $3-$dimensional trans-Sasakian manifold $(M^3,\varphi,\zeta,\eta,g)$*
(i) of type $(0,0)$, i.e. cosymplectic, is always steady,
(ii) of type $(\alpha,0)$, i. e. $\alpha-$Sasakian, is always shrinking,
(iii) of type $(0,\beta)$, i. e. $\beta-$Kenmotsu, is always expanding.

4 Generalized recurrent trans-Sasakian manifolds

In this section we illustrate following theorems that are related to generalized recurrent trans-Sasakian manifolds.

Theorem 4.1. *If (M,g) is a generalized recurrent trans-Sasakian manifold, then $K_\zeta\psi + \chi = 0$.*

Proof. Let (M,g) be is a generalized recurrent trans-Sasakian manifold. From (2.22) taking $Y = U = \zeta$ we obtain

$$(\nabla_X R)(\zeta,Z)\zeta = \psi(X)R(\zeta,Z)\zeta + \chi(X)\{g(Z,\zeta)\zeta - g(\zeta,\zeta)Z\}. \tag{4.1}$$

It is clear that

$$(\nabla_X R)(\zeta,Z)\zeta = \nabla_X R(\zeta,Z)\zeta - R(\nabla_X\zeta,Z)\zeta - R(\zeta,\nabla_X Z)\zeta - R(\zeta,Z)\nabla_X\zeta.$$

In view of (2.1), (2.2), (2.4), (2.5), (2.12) and (2.13), above equation takes the form

$$(\nabla_X R)(\zeta,Z)\zeta = 0. \tag{4.2}$$

Keeping in mind (4.2), equation (4.1) has the result

$$\left\{(\alpha^2-\beta^2)\psi(X)+\chi(X)\right\}\ (\eta(Z)\zeta-Z)=0.$$

It is remarkable that the equality $\eta(Z)\zeta - Z = 0$ does not hold on (M, g), provided $n \geq 1$. Thus we have

$$(\alpha^2-\beta^2)\psi(X)+\chi(X)=0, \tag{4.3}$$

for all $X \in T(M)$. That is independent of choice of the vector field X. Thus we yield $K_\zeta\psi + \chi = 0$. Proof is completed. □

Theorem 4.2. *If (M, g) is a generalized Ricci recurrent trans-Sasakian manifold, then $K_\zeta\psi + \chi$ is everywhere zero.*

Proof. It is clear that if (M, g) be a generalized Ricci recurrent trans-Sasakian manifold. Then from (2.24) taking $Y = Z = \zeta$ and adopting $(\nabla_X S)(\zeta, \zeta) = 0$, we obtain that $K_\zeta\psi + \chi = 0$. □

Corollary 4.3. *If (M, g) is a generalized recurrent trans-Sasakian manifold, then the scalar curvature κ of (M, g) satisfies the following restriction*

$$\kappa = n(n-1)K_\zeta.$$

Proof. Making use of Bianchi's identity in (2.22), we get

$$\begin{aligned}&\psi(X)R(Y,Z)W+\chi(X)\{g(Z,W)Y-g(Y,W)Z\}+\chi(Y)\{g(X,W)Z-g(Z,W)X\}\\&+\psi(Y)R(Z,X)W+\psi(Z)R(X,Y)W+\chi(Z)\{g(Y,W)X-g(X,W)Y\}=0.\end{aligned}$$

The contraction of above equation along the vector field Y gives

$$\begin{aligned}&\psi(X)S(Z,W)+(n-1)\chi(X)g(Z,W)+\psi(R(Z,X)W)+\chi(Z)g(X,W)\\&-\chi(X)g(Z,W)-\psi(Z)S(X,W)-(n-1)\chi(Z)g(X,W)=0,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\psi(X)QW+(n-2)\chi(X)W-R(W,A)X\\&-S(X,W)A-(n-2)g(X,W)B=0.\end{aligned}$$

Again contracting last expression along W, we find that

$$\kappa\psi(X)+n(n-2)\chi(X)-2S(A,X)-(n-2)g(X,B)=0.$$

Putting $X = \zeta$ in last equation and then use of (2.1), (2.2), (2.23), (2.17) and (4.3) in it, we find

$$[\kappa-n(n-1)K_\zeta]\eta(A)=0.$$

Since $\eta \neq 0$ on almost contact metric manifold (in general), therefore above equation gives the statement of the corollary. □

Theorem 4.4. *If (M, g) be a generalized concircular recurrent trans-Sasakian manifold, then following restriction*

$$\left[\left(K_\zeta - \frac{\kappa}{n(n-1)}\right)\psi(X) + \chi(X) + \frac{X[\kappa]}{n(n-1)}\right] = 0,$$

holds for every vector field X on (M, g).

