

Basic structural equations for almost Ricci-harmonic solitons and applications

Abimbola Abolarinwa

Abstract. This paper studies gradient almost Ricci-harmonic soliton with respect to a fixed metric. Analytic technique is applied to establish some basic elliptic and integral equations for the structure of almost Ricci-harmonic soliton, which generalize those of Ricci-harmonic solitons on one hand and those of almost Ricci soliton on the other hand. The consequences of these formulas in relation to classification and rigidity results for gradient almost Ricci-harmonic solitons are also discussed.

M.S.C. 2010: 53C21, 53C25, 53C44, 53C65.

Key words: Ricci soliton, harmonic maps, Ricci harmonic soliton, rigidity.

1 Introduction and the setting

Let (M, g) be a complete Riemannian manifold. (M, g) is said to have Ricci soliton structure if it satisfies the soliton equation, for any smooth vector field X and some constant $\sigma \in \mathbb{R}$,

$$(1.1) \quad Rc + \frac{1}{2}\mathcal{L}_X g = \sigma g,$$

where Rc is the Ricci tensor of M , g is the metric and \mathcal{L}_X is the Lie derivative in the direction of X . In this case, the triple (g, X, σ) is called Ricci soliton and said to be shrinking, steady or expanding if $\sigma > 0, \sigma = 0$ or $\sigma < 0$, respectively. If X is a gradient of some smooth function f on M , that is, $X = \nabla f$, Ricci soliton equation (1.1) is then written as

$$(1.2) \quad Rc + \text{Hess}f = \sigma g,$$

where $\text{Hess}f$ is the Hessian of function f . Then the triple (g, f, σ) is called gradient Ricci soliton and f is called the potential function. If either X or ∇f is a Killing vector field or f is a constant, then the soliton is said to be trivial and the underlying metric is Einstein. Compact Ricci solitons are special solutions to Hamilton's Ricci flow [21] and they occur as the flow's singularity models or blow-up limits [9, 27]. Interested readers are referred to [9, 13, 14] for background on Ricci solitons and their connection

to the Ricci flow. Ricci solitons are also very useful in theoretical physics as they have connection with renormalization group flow, Einstein field and string theories [11, 19]. A generalization of Ricci solitons is the Ricci-harmonic solitons which serve as special solutions of the Ricci-harmonic flow.

1.1 The Ricci-harmonic flow

Let (M, g) and (N, h) be compact Riemannian manifolds of dimensions m and n , respectively. Let a smooth map $u : M \rightarrow N$ be a critical point of the energy integral $E(u) = \int_M |\nabla u|_g^2 d\mu_g$, where N is isometrically embedded in \mathbb{R}^d , $d \geq n$, by the Nash embedding theorem. These critical points are called harmonic maps between M and N . Recall that harmonic maps are generalisation of harmonic functions such as identity and constant maps. The configuration $(g(x, t), u(x, t)), t \in [0, T)$ of a one parameter family of Riemannian metrics $g(x, t)$ and a family of smooth maps $u(x, t)$ is defined to be Ricci-harmonic flow if it satisfies the coupled system of nonlinear parabolic equations

$$(1.3) \quad \begin{aligned} \frac{\partial}{\partial t} g(x, t) &= -2Rc(x, t) + 2\alpha(t)\nabla u(x, t) \otimes \nabla u(x, t) \\ \frac{\partial}{\partial t} u(x, t) &= \tau_g u(x, t). \end{aligned}$$

Here $\alpha(t) \equiv \alpha > 0$ is a time-dependent coupling constant and $\tau_g u$ is the intrinsic Laplacian of u which denotes the tension field of map u . The system (1.3) was first studied by List [24] in a special case $(N, h) = (\mathbb{R}, dr^2)$, with the motivation coming from general relativity. When u is a constant function, List's flow becomes the well known Hamilton's Ricci flow. List's flow was generalized by Müller [25] to the general case $N \hookrightarrow \mathbb{R}^d$ for sufficiently large d , as we see in system (1.3). Precisely, the system couples together the Hamilton's Ricci flow [21] and Eells and Sampson's heat flow for harmonic maps [15]. The Ricci-harmonic flow is closer to the Ricci flow in behaviours, though less singular. For a detailed introduction to the study of the Ricci flow see the book by Chow and Knopf [13] or Chow, Lu and Ni [14]. The note [16] provides necessary background on harmonic map theory.

