On the concircular curvature tensor of a semi-symmetric metric connection

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Abstract. The object of the present paper is to study Lorentzian concircular structure manifolds (briefly $(LCS)_{2n+1}$-manifolds) admitting a semi-symmetric metric connection, whose concircular curvature tensor satisfies certain conditions.

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1 Introduction

We say that a $(2n+1)$-dimensional Lorentzian manifold M is a smooth connected para-contact Hausdorff manifold with a Lorentzian metric g, if M admits a smooth symmetric tensor field g of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature$(-,+,...,+)$, where T_pM denotes the tangent space of M at p.

In a Lorentzian manifold (M,g) a vector field P defined by

$$g(X,P) = A(X),$$

for any vector field $X \in \chi(M)$ is said to be concircular vector field [17] if

$$(\nabla_X A)(Y) = \alpha \left[g(X,Y) + \omega(X)A(Y) \right],$$

where α is a non zero scalar function, A is a 1-form and ω is a closed 1-form.

Let M be a Lorentzian manifold admitting a unit time like concircular vector field ξ, called *the characteristic vector field of the manifold*; we have

$$g(\xi,\xi) = -1. \tag{1.1}$$

Since ξ is the unit concircular vector field, there exists a non zero 1-form η such that

$$g(X,\xi) = \eta(X), \tag{1.2}$$

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and an equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha\,[g(X,Y) + \eta(X)\eta(Y)] \quad (\alpha \neq 0), \tag{1.3}$$

for all vector field X, Y, where ∇ denotes the operator of covariant differentiation with respect to Lorentzian metric g and α is a non zero scalar function satisfying

$$(\nabla_X \alpha) = (X\alpha) = \rho\eta(X), \tag{1.4}$$

where ρ is a scalar function. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{1.5}$$

in (1.3), we obtain

$$\phi^2 X = X + \eta(X)\xi, \tag{1.6}$$

from which it follows that ϕ is a symmetric $(1,1)$-tensor. Thus the Lorentzian manifold M together with unit time like concircular vector field ξ, its associated 1-form η and the $(1,1)$-tensor field ϕ is said to be *Lorentzian concircular structute manifold* (briefly $(LCS)_{2n+1}$-manifold) [17]. In particular if $\alpha = 1$, then we obtain the LP-Sasakian structure of Matsumoto [15]. The properties of $(LCS)_{2n+1}$-manifolds have been intensively studied (e.g., in [20, 21, 22, 24, 26]). Let M be an n-dimensional Riemannian manifold of class C^∞ endowed with the Riemannian metric g and let D be the Levi-Civita connection on (M^n, g). A linear connection ∇ defined on (M^n, g) is said to be *semi-symmetric* [11] if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{1.7}$$

where η is an 1-form and ξ is a vector field defined by

$$g(X,\xi) = \eta(X), \tag{1.8}$$

for all vector fields $X \in \chi(M^n)$; $\chi(M^n)$ is the set of all differentiable vector fields on M^n. A semi-symmetric connection ∇ is called *a semi-symmetric metric connection* [12], if it further satisfies

$$\nabla g = 0. \tag{1.9}$$

A relation between the semi-symmetric metric connection ∇ and the Levi-Civita connection D on (M^n, g) has been obtained by Yano [28], which is given by

$$\nabla_X Y = D_X Y + \eta(Y)X - g(X,Y)\xi; \tag{1.10}$$

we also have

$$(\nabla_X \eta)(Y) = (D_X \eta)Y - (X)(Y) + \eta(\xi)g(X,Y). \tag{1.11}$$

Further, a relation between the curvature tensor R of the semi-symmetric connection ∇ and the curvature tensor K of the Levi-Civita connection D is given by

$$R(X,Y)W = K(X,Y)W + \alpha(X,W)Y - \alpha(Y,W)X + g(X,W)QY - g(Y,W)QX, \tag{1.12}$$

where α is a tensor field of type $(0,2)$ and Q is a tensor field of type $(1,1)$, defined by

$$\alpha(Y,W) = g(QY,W) = (D_Y\eta)(W) - \eta(Y)\eta(W) + \frac{1}{2}\eta(\xi)g(Y,W). \tag{1.13}$$