Proof. Assume that (M, g) is a generalized concircular recurrent trans-Sasakian manifold. Then replacing $Y = U = \zeta$ in (2.26), we have

$$(\nabla_X \breve{C})(\zeta, Z)\zeta = \psi(X)\breve{C}(\zeta, Z)\zeta + \chi(X)\{g(Z, \zeta)\zeta - g(\zeta, \zeta)Z\}. \tag{4.4}$$

By definition of covariant derivative and then use of (2.1), (2.2), (2.4), (2.5), (2.8), (2.12), (2.13) and (2.25), we have

$$(\nabla_X \breve{C})(\zeta, Z)\zeta = -\left\{\frac{X[\kappa]}{n(n-1)}\right\}[\eta(Z)\zeta - Z], \tag{4.5}$$

where $X[\kappa]$ indicates the derivative of κ with respect to the vector field X. In view of (4.5) and (2.17), equation (4.4) takes the form

$$\left[\left(K_\zeta - \frac{\kappa}{n(n-1)}\right)\psi(X) + \chi(X) + \frac{X[\kappa]}{n(n-1)}\right]\{\eta(Z)\zeta - Z\} = 0. \tag{4.6}$$

It is remarkable that the equality $\eta(Z)\zeta - Z = 0$ does not hold on (M, g). Thus (4.6) complete the proof. □

5 Ricci Soliton in Weyl semisymmetric trans-Sasakian manifolds

We suppose that the trans-Sasakian manifold (M, g) is Weyl semisymmetric. Then from definition 2.4, we get

$$(R(X, Y) \cdot \breve{W})(U, V)Z = 0. \tag{5.1}$$

With reference to (7), we write it as

$$R(X, Y)\breve{W}(U, V)Z - \breve{W}(R(X, Y)U, V)Z - \breve{W}(U, R(X, Y)V)Z - \breve{W}(U, V)R(X, Y)Z = 0. \tag{5.2}$$

Replacing $X = \zeta$ in (5.2) and keeping in mind the equation (2.13), we have

$$\begin{aligned}&'\breve{W}(U, V, Z, Y)\zeta - \eta(\breve{W}(U, V)Z)Y - g(Y, U)\breve{W}(\zeta, V)Z + \eta(U)\breve{W}(Y, V)Z \\ &-g(Y, V)\breve{W}(U, \zeta)Z + \eta(V)\breve{W}(U, Y)Z - g(Y, Z)\breve{W}(U, V)\zeta + \eta(Z)\breve{W}(U, V)Y = 0,\end{aligned}$$

which gives

$$\begin{aligned}&'\breve{W}(U, V, Z, Y) - \eta(\breve{W}(U, V)Z)\eta(Y) - g(Y, U)\eta(\breve{W}(\zeta, V)Z) \\ &+\eta(U)\eta(\breve{W}(Y, V)Z) - g(Y, V)\eta(\breve{W}(U, \zeta)Z) + \eta(V)\eta(\breve{W}(U, Y)Z) \\ &-g(Y, Z)\eta(\breve{W}(U, V)\zeta) + \eta(Z)\eta(\breve{W}(U, V)Y) = 0,\end{aligned} \tag{5.3}$$

where equations (2.1) and (2.2) are used. Let $\{e_i, i = 1, 2, ..., n\}$ be a set of orthonormal basis of the tangent space each point of the manifold. Setting $Y = U = e_i$ in (5.3) and then taking the summation over i, $1 \le i \le n$, we find that

(5.4)
$$\sum_{i=1}^{n} {}'\breve{W}(e_i, V, Z, e_i) + \eta(Z) \sum_{i=1}^{n} {}'\breve{W}(e_i, V, \zeta, e_i) = (n-1)\eta(\breve{W}(\zeta, V)Z) + \eta(\breve{W}(Z, V)\zeta).$$

In consequence of (2.1), (2.2), (2.12), (2.13), (2.14), (2.15), (2.16), (2.18) and (2.19), we can find that

$$\begin{aligned} \eta(\breve{W}(X,Y)Z) &= \frac{1}{n-2}\left[\frac{\kappa}{n-1} - (\alpha^2-\beta^2)\right]\{\eta(X)g(Y,Z) - \eta(Y)g(X,Z)\} \\ &\quad -\frac{1}{n-2}\{\eta(X)S(Y,Z) - \eta(Y)S(X,Z)\}, \end{aligned} \tag{5.5}$$

$$\begin{aligned} \eta(\breve{W}(\zeta,Y)Z) &= \frac{1}{n-2}\left[-S(Y,Z) + \{\frac{\kappa}{n-1} - (\alpha^2-\beta^2)\}g(Y,Z)\right] \\ &\quad -\frac{1}{n-2}\left[\frac{\kappa}{n-1} - n(\alpha^2-\beta^2)\right]\eta(Y)\eta(Z), \end{aligned} \tag{5.6}$$

$$\eta(\breve{W}(X,Y)\xi) = 0, \tag{5.7}$$

and

$$\sum_{i=1}^{n} {}'\breve{W}(e_i, V, Z, e_i) = 0. \tag{5.8}$$

In view of (5.7) and (5.8), (5.4) gives

$$\eta(\breve{W}(\zeta,Y)Z) = 0. \tag{5.9}$$

Thus equation (5.6) becomes

$$S(Y,Z) = \left[\frac{\kappa}{n-1} - (\alpha^2-\beta^2)\right]g(Y,Z) - \left[\frac{\kappa}{n-1} - n(\alpha^2-\beta^2)\right]\eta(Y)\eta(Z), \tag{5.10}$$

which is an $\eta-$Einstein trans-Sasakian manifold. Putting $U = \zeta$ in (5.3) and then considering the equation (5.9), we observe that

$$\eta(\breve{W}(X,Y)Z) = 0 \tag{5.11}$$

and therefore

$$\breve{W}(X,Y)Z) = 0. \tag{5.12}$$

That is the manifold under consideration is conformally flat. Converse part is obvious. Thus we can state

Lemma 5.1. *A trans-Sasakian manifold (M, g) of dimension $n > 2$ is Weyl semisymmetric if and only if it is conformally flat and the Ricci tensor satisfies the relation (5.10).*

Let us suppose that the scalar curvature of the manifold consider the form $\kappa = n(n-1)(\alpha^2 - \beta^2)$ and therefore equation (5.10) takes the form

$$S(X, Z) = (n-1)(\alpha^2 - \beta^2)g(X, Z). \tag{5.13}$$

Therefore we cite the result as the lemma.