In view of (1.12) and (1.13), we get

$$\begin{aligned}\tilde{R}(X,Y,W,U) &= \tilde{K}(X,Y,W,U) - \alpha(Y,W)g(X,U)\\ &+ \alpha(X,W)g(Y,U) - g(Y,W)\alpha(X,U) + g(X,W)\alpha(Y,U),\end{aligned} \tag{1.14}$$

where $\tilde{R}(X,Y,W,U) = g(R(X,Y)W,U)$, $\tilde{K}(X,Y,W,U) = g(K(X,Y)W,U)$.

The properties of semi-symmetric connections have been studied in detail, in [5, 6, 7, 8, 9, 10, 14, 16, 23, 25]. A transformation of an n-dimensional Riemannian manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a *concircular transformation* ([13, 27]). A concircular transformation is always a conformal transformation ([13]). Here geodesic circle means a curve in M whose first curvature is constant and the second curvature is identically zero. Thus the geometry of concircular transformations (the concircular geometry) is the generalization of inverse geometry in the sense that the change of metric is more general then the induced by circle preserving diffeomorphisms [2]. An invariant of a concircular transformations the concircular curvature tensor C, is defined by ([13, 27]).

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{2n(2n+1)}\left[g(Y,Z)X - g(X,Z)Y\right], \tag{1.15}$$

In view of (1.15), it follows that

$$\tilde{C}(X,Y,Z,U) = \tilde{R}(X,Y,Z,U) - \frac{r}{2n(2n+1)}\left[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\right], \tag{1.16}$$

and $\tilde{C}(X,Y,Z,U) = g(C(X,Y)Z,U)$, where $X,Y,Z,U \in \chi(M)$, C is the concircular curvature tensor and r is the scalar curvature tensor with respect to the semi-symmetric metric connection. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

The paper is organized as follows: after an introduction in section 2, we define the $(LCS)_{2n+1}$-manifolds. Section 3 is devoted to the study of ϕ-concircularly flat $(LCS)_{2n+1}$-manifolds with respect to the semi-symmetric metric connection and also determines the ϕ-sectional curvature of the plane by two vectors. Sections 4 and 5 deal with $C \cdot S = 0$ and $C \cdot C = 0$ in $(LCS)_{2n+1}$-manifolds with respect to such connections and prove that the manifold is η-Einstein. Finally, in section 6, it is also shown that concircular ϕ-recurrent $(LCS)_{2n+1}$-manifolds with such connections are η-Einstein manifolds and that the characteristic vector filed ξ and the vector field ρ associated to the 1-form A have opposite directions.

2 $(LCS)_{2n+1}$-manifolds

A differentiable manifold M of dimension $(2n+1)$ is called $(LCS)_{2n+1}$-manifold if it admits a $(1,1)$-tensor ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g, which satisfy the following

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 = I + \eta \otimes \xi, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

$$\phi\xi = 0,\ \eta(\phi X) = 0, \tag{2.5}$$

For all $X, Y \in TM$, also in an $(LCS)_{2n+1}$-manifold, the following relations are satisfied [18].

$$\eta(K(X,Y)Z) = (\alpha^2 - \rho)\left[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\right], \tag{2.6}$$

$$K(X, Y)\xi = (\alpha^2 - \rho)\left[\eta(Y)X - \eta(X)Y\right], \tag{2.7}$$

$$K(\xi, X)Y = (\alpha^2 - \rho)\left[g(X,Y)\xi - \eta(Y)X\right], \tag{2.8}$$

$$R(\xi, X)\xi = (\alpha^2 - \rho)\left[\eta(X)\xi + X\right], \tag{2.9}$$

$$(\nabla_X \phi)(Y) = \alpha\left[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\right], \tag{2.10}$$

$$\tilde{S}(X, \xi) = 2n(\alpha^2 - \rho)\eta(X), \tag{2.11}$$

$$\tilde{S}(\phi X, \phi Y) = S(X, Y) + 2n(\alpha^2 - \rho)\eta(X)\eta(Y), \tag{2.12}$$

where K is the curvature tensor and $\tilde{S}$ is the Ricci curvature tensor of the manifold, with respect to the Levi-Civita connection.

3 ϕ-concircurlarly flat $(LCS)_{2n+1}$-manifold with respect to semi-symmetric metric connection

Let C be the Weyl conformal curvature tensor of a $(2n+1)$–dimensional manifold M. Since at each point $p \in M$ the tangent space $\chi_p(M)$ can be decomposed into the direct sum $\chi_p(M) = \phi(\chi_p(M)) \oplus L(\xi_p)$, where $L(\xi_p)$ is an 1-dimensional linear subspace of $\chi_p(M)$ generated by ξ_p, there exists the map:

$$C : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \to \phi(\chi_p(M)) \oplus L(\xi_p).$$

We may consider the following particular cases

1. $C : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \to L(\xi_p)$, i.e. the projection of the image of C in$\phi(\chi_p(M))$ is zero.
2. $C : \chi_p(M) \times \chi_p(M) \times \chi_p(M) \to \phi(\chi_p(M))$, i.e. the projection of the image of C in $L(\xi_p)$is zero.

   $$C(X,Y)\xi = 0. \tag{3.1}$$

3. $C : \phi(\chi_p(M)) \times \phi(\chi_p(M)) \times \phi(\chi_p(M)) \to L(\xi_p)$, i.e. when C is restricted to$\phi(\chi_p(M)) \times \phi(\chi_p(M)) \times \phi(\chi_p(M))$, the projection of the image of C in $\phi(\chi_p(M))$ is zero. The condition is equivalent to

   $$\phi^2 C(\phi X, \phi Y)\phi Z = 0. \tag{3.2}$$

Here the cases 1, 2 and 3 are conformally symmetric, ξ-conformally flat and ϕ-conformaly flat, respectively. The cases (1.1) and (1.2) where considered in [29] and [4], respectively. A case (1.3) was considered in [3] when M is a K-contact manifold. Furthermore, in [1], the author studied such contact metric manifolds. Similar to definition (1.3) of ξ-conformally and ϕ-coformally flatness, we may introduced the following definitions:

Definition 3.1. A $(LCS)_{2n+1}$-manifold is said to be ϕ-concircurlarly flat with respect to the semi-symmetric metric connection if

$$g\left(C(\phi X, \phi Y)\phi Z, \phi W\right) = 0, \tag{3.3}$$

where $X, Y, Z, W \in \chi(M)$.

Definition 3.2. A $(LCS)_{2n+1}$-manifold is said to be an η-Einstein manifold if its Ricci tensor $\tilde{S}$ of the Levi-Civita connection is of the form

$$\tilde{S}(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a and b are smooth function on the manifold.

Theorem 3.1. *If a $(LCS)_{2n+1}$-manifold is ϕ-concircularly flat with respect to the semi-symmetric metric connection then the manifold is an η–Einstein manifold.*

Proof. In view of equation (1.3), (1.2) and (1.13), we obtain

$$\begin{aligned} R(X,Y)W = K(X,Y)W &+ \left[2n(\alpha^2-\rho)+\alpha-\tfrac{1}{2}\right]\{g(X,W)Y - g(Y,W)X\} \\ &+ [(1-\alpha)]\{\eta(Y)\eta(W)X - \eta(X)\eta(W)Y\} \end{aligned}. \tag{3.4}$$