Lemma 5.2. *Every Weyl semisymmetric trans-Sasakian manifold (M, g) is an Einstein manifold, provided $\kappa = n(n-1)(\alpha^2 - \beta^2)$.*

Let (M, g) be a Weyl semisymmetric trans Sasakian manifold and $\kappa = n(n-1)(\alpha^2 - \beta^2)$. With the help of (1.2) and (5.13), we conclude that

$$(L_V g)(X, Z) = -2[\lambda + (n-1)(\alpha^2 - \beta^2)]g(X, Z). \tag{5.14}$$

Before going to state our result, we will recall the following definition

Definition 5.1. A vector field V on a Riemannian manifold of dimension n is said to be conformal Killing if

$$L_V g = \vartheta g, \tag{5.15}$$

for some scalar function ϑ on M.

From equations (5.14) and (5.15), it is clear that the vector field V in the triplet (g, V, λ) is a conformal Killing vector field for $\vartheta = -2[\lambda + (n-1)(\alpha^2 - \beta^2)]$.

Theorem 5.3. *Let (g, V, λ) be a generator of a Ricci soliton in a Weyl semisymmetric trans-Sasakian manifold (M, g) with $\kappa = n(n-1)(\alpha^2 - \beta^2)$. Then V is conformal Killing.*

Remark 5.2. The authors in [2] studied the properties of Weyl semisymmetric trans-Sasakian manifold but their expression for Ricci tensor was different from our expression (5.10).

6 Trans Sasakian manifolds satisfy $R \cdot R = R \cdot \breve{W}$

Now we are going to prove the following:

Theorem 6.1. *If (g, V, λ) is a generator of a Ricci soliton in trans-Sasakian manifold (M, g), then (M, g) satisfies $R \cdot R = R \cdot \breve{W}$ if and only if V is conformal Killing and (g, ζ, λ) is shrinking and expanding according as $K_\zeta > 0$ and < 0 respectively.*

Proof. Let (g, V, λ) be generator of a Ricci soliton and V is a conformal Killing vector field on (M, g), then from(1.2) we obtain

$$S = -\left(\lambda g + \frac{1}{2}L_V g\right). \tag{6.1}$$

In view of (5.15) and (6.1), we get

$$S(X,Y) = -\left(\lambda + \frac{\vartheta}{2}\right) g(X,Y). \tag{6.2}$$

In consequence of (2.18) and (6.2), we find that

$$\breve{W}(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}(\lambda + \frac{\vartheta}{2})\{g(Y,Z)X - g(X,Z)Y\}. \tag{6.3}$$

It is well known that

$$\begin{aligned}(R(X,Y)\cdot \breve{W})(U,V)Z &= R(X,Y)\breve{W}(U,V)Z - \breve{W}(R(X,Y)U,V)Z \\ &\quad -\breve{W}(U,R(X,Y)V)Z - \breve{W}(U,V)R(X,Y)Z.\end{aligned} \tag{6.4}$$

In view of (6.3) and (2.11), we can easily find that

$$R\cdot \breve{W} = R\cdot R. \tag{6.5}$$

Conversely we suppose that (M,g) satisfies $R\cdot R = R\cdot \breve{W}$, that is

$$\begin{aligned}&R(X,Y)\breve{W}(U,V)Z - \breve{W}(R(X,Y)U,V)Z - \breve{W}(U,R(X,Y)V)Z \\ &-\breve{W}(U,V)R(X,Y)Z = R(X,Y)R(U,V)Z - R(R(X,Y)U,V)Z \\ &-R(U,R(X,Y)V)Z - R(U,V)R(X,Y)Z.\end{aligned} \tag{6.6}$$

Changing X with ζ in (6.6) and then utilizing equation (2.13) in it, we have

$$\begin{aligned}&'\breve{W}(U,V,Z,Y)\zeta - \eta(\breve{W}(U,V)Z)Y - g(Y,U)\breve{W}(\zeta,V)Z + \eta(U)\breve{W}(Y,V)Z \\ &-g(Y,V)\breve{W}(U,\zeta)Z + \eta(V)\breve{W}(U,Y)Z - g(Y,Z)\breve{W}(U,V)\zeta + \eta(Z)\breve{W}(U,V)Y \\ &=' R(U,V,Z,Y)\zeta - \eta(R(U,V)Z)Y - g(Y,U)R(\zeta,V)Z + \eta(U)R(Y,V)Z \\ &-g(Y,V)R(U,\zeta)Z + \eta(V)R(U,Y)Z - g(Y,Z)R(U,V)\zeta + \eta(Z)R(U,V)Y,\end{aligned}$$