From (3.4), it is clear that

$$\begin{aligned} \tilde{R}(X,Y,W,U) = \tilde{K}(X,Y,W,U) &+ \left[2n(\alpha^2-\rho)+\alpha-\tfrac{1}{2}\right]\left\{\begin{array}{c} g(X,W)g(Y,U) \\ -g(Y,W)g(X,U) \end{array}\right\} \\ &+ [(1-\alpha)]\{\eta(Y)\eta(W)g(X,U) - \eta(X)\eta(W)g(Y,U)\}; \end{aligned} \tag{3.5}$$

contracting X in (3.5), we get

$$S(Y,W) = \tilde{S}(Y,W) - 2n\left[2n(\alpha^2-\rho)+\alpha-\frac{1}{2}\right]g(Y,W) + 2n(1-\alpha)\eta(Y)\eta(W); \tag{3.6}$$

substituting $W=\xi$ in (3.7) and using (2.1) and (2.11), we have

$$S(Y,\xi)=\{(2n(1-2n)(\alpha^2-\rho)-n\}\eta(Y); \tag{3.7}$$

again, contracting Y and W in (3.6), it follows that

$$r=\tilde{r}-4n^2(2n+1)(\alpha^2-\rho)-4n\alpha(n-1)+n(2n+3), \tag{3.8}$$

where r,S and $\tilde{r}$,$\tilde{S}$ are the scalar curvature, the Ricci tensor with respect to the semi-symmetric metric connection and the Levi-Civita connection, respectively.

By substituting $X=\phi X, Y=\phi Y, W=\phi W$ and $U=\phi U$ in (1.16) and using (1.14), we obtain

(3.9)

$$g(C(\phi X,\phi Y,\phi W,\phi U)=\tilde{K}(\phi X,\phi Y,\phi W,\phi U)+\left[2n(\alpha^2-\rho)+\alpha-\tfrac{1}{2}\right]$$
$$\left\{\begin{array}{l} g(\phi X,\phi W)g(\phi Y,\phi U) \\ -g(\phi Y,\phi W)g(\phi X,\phi U)\end{array}\right\}-\tfrac{r}{2n(2n+1)}\left\{\begin{array}{l} g(\phi Y,\phi W)g(\phi X,\phi U) \\ -g(\phi X,\phi W)g(\phi Y,\phi U)\end{array}\right\};$$

again, using (3.3) in (3.9), we get

$$\tilde{K}(\phi X,\phi Y,\phi W,\phi U)=\left[2n(\alpha^2-\rho)+\alpha-\frac{1}{2}+\frac{r}{2n(2n+1)}\right]$$

$$\times\left\{\ g(\phi Y,\phi W)g(\phi X,\phi U)-g(\phi X,\phi W)g(\phi Y,\phi U)\ \right\}. \tag{3.10}$$

Let $\{e_{1,\dots}e_{2n,}\xi\}$ be an orthonormal basis of vector fields in M; then $\{\phi e_1,\dots,\phi e_{2n,\xi}\}$ is also a local orthonormal basis. Putting $X=U=e_i$ in (3.10) and summing over $i=1$ to $2n$, we get

$$\sum_{i=1}^{2n}\tilde{K}(\phi e_i,\phi Y,\phi W,\phi e_i)=\sum_{i=1}^{2n}\left[2n(\alpha^2-\rho)+\alpha-\frac{1}{2}+\frac{r}{2n(2n+1)}\right]$$

$$\times\left\{\begin{array}{l} g(\phi Y,\phi W)g(\phi e_i,\phi e_i) \\ -g(\phi e_i,\phi W)g(\phi Y,\phi e_i)\end{array}\right\}, \tag{3.11}$$

and

$$\tilde{S}(\phi Y,\phi W)=(2n+1)\left[2n(\alpha^2-\rho)+\alpha-\frac{1}{2}+\frac{r}{2n(2n+1)}\right]g(\phi Y,\phi W). \tag{3.12}$$

Using (2.3) and (2.12) in (3.12), we obtain

$$\tilde{S}(Y,W)=a\,g(Y,W)+b\eta(Y)\eta(W),$$

where

$$a=\left[(2n-1)\left\{\frac{4n^2(2n+1)(\alpha^2-\rho)+2n(2n+1)(2\alpha-1)+2r}{4n(2n+1)}\right\}\right],$$

$b=((2n-1))$

$$\times\left\{\frac{4n^2(2n+1)(\alpha^2-\rho)+2n(2n+1)(2\alpha-1)+2\alpha-4n^2(2n-1)(\alpha^2-1)}{4n(2n+1)}\right\}.$$

These results show that the manifold is an η-Einstein manifold, and proves Theorem 3.1. □

Definition 3.3. A plane section in $\chi(M)$ is called ϕ-sectional if there exists a unit vector X in $\chi(M)$ orthogonal to ξ, such that $\{X, \phi X\}$ is an orthogonal basis of the plane section. Then the sectional curvature ${}^{/}K(X, \phi X)$ is called a ϕ-sectional curvature.

Theorem 3.2. *If a $(LCS)_{2n+1}$-manifold admits a semi-symmetric metric connection whose curvature tensor vanishes, then the ϕ-sectional curvature of the plane determined by two vectors $X, \phi X \in \xi^{\perp}$ is $2n(\alpha^2 - \rho) + \alpha - \frac{1}{2}$.*

Proof. Let $\xi^{\perp}$ denote the $(2n+1)$-dimensional distribution orthogonal to ξ in a $(LCS)_{2n+1}$-manifold admitting a semi-symmetric metric connection whose curvature tensor vanishes for any $X \in \xi^{\perp}, g(X, \xi) = 0$. Now we shall determine the ϕ-sectional curvature ${}^{/}K$ at the plane determined by the vectors $X, \phi X \in \xi^{\perp}$. Putting $Y = \phi X$, $W = \phi X$ and $U = X$ in (3.5) it can be seen that

$$\tilde{K}(X, \phi X, \phi X, X) = \left[2n(\alpha^2 - \rho) + \alpha - \frac{1}{2}\right]\{g(X, X)g(\phi X, \phi X)\}]$$

$$-g(X, \phi X)g(\phi X, X). \tag{3.13}$$

Therefore

$$ {}^{/}K(X, \phi X) = \frac{\tilde{K}(X, \phi X, \phi X, X)}{g(X, X)g(\phi X, \phi X) - g(X, \phi X)^2} = \left\{2n(\alpha^2 - \rho) + \alpha - \frac{1}{2}\right\}. \tag{3.14}$$

This proves the Theorem 3.2. □

4 A $(LCS)_{2n+1}$-manifolds with respect to semi-symmetric metric connections satisfying $C \cdot S = 0$.

Theorem 4.1. *If a $(LCS)_{2n+1}$-manifold with respect to a semi-symmetric connection satisfying $C \cdot S = 0$, then the manifold is an η-Einstein manifold.*

Proof. We consider an $(LCS)_{2n+1}$-manifold with respect to a semi-symmetric metric connection satisfying the condition

$$(C(U, Y) \cdot S)(W, X) = 0.$$

Then we have

$$S(C(U, Y)W, X) + S(W, C(U, Y)X) = 0; \tag{4.1}$$

substituting $U = \xi$ in (4.1), it follows that

$$S(C(\xi, Y)W, X) + S(W, C(\xi, Y)X) = 0; \tag{4.2}$$

putting $X = \xi$ in (1.15) and using (2.4), (2.8) and (3.8), we get

$$\begin{aligned} C(\xi, Y, W) &= \left[(2n-1)(\alpha^2 - \rho) - \alpha + \tfrac{1}{2} - \tfrac{r}{2n(2n+1)}\right] g(Y, W)\xi \\ &+ \left[\begin{array}{l} (2n-1)(\alpha^2 - \rho) + \alpha - \frac{1}{2} \\ +(1-\alpha) + \frac{r}{2n(2n+1)} \end{array}\right] \eta(W)Y + (1-\alpha)\eta(Y)\eta(W)\xi. \end{aligned} \tag{4.3}$$