provided $\alpha^2 - \beta^2 \neq 0$. Inner product of above equation with ζ gives

$$\begin{aligned}&'\breve{W}(U,V,Z,Y) - \eta(\breve{W}(U,V)Z)\eta(Y) - g(Y,U)\eta(\breve{W}(\zeta,V)Z) + \eta(U)\eta(\breve{W}(Y,V)Z) \\ &\quad - g(Y,V)\eta(\breve{W}(U,\zeta)Z) + \eta(V)\eta(\breve{W}(U,Y)Z) - g(Y,Z)\eta(\breve{W}(U,V)\zeta) \\ &\quad + \eta(Z)\eta(\breve{W}(U,V)Y) =' R(U,V,Z,Y) - \eta(R(U,V)Z)\eta(Y) \\ &\quad - g(Y,U)\eta(R(\zeta,V)Z) + \eta(U)\eta(R(Y,V)Z) - g(Y,V)\eta(R(U,\zeta)Z) \\ &\quad + \eta(V)\eta(R(U,Y)Z) - g(Y,Z)\eta(R(U,V)\zeta) + \eta(Z)\eta(R(U,V)Y).\end{aligned} \tag{6.7}$$

Putting $Y = U = e_i$ in (6.7) and then taking the summation over i, $1 \le i \le n$, we find that

$$S(V,Z) = \{\kappa - (n-1)^2(\alpha^2 - \beta^2)\}g(V,Z) - \{\kappa - n(n-1)(\alpha^2-\beta^2)\}\eta(V)\eta(Z). \tag{6.8}$$

Again taking $Z = V = e_i$ in (6.8) and then taking the summation over i, $1 \le i \le n$, we obtain

$$\kappa = n(n-1)(\alpha^2 - \beta^2). \tag{6.9}$$

Putting the value of κ from (6.9) into (6.8), we obtain

$$S(V,Z) = (n-1)(\alpha^2-\beta^2)g(V,Z), \tag{6.10}$$

which shows that the manifold under consideration is an Einstein manifold. From equations (1.2) and (6.8), we find that

$$(L_V g)(X,Z) = \vartheta g(X,Z), \tag{6.11}$$

where $\vartheta = -2[(n-1)(\alpha^2-\beta^2)+\lambda]$. Thus the vector field V on M is conformal Killing. Again equations (3.3) and (6.10) give

$$\{(n-1)(\alpha^2-\beta^2)+\lambda+\beta\}g(Y,Z) - \beta\eta(Y)\eta(Z) = 0. \tag{6.12}$$

Putting $Y=\zeta$ in (6.12), we get the result $\lambda = -(n-1)K_\zeta$. Thus the Ricci solitons (g,ζ,λ) on (M,g) are expanding or shrinking as $K_\zeta < 0$ or > 0 respectively. Hence proof is completed. □

7 Solitons in Einstein semisymmetric trans-Sasakian manifolds

We consider the Einstein semisymmetric trans-Sasakian manifold (M,g). From (2.27), we get

$$(R(X,Y)\cdot E)(Z,U) = 0.$$

Above equation can be written as

$$E(R(X,Y)Z,U) + E(Z,R(X,Y)U) = 0. \tag{7.1}$$

In view of (2.27) and (7.1), we obtain

$$S(R(X,Y)Z,U) + S(Z,R(X,Y)U) = \frac{\kappa}{n}\{g(R(X,Y)Z,U) + g(Z,R(X,Y)U\}. \tag{7.2}$$

Replacing $X=\zeta$ in (7.2) and then using $Z=\zeta$ and equations (2.1), (2.2), (2.13) and (2.15) in it, we can easily find

$$S(Y,U) = (n-1)(\alpha^2-\beta^2)g(Y,U), \tag{7.3}$$

provided $\alpha^2-\beta^2 \neq 0$. Therefore we cite the result as the lemma.

Lemma 7.1. *Every Einstein semisymmetric trans-Sasakian manifold (M,g) is Einstein manifold.*

Theorem 7.2. *Let (g,V,λ) be a generator of a Ricci soliton in a trans-Sasakian manifold (M,g). Then (M,g) is Einstein semisymmetric if and only if V is conformal Killing and the triplet (g,ζ,λ) is shrinking or expanding as $K_\zeta > 0$ or < 0 respectively.*

Proof. Let (g,V,λ) be a generator of a Ricci soliton and V is conformal Killing vector field on (M,g). From (2.27) and (7.2), we have

$$\begin{aligned}(R(X,Y)\cdot E)(Z,U) &= S(R(X,Y)Z,U) + S(Z,R(X,Y)U) \\ &\quad -\frac{\kappa}{n}\{g(R(X,Y)Z,U) + g(Z,R(X,Y)U\}.\end{aligned} \tag{7.4}$$

In view of (6.2), we have from (7.4)

$$(R(X,Y)\cdot E)(Z,U)=0. \tag{7.5}$$

This implies that the trans-Sasakian manifold (M,g) is an Einstein semisymmetric. Conversely, if (M,g) be an Einstein semisymmetric trans-Sasakian manifold and (g,V,λ) be a Ricci soliton on (M,g). Then from (1.2) and (7.3), we get

$$(L_V g)(X,Z)=\vartheta g(X,Z), \tag{7.6}$$

where $\vartheta=-2\left(\frac{\kappa}{n}+\lambda\right)$, i. e., V is conformal Killing. Also from (3.3) and (7.3), we have

$$\left(\frac{\kappa}{n}+\lambda+\beta\right)g(X,Z)-\beta\eta(X)\eta(Z)=0. \tag{7.7}$$

Putting $X=\zeta$ in (7.7), we get the result $\lambda=-(n-1)K_\zeta$. Thus the Ricci soliton (g,ζ,λ) to be expanding or shrinking as $K_\zeta<0$ or >0 respectively. Proof is completed. □

8 Ricci soliton in Weyl pseudosymmetric trans-Sasakian manifolds

We consider the Weyl pseudosymmetric trans-Sasakian manifold (M,g), then by definition 2.6 we have

$$(R(X,Y)\cdot\breve{W})(U,V)Z=L_{\breve{W}}\{\,Q(g,W)(U,V,Z;X,Y\,)\}.$$

Above equation can be written as

$$\begin{aligned}&R(X,Y)\breve{W}(U,V)Z-\breve{W}(R(X,Y)U,V)Z-\breve{W}(U,R(X,Y)V)Z\\&-\breve{W}(U,V)R(X,Y)Z=L_{\breve{W}}\{(X\wedge_g Y)\breve{W}(U,V)Z\\&-\breve{W}((X\wedge_g Y)U,V)Z-\breve{W}(U,X\wedge_g Y)V)Z-\breve{W}(U,V)(X\wedge_g Y)Z\}.\end{aligned} \tag{8.1}$$

Assuming $X=U=\zeta$ in (8.1) and using (2.1), (2.2), (2.13) and (2.32), we yield

$$\begin{aligned}&\{L_{\breve{W}}-(\alpha^2-\beta^2)\}[g(Y,\breve{W}(\zeta,V)Z)\zeta-\eta(\breve{W}(\zeta,V)Z)Y+\breve{W}(Y,V)Z-\eta(Y)\breve{W}(\zeta,V)Z\\&+\eta(V)\breve{W}(\zeta,Y)Z-g(Y,V)\breve{W}(\zeta,\zeta)Z+\eta(Z)\breve{W}(\zeta,V)Y-g(Y,Z)\breve{W}(\zeta,V)\zeta\}]=0,\end{aligned}$$

which shows that either $L_{\breve{W}}=(\alpha^2-\beta^2$ or

$$\begin{aligned}&g(Y,\breve{W}(\zeta,V)Z)\zeta-\eta(\breve{W}(\zeta,V)Z)Y+\breve{W}(Y,V)Z-\eta(Y)\breve{W}(\zeta,V)Z\\&+\eta(V)\breve{W}(\zeta,Y)Z-g(Y,V)\breve{W}(\zeta,\zeta)Z+\eta(Z)\breve{W}(\zeta,V)Y-g(Y,Z)\breve{W}(\zeta,V)\zeta\}=0.\end{aligned} \tag{8.2}$$

Contracting (8.2) along the vector field Y, we get

$$\eta(\breve{W}(\zeta,V)Z)=0. \tag{8.3}$$

In view of (5.6) and (8.3), we conclude that

$$S(V,Z)=\left[\frac{\kappa}{n-1}-(\alpha^2-\beta^2)\right]g(V,Z)-\left[\frac{\kappa}{n-1}-n(\alpha^2-\beta^2)\right]\eta(V)\eta(Z), \tag{8.4}$$

which shows that the manifold under consideration is an $\eta-$Einstein manifold. Therefore we cite the result as:

Theorem 8.1. *A Weyl pseudosymmetric trans-Sasakian manifold* (M, g) *is either* $L_{\breve{W}} = K_\zeta$ *or an* $\eta-$*Einstein manifold.*

Consequently we will prove the following corollary as:

Corollary 8.2. *A generator* (g, V, λ) *of Ricci soliton in Weyl pseudosymmetric trans-Sasakian manifold* (M, g) *with* $\kappa = n(n-1)K_\zeta$ *is shrinking or expanding accordingly* $K_\zeta > 0$ *or* < 0*, if* $L_{\breve{w}} \neq K_\zeta$*.*

Proof. We consider that (M, g) is a Weyl pseudosymmetric trans-Sasakian manifold equipped with Ricci soliton (g, V, λ) and $\kappa = n(n-1)(\alpha^2 - \beta^2)$. In particular, we suppose that $L_{\breve{w}} \neq K_\zeta$ and $\kappa = n(n-1)(\alpha^2 - \beta^2)$, then equation (8.4) gives

$$S(Y, Z) = n(n-1)(\alpha^2 - \beta^2)g(Y, Z). \tag{8.5}$$

Again from equations (3.3) and (8.5), we get

$$\{(n-1)(\alpha^2 - \beta^2) + \lambda + \beta\}g(Y, U) - \beta\eta(Y)\eta(U) = 0. \tag{8.6}$$

Putting $Y = \zeta$ in (8.6), we get the result $\lambda = -(n-1)K_\zeta$. The Ricci soliton (g, ζ, λ) of M to be shrinking or expanding if $K_\zeta > 0$ *or* < 0. Thus the proof is completed. □