In view of (4.2) and (4.3), it follows that

$$
\begin{aligned}
&\left[(2n-1)(\alpha^2-\rho)-\alpha+\tfrac{1}{2}-\tfrac{r}{2n(2n+1)}\right]\{g(Y,\xi)S(\xi,X)+g(Y,X)S(W,\xi)\}\\
&+\left[\begin{array}{l}(2n-1)(\alpha^2-\rho)+\alpha-\frac{1}{2}\\ +(1-\alpha)+\frac{r}{2n(2n+1)}\end{array}\right]\{\eta(W)S(X,Y)+\eta(X)S(W,Y)\}\\
&+(1-\alpha)\{\eta(Y)\eta(W)S(\xi,X)+\eta(Y)\eta(X)S(W,\xi)\}=0;
\end{aligned}
\tag{4.4}
$$

substituting $W=\xi$ in (4.4) and using (2.11) and (3.6), we get

$$\tilde{S}(X,Y)=ag(X,Y)+b\eta(X)\eta(Y),$$

where

$$
a=\left[\begin{array}{l}\frac{\{8n^3\alpha-8n^3\rho-2n\alpha^2+2n\rho-4n^2\alpha-2n\alpha+2n^2+n-r\}\{4n^2\alpha^2-4n^2\rho-2n\alpha^2+2n\rho+n\}}{\{8n^3\alpha-8n^3\rho-2n\alpha^2+2n\rho+2n^2+n-r\}}\\ +n-2n\alpha-4n^2\alpha+4n^2\rho\end{array}\right],
$$

$$
b=\left[\begin{array}{l}\frac{\{8n^3\alpha-8n^3\rho-2n\alpha^2+2n\rho-4n^2\alpha-2n\alpha+2n^2+n-r\}\{4n^2\alpha^2-4n^2\rho-2n\alpha^2+2n\rho-r\}}{\{8n^3\alpha-8n^3\rho-2n\alpha^2+2n\rho+2n^2+n+r\}}\\ +\frac{2n(2n+1)(1-\alpha)\{4n+8n^2(\alpha^2-\rho)-4n^2\alpha^2+4n^2\rho\}}{\{8n^3\alpha-8n^3\rho-2n\alpha^2+2n\rho+2n^2+n+r\}}\\ +2n\alpha^2-2n\rho-2n\alpha-4n^2\alpha^2+4n^2\rho+n\end{array}\right].
$$

This proves Theorem 4.1. □

5 A $(LCS)_{2n+1}$-manifolds with respect to semi symmetric metric connection satisfying $C\cdot C=0$.

Theorem 5.1. *Any $(LCS)_{2n+1}$-manifold with respect to a semi-symmetric connection satisfying $C\cdot C=0$, is an η-Einstein manifold.*

Proof. We consider an $(LCS)_{2n+1}$-manifold with respect to a semi-symmetric metric connection satisfying the condition

$$(C(X,Y)\cdot C)(U,V)W=0.$$

Then we have

$$(C(X,Y)C)(U,V)W-C(C(X,Y)U,V)W-C(U,C(X,Y)V)W-C(U,V)C(X,Y)W=0. \tag{5.1}$$

By substituting $X=\xi$ in (5.1), it follows that

$$(C(\xi,Y)C)(U,V)W-C(C(\xi,Y)U,V)W-C(U,C(\xi,Y)V)W-C(U,V)C(\xi,Y)W=0; \tag{5.2}$$

again, by replacing $U=\xi$ in (5.2), we get

$$(C(\xi,Y)C)(\xi,V)W-C(C(\xi,Y)\xi,V)W-C(\xi,C(\xi,Y)V)W-C(\xi,V)C(\xi,Y)W=0. \tag{5.3}$$