9 Ricci soliton in partially Ricci pseudosymmetric trans-Sasakian manifold

We suppose that (M, g) be a partially Ricci pseudo symmetric trans Sasakian manifold. Then by definition 2.7

$$(R(X, Y) \cdot S(Z, U) = f(q)\{(X \wedge_g Y) \cdot S)(Z, U)\}. \tag{9.1}$$

In view of (2.30) and (2.31), equation (9.1) reduces to

$$S(R(X, Y)Z, U) + S(Z, R(X, Y)U) = f(q)[S((X \wedge_g Y)Z, U) + S(Z, (X \wedge_g Y)U)]. \tag{9.2}$$

Taking $X = U = \zeta$ in (9.2), we have

$$S(R(\zeta, Y)Z, \zeta) + S(Z, R(\zeta, Y)\zeta) = f(q)[S((\zeta \wedge_g Y)Z, \zeta) + S(Z, (\zeta \wedge_g Y)\zeta)]. \tag{9.3}$$

Applying (2.1), (2.2), (2.13), (2.15) and (2.32) in (9.3), we yield

$$\{f(q) - (\alpha^2 - \beta^2)\}[(n-1)(\alpha^2 - \beta^2)g(Y, Z) - S(Y, Z)] = 0. \tag{9.4}$$

That implies that either $f(q) = (\alpha^2 - \beta^2)$ or

$$S(Y, Z) = (n-1)(\alpha^2 - \beta^2)g(Y, Z). \tag{9.5}$$

Therefore we cite the result as:

Theorem 9.1. *A partially Ricci pseudosymmetric trans-Sasakian manifold* (M, g) *is an Einstein manifold provided* $f(q) \neq K_\zeta$*.*

Theorem 9.2. *A generator (g, ζ, λ) of a Ricci soliton in a partially Ricci pseudosymmetric trans Sasakian manifold (M, g) is shrinking or expanding as $K_\zeta > 0$ or < 0 respectively if $f(q) \neq K_\zeta$.*

Proof. Assume that (M, g) is a partially Ricci pseudosymmetric trans Sasakian manifold equipped with Ricci soliton (g, ζ, λ). Then from (3.3) and (9.5), we get

$$\{(n-1)(\alpha^2 - \beta^2) + \lambda + \beta\}g(X, Z) - \beta\eta(X)\eta(Z) = 0. \tag{9.6}$$

Putting $X = \zeta$ in (9.6), we get the result $\lambda = -(n-1)K_\zeta$. Hence the Ricci soliton on (M, g) to be shrinking or expanding as $K_\zeta > 0$ *or* < 0 respectively. Proof is completed. □

10 Weyl Ricci pseudosymmetric trans-Sasakian manifolds

Let (M, g) be a Weyl Ricci pseudosmmetric trans Sasakian manifold. Then from (2.33) we have

$$(\breve{W}(X, Y) \cdot S)(U, V) = L_S Q(g, S)(U, V; X, Y). \tag{10.1}$$

Equation (10.1) can be written as

$$S(\breve{W}(X, Y)U, V) + S(U, \breve{W}(X, Y)V) = L_S[S((X \wedge_g Y)U, V) + S(U, (X \wedge_g Y)V)]. \tag{10.2}$$

Adopting $X = V = \zeta$ in (10.2), we yield

$$S(\breve{W}(\zeta, Y)U, \zeta) + S(U, \breve{W}(\zeta, Y)\zeta) = L_S[S((\zeta \wedge_g Y)U, \zeta) + S(U, (\zeta \wedge_g Y)\zeta)]. \tag{10.3}$$

In view of (2.15), (2.19), (2.20) and (2.32), equation (10.3) reduces to

$$\begin{aligned} &\left[\frac{\kappa - (n-1)(\alpha^2 - \beta^2)}{(n-1)(n-2)} - L_S\right]\left[(n-1)(\alpha^2 - \beta^2)g(Y, U) - S(Y, U)\right] \\ &-\frac{1}{n-2}\{(n-1)(\alpha^2 - \beta^2)S(U, Y) - S(U, QY)\} = 0. \end{aligned} \tag{10.4}$$

It can be hold only if

$$S(Y, U) = (n-1)(\alpha^2 - \beta^2)g(Y, U). \tag{10.5}$$

Conversely, if we suppose that the trans Sasakian manifold satisfies equation (10.5), then from equations (10.1), (10.2) and (10.5) it is obvious that the manifold to be Weyl Ricci pseudosymmetric trans Sasakian manifold. Therefore we cite the result as:

Theorem 10.1. *A trans Sasakian manifold (M, g) is the Weyl Ricci pseudosymmetric if and only if it is an Einstein manifold.*

Theorem 10.2. *A generator (g, ζ, λ) of a Ricci soliton in Weyl Ricci pseudosymmetric trans Sasakian manifold (M, g) is shrinking or expanding as $K_\zeta > 0$ or < 0 respectively if $f(q) \neq K_\zeta$.*

Proof. Assuming that (M, g) is a Weyl Ricci pseudosymmetric trans Sasakian manifold equipped with Ricci soliton (g, ζ, λ). Then from (3.3) and (10.5), we get

$$\{(n-1)(\alpha^2 - \beta^2) + \lambda + \beta\}g(Y, U) - \beta\eta(Y)\eta(U) = 0. \tag{10.6}$$

Putting $Y = \zeta$ in (10.6), we get the result $\lambda = -(n-1)K_\zeta$. Thus the proof of the theorem. □

11 Examples of 3−dimensional trans-Sasakian manifolds

Example 11.1. We consider a 3−dimensional manifold $M = \{(x, y, z) \in \Re^3 : z \neq 0\}$, where (x, y, z) are the standard coordinates of $\Re^3$. Let the vector fields

$$E_1 = e^{-2z}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \qquad E_2 = -e^{-2z}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right), \qquad E_3 = \frac{\partial}{\partial z}.$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(E_i, E_j) = \begin{cases} 0 & \text{for} \quad i \neq j \\ 1 & \text{for} \quad i = j \end{cases}.$$

and η be the 1−form defined by $\eta(V) = g(V, E_3)$ for any $V \in T(M)$. Let ϕ be the $(1, 1)$ tensor field defined by $\phi E_1 = E_2$, $\phi E_2 = -E_1$, $\phi E_3 = 0$. Then we have

$$\eta(E_3) = g(E_3, E_3) = 1, \quad \phi^2 V = -V + \eta(V)E_3, \; g(\phi V, \phi W) = g(V, W) - \eta(V)\eta(W),$$

for any $V, W \in T(M)$. Let ∇ be the Riemannian connection with respect to the metric g. Then we obtain

$$[E_1, E_2] = 0, \qquad [E_1, E_3] = 2E_1, \qquad [E_2, E_3] = 2\,E_2.$$

The Riemannian connection ∇ of the metric g is given by Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{11.1}$$

From (11.1) we have

$$2g(\nabla_{E_1} E_3, E_1) = 4, \; 2g(\nabla_{E_1} E_3, E_2) = 0, \; 2g(\nabla_{E_1} E_3, E_3) = 0. \tag{11.2}$$

Thus $\nabla_{E_1} E_3 = 2E_1$. Also from (11.1) we get $2g(\nabla_{E_2} E_3, E_1) = 0$, $\quad 2g(\nabla_{E_2} E_3, E_2) = 4$, $\quad 2g(\nabla_{E_3} E_3, E_3) = 0$. Therefore $\nabla_X \zeta = -\alpha\,\phi\,X + \beta\,(X - \eta(X)\,\zeta)$ holds on M for $\alpha = 0$, $\beta = 2$. Thus (M, g) is a 3−dimensional trans Sasakian manifold of type $(0, \beta)$.

Further we get from (11.1)

$$\nabla_{E_1} E_3 = 2E_1, \qquad \nabla_{E_1} E_1 = -2E_3, \qquad \nabla_{E_1} E_2 = 0,$$

$$\nabla_{E_2} E_3 = 2E_2, \qquad \nabla_{E_3} E_2 = 0, \qquad \nabla_{E_2} E_1 = 0,$$

$$\nabla_{E_3} E_3 = 0, \qquad \nabla_{E_2} E_2 = -2E_3, \qquad \nabla_{E_3} E_1 = 0.$$

Using the above relations, we can easily calculate the components of the curvature tensor R and Ricci tensor S as follows:

$$\begin{aligned} &R(E_1, E_2)E_2 = -4E_1,\ \ R(E_2, E_3)E_2 = 4E_3,\ \ R(E_1, E_3)E_1 = 4E_3,\\ &R(E_2, E_3)E_1 = 0,\ \ R(E_2, E_3)E_3 = -4E_2,\ \ R(E_1, E_2)E_1 = 4E_2,\\ &R(E_1, E_3)E_2 = 0,\ \ R(E_1, E_3)E_3 = -4E_1,\ \ R(E_1, E_2)E_3 = 0,\\ &S(E_1, E_1) = -8, \qquad S(E_2, E_2) = -8, \qquad S(E_3, E_3) = -8. \end{aligned} \tag{11.3}$$

Since $\{E_1, E_2, E_3\}$ form a basic for $M^3(\phi, \xi, \eta, g)$, then any vector $X, Y, Z, U \in T(M)$ can be written as

$$\begin{aligned} &X = a_1E_1 + b_1E_2 + c_1E_3,\ \ Y = a_2E_1 + b_2E_2 + c_2E_3,\\ &Z = a_3E_1 + b_3E_2 + c_3E_3,\ \ U = a_4E_1 + b_4E_2 + c_4E_3. \end{aligned} \tag{11.4}$$

where $a_i, b_i, c_i \in \Re^+$ for all $i = 1, 2, 3$ such that a_i, b_i, c_i are not proportional. Then

$$R(X, Y)Z = 4c_3(b_1b_2 - a_1b_2)E_1 - 2(b_1b_2c_3)E_2 - 2(b_1b_2b_3)E_3. \tag{11.5}$$

$$R(X, Y)U = 4c_4(b_1b_2 - a_1b_2)E_1 - 2(b_1b_2c_4)E_2 - 2(b_1b_2b_4)E_3 \tag{11.6}$$