In view of (4.3), we get

$$C(\xi,Y)\xi=\left[(2n-1)(\alpha^2-\rho)+\alpha-\frac{1}{2}+(1-\alpha)+\frac{r}{2n(2n+1)}\right]\{-Y-\eta(Y)\xi\}; \tag{5.4}$$

using (4.3) and (5.4) in (5.3), it follows that

$$\tilde{S}(V,W)=\left[\begin{array}{l}2n(2n+1)(\alpha^2-\rho)+\frac{r}{2n}\\-\frac{r}{2n(2n+1)}+2n\alpha-n\end{array}\right]g(V,W)-[2n(1-\alpha)]\ \eta(V)\eta(W). \tag{5.5}$$

This proves the Theorem 5.1. □

6 Concircular ϕ-recurrent $(LCS)_{2n+1}$-manifolds with respect to semi-symmetric metric connections

Definition 6.1. A $(LCS)_{2n+1}$-manifold is said to be concircular ϕ-recurrent [19] if there exists a non-zero 1-form A such that

$$\phi^2\left((\nabla_W C)(X,Y)Z\right)=A(W)C(X,Y)Z,$$

where A is defined by $A(W)=g(W,\rho)$ and ρ is a vector field associated with the 1 -form A.

Theorem 6.1. *A concircular ϕ-recurrent$(LCS)_{2n+1}$-manifold with respect to semi-symmetric metric connection is an η-Einstein manifold.*

Proof. Let us consider a concircular ϕ-recurrent $(LCS)_{2n+1}$-manifold with respect to the semi-symmetric metric connection defined by

$$\phi^2\left((\nabla_W C)(X,Y)Z\right)=A(W)C(X,Y)Z. \tag{6.1}$$

In view of (2.1) and (6.1), we get

$$(\nabla_W C)(X,Y)Z+\eta((\nabla_W C)(X,Y)Z)\xi=A(W)C(X,Y)Z. \tag{6.2}$$

From (6.2), it follows that

$$g((\nabla_W C)(X,Y)Z,U)+\eta((\nabla_W C)(X,Y)Z)g(\xi,U)=A(W)g(C(X,Y)Z,U), \tag{6.3}$$

where

$$\begin{array}{l}(\nabla_W C)(X,Y)Z+((\nabla_W K)(X,Y)Z)=\left[(4n\alpha+1)\rho\eta(W)-2n\rho\right]g(X,Z)Y\\=g(Y,Z)X-\rho\left\{\eta(Y)\eta(Z)\eta(W)X-\eta(X)\eta(Z)\eta(W)Y\right\}\\+\alpha(1-\alpha)\left[\begin{array}{l}g(W,Y)\eta(Z)X-g(W,Z)\eta(X)Y\\-g(W,X)\eta(Z)Y+g(W,Z)\eta(Y)X\\-2\eta(Y)\eta(Z)\eta(W)X-2\eta(X)\eta(Z)\eta(W)Y\end{array}\right]\\-\frac{\nabla_W r}{2n(2n+1)}\left[g(Y,Z)X-g(X,Z)Y\right]\end{array}. \tag{6.4}$$

Let $\{e_i\}, i=1,2,...,2n+1$ be an orthonormal basis of the tangent space at some point of the manifold. Then putting $X=U=e_i$ in (6.4)and taking summation over $i, 1\leq i\leq 2n+1$, we have

$$\begin{array}{r}-(\nabla_W S)(Y,Z)+\frac{\nabla_W r}{(2n+1)}g(Y,Z)=-\frac{\nabla_W r}{2n(2n+1)}\left[g(Y,Z)+\eta(Y)\eta(Z)\right]\\+A(W)\left[S(Y.Z)-\frac{r}{(2n+1)}g(Y,Z)\right]\end{array}; \tag{6.5}$$