In view of (2.18) and (11.5), we have

$$\begin{aligned} \breve{W}(X, Y)Z = {} & 4\{(b_1 - a_1)b_2b_3 + (2c_1 + c_2)a_2c_3 + (2b_1b_3 - a_1a_3)a_2 - a_1b_2b_3\}E_1\\ & + 4\{(4b_2 - 3c_2)c_1c_3 + (b_3 - 2c_3)b_1b_2 + (2a_1b_2 - 3b_1a_2)a_3 + 2b_1c_2c_3\}E_2\\ & + 4\{(3c_1c_3 + 2a_1a_3)c_2 + (2c_2b_1 - c_1b_2)b_3 - 3c_1a_2a_3\}E_3. \end{aligned} \tag{11.7}$$

and hence

$$\begin{aligned} S(\breve{W}(X, Y)Z, U) \ = \ & -32\{(4b_2 - 3c_2)c_1c_3b_4 + (b_3 - 2c_3)b_1b_2b_4\\ & +(2a_1b_2 - 3b_1a_2)a_3b_4 + 2b_1b_4c_2c_3\} - 32\{(3c_1c_3\\ & +2a_1a_3)c_2c_4 + (2c_2b_1 - c_1b_2)b_3c_4 - 3c_1c_4a_2a_3\}. \end{aligned} \tag{11.8}$$

Similarly we can easily calculate

$$\begin{aligned} S(Z, \breve{W}(X, Y)U) \ = \ & -32\{(4b_2 - 3c_2)c_1c_4b_3 + (b_4 - 2c_4)b_1b_2b_3\\ & +(2a_1b_2 - 3b_1a_2)a_4b_3 + 2b_1b_3c_2c_4\} - 32\{(3c_1c_4\\ & +2a_1a_4)c_2c_3 + (2c_2b_1 - c_1b_2)b_4c_3 - 3c_1a_2a_4c_3\}. \end{aligned} \tag{11.9}$$

Also, we have

$$\begin{aligned} &g(X, Z) = a_1a_3 + b_3b_1 + c_1c_3,\ \ g(Y, U) = a_2a_4 + b_4b_2 + c_2c_4,\\ &g(X, U) = a_1a_4 + b_4b_1 + c_1c_4,\ \ g(Y, Z) = a_2a_3 + b_3b_2 + c_2c_3. \end{aligned} \tag{11.10}$$

and

$$\begin{aligned} &S(Y, Z) = -8(2b_3b_2 + c_2c_3),\ \ S(X, Z) = -8(2b_1b_2 + c_1c_3)\\ &S(Y, U) = -8(2b_4b_2 + c_2c_4),\ \ S(X, U) = -8(2b_1b_4 + c_1c_4). \end{aligned} \tag{11.11}$$

Also from (11.10) and (11.11), we obtain

$$
\begin{aligned}
&g(Y,Z)S(X,U)-g(X,Z)S(Y,U)+g(Y,U)S(X,Z)-g(X,U)S(Y,Z)\\
&=8\left\{\begin{array}{l}-(2a_2a_3+c_1c_3)b_1b_4+(b_1b_3+a_1a_3+c_1c_3+2a_1a_2)c_2c_4\\ -2(a_2a_4+c_2c_4)b_1b_3+(2b_2b_3-2a_2a_3)c_1c_4\\ -(b_3b_2+a_2a_4)c_1c_3+(2b_2b_3+c_2c_3)a_1a_4\end{array}\right\}\neq 0.
\end{aligned}\tag{11.12}
$$

Since a_i, b_i, c_i are not proportional and assume that

$$
\begin{aligned}
&-(2a_2a_3+c_1c_3)b_1b_4+(b_1b_3+a_1a_3+c_1c_3+2a_1a_2)c_2c_4-2(a_2a_4+c_2c_4)b_1b_3\\
&\quad+(2b_2b_3-2a_2a_3)c_1c_4-(b_3b_2+a_2a_4)c_1c_3+(2b_2b_3+c_2c_3)a_1a_4\neq 0.
\end{aligned}
$$

Again from (11.8) and (11.9), we have

$$
\begin{aligned}
&S(\breve{W}(X,Y)Z,U)+S(Z,\breve{W}(X,Y)U)\\
&=32\left\{\begin{array}{l}(2a_2a_3+c_1c_3)b_1b_4-(b_1b_3+a_1a_3+c_1c_3+2a_1a_2)c_2c_4\\ +2(a_2a_4+c_2c_4)b_1b_3-(2b_2b_3-2a_2a_3)c_1c_4\\ +(b_3b_2+a_2a_4)c_1c_3-(2b_2b_3+c_2c_3)a_1a_4\end{array}\right\}.
\end{aligned}\tag{11.13}
$$

Let we assume the function

$$
L_S=-4. \tag{11.14}
$$

In view of (11.12), (11.13) and (11.14), we conclude that

$$
\begin{aligned}
&S(\breve{W}(X,Y)Z,U)+S(Z,\breve{W}(X,Y)U)\\
&=L_S[g(Y,Z)S(X,U)-g(X,Z)S(Y,U)+g(Y,U)S(X,Z)-g(X,U)S(Y,Z)].
\end{aligned}
$$

Thus the structure $M^3(\phi,\xi,\eta,g)$ under consideration is Weyl pseudosymmetric trans Sasakian manifold. Also equation (11.3) tell us that (M^3,g) is an Einstein manifold. Thus theorems 10.1 and 10.2 verified. In the similar way, we can also verify the theorem 9.1.

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