by replacing $Z = \xi$in (6.5), we obtain

$$(\nabla_W S)(Y,\xi) = \frac{\nabla_W r}{(2n+1)}\eta(Y) - A(W)\left[S(Y.Z) - \frac{r}{(2n+1)}\eta(Y)\right]; \tag{6.6}$$

on the other hand, we have

$$(\nabla_W S)(Y,\xi) = \nabla_W S(Y,\xi) - S(\nabla_W Y,\xi) - S(Y,\nabla_W \xi);$$

using (1.3), (1.5), (2.11) and (3.6), the above equation reduces to

$$\begin{aligned}(\nabla_W S)(Y,\xi) = [2n\alpha(1-2n) - n\alpha]\{g(Y,W) + \eta(Y)\eta(W)\} - \alpha S(Y,\phi W)\\ +2n\alpha\left[2n(\alpha^2-\rho)+\alpha-\tfrac{1}{2}\right]g(Y,\phi W).\end{aligned} \tag{6.7}$$

In view of (6.6) and (6.7), upon simplification, we get

$$\tilde{S}(Y,W) = \left[4n^2(\alpha^2-\rho)+2n\alpha-n\right]g(Y,W) + \left[4n^2(\alpha^2-\rho)+2n\alpha-n\right]\eta(Y)\eta(W).$$

This proves the Theorem 6.1. □

Theorem 6.2. *In a concircular ϕ-recurrent $(LCS)_{2n+1}$-manifold admitting a semi-symmetric metric connection, the characteristic vector field ξ and the vector ρ associated to the $1-$ form A have opposite directions, and the 1-form A is given by (6.12).*

Proof. Considering (6.2) as well, one has

$$(\nabla_W C)(X,Y)Z = -\eta((\nabla_W C)(X,Y)Z)\xi + A(W)C(X,Y).Z; \tag{6.8}$$

now using (3.4), (6.4) and Bianchi's identity in (6.8), we obtain

$$\begin{aligned}&A(W)\eta(K(X,Y)Z) + A(X)\eta(K(Y,W)Z) + A(W)\eta(K(W,X)Z = \\ &\quad A(W)\left[2n(\alpha^2-\rho)+\alpha-\tfrac{1}{2}\right]\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\\ &\quad +A(X)\left[2n(\alpha^2-\rho)+\alpha-\tfrac{1}{2}\right]\{g(Y,Z)\eta(W) - g(W,Z)\eta(Y)\}\\ &\quad +A(Y)\left[2n(\alpha^2-\rho)+\alpha-\tfrac{1}{2}\right]\{g(W,Z)\eta(X) - g(X,Z)\eta(W)\}\\ &\quad +\tfrac{r}{2n(2n+1)}\begin{bmatrix} A(W)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\\ +A(X)\{g(W,Z)\eta(Y) - g(Y,Z)\eta(W)\}\\ +A(Y)\{g(X,Z)\eta(W) - g(W,Z)\eta(X)\}\end{bmatrix}\end{aligned}; \tag{6.9}$$

substituting $Y = Z = \xi$ in (6.9) and taking summation over $i, 1 \le i \le 2n+1$, we infer (6.10)

$$\begin{aligned}2n\left\{2n(\alpha^2-\rho)+\alpha-\tfrac{1}{2}\right\}A(W)\eta(X) = 2n\left\{2n(\alpha^2-\rho)+\alpha-\tfrac{1}{2}\right\}A(X)\eta(W)\\ +\tfrac{r}{2n(2n+1)}\{-2nA(X)\eta(W) + 2nA(W)\eta(X)\};\end{aligned}$$

again, replacing X by ξ in (6.10), we obtain

$$\left[4n^2(\alpha^2-\rho)+2n\alpha-n-\frac{r}{2n(2n+1)}\right]\{A(W) + A(\xi)\eta(W)\} = 0. \tag{6.11}$$

Therefore, we have

$$A(W) = -\eta(W)\eta(\rho), \tag{6.12}$$

for any vector field W. This proves Theorem 6.2. □

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