The short-time existence, uniqueness and obstruction to long-time existence of Ricci-harmonic flow are discussed in [25], see also [23]. In recent time, several researchers have extended fundamental results such as Perelman's entropy formulas, no breather theorems, noncollapsing theorems, volume growth, gradient estimates and Harnack inequalities in the study of Ricci flow to the setting of Ricci-harmonic flow, for example, see [1, 2, 3, 20, 24, 26, 30, 31, 33, 35].

1.2 Gradient Ricci-harmonic solitons

Let $f : M \rightarrow \mathbb{R}$ be a smooth function and $u : (M, g) \rightarrow (N, h)$ a smooth map (not necessarily harmonic map), where (M, g) and (N, h) are static Riemannian manifolds. Then, the quadruple (g, u, f, σ) is called a gradient Ricci-harmonic soliton if it satisfies

the coupled system of elliptic partial differential equations

$$(1.4) \quad \begin{aligned} Rc - \alpha \nabla u \otimes \nabla u + \text{Hess} f &= \sigma g \\ \tau_g u - \langle \nabla u, \nabla f \rangle &= 0, \end{aligned}$$

where $\alpha > 0$ is a positive constant depending on m and $\sigma \in \mathbb{R}$. The function f is called the potential. The gradient Ricci-harmonic soliton is said to be shrinking, steady or expanding depending on whether $\sigma > 0$, $\sigma = 0$ or $\sigma < 0$. We say that the soliton is trivial if f is set to be constant. Here are some examples as they appear in certain contexts:

- (1) If u is a constant map, then (1.4) becomes

$$Rc + \text{Hess} f = \sigma g,$$

which is the equation of gradient Ricci soliton.

- (2) If $(N, h) = (\mathbb{R}, dr^2)$ and u, f are constants, then (1.4) reduces to

$$Rc = \sigma g,$$

which says that (M, g) is Einstein.

- (3) If f is a constant and u is a harmonic map, then (1.4) reduces to

$$\begin{aligned} Rc - \alpha \nabla u \otimes \nabla u &= \sigma g \\ \tau_g u &= 0, \end{aligned}$$

which is usually called harmonic-Einstein and generalises the notion of Einstein manifold.

For generalisation and some extension of results on gradient Ricci solitons see [4, 8, 12, 17, 22, 29, 34]. The paper [9] (see also the references therein) provides details on the geometry of Ricci solitons. Recently, a particular generalisation of gradient Ricci soliton, called gradient almost Ricci soliton, was proposed in [29]. See [29, 5, 6, 7, 18] for detail discussion on existence, rigidity, triviality, classification and so on for gradient almost Ricci soliton. In this paper however, gradient almost Ricci-harmonic solitons are considered, in which case, σ appearing in (1.4) is not necessarily a constant but a smooth function on M . In this setting similar terminology is adopted, where we say gradient almost Ricci-harmonic soliton is shrinking, steady or expanding depending on whether σ is positive, null or negative on M . The gradient almost Ricci-harmonic soliton is said to be trivial if f is a constant.

The organisation of the paper is as follows: In Section 2, we give detail of notation adopted in the paper and statements of results to be proved later. Sections 3 discusses basic equations which will be used to prove some classification and rigidity results for almost Ricci harmonic solitons. The last section is devoted to the proofs of main results.

2 Notation and Statement of results

2.1 Notation

In the sequel, Ricci harmonic soliton will be abbreviated as $(RH)_\alpha$ soliton. We write $(0, 2)$ -tensor $Sc := Rc - \alpha \nabla u \otimes \nabla u$, its components as $S_{ij} := R_{ij} - \alpha \nabla_i u \nabla_j u$ in local coordinates while its metric trace $g^{ij} S_{ij}$ is denoted by $S := R - \alpha |\nabla u|^2$, where Rc , R_{ij} and R are the Ricci curvature tensor, its components and scalar curvature, respectively. div is the divergence operator, i.e., $(\text{div} Sc)_k = g^{ij} \nabla_i S_{jk}$. For a smooth map $u(x) : (M, g) \rightarrow (N, h)$, where $g = g_{ij} dx^i dx^j$ and $h = h_{\nu\gamma} du^\nu du^\gamma$, the energy density of $u(x)$ at a point $x \in M$ is $|\nabla u|_g^2 = g^{ij} h_{\nu\gamma} \partial_i u^\nu \partial_j u^\gamma$, i.e., the $\text{trace}_g u^* h$ and $\nabla u \otimes \nabla u = u^* h$ is the pullback of the metric tensor h on N via the map u . The tension field of map u is denoted by $\tau_g u = \text{trace}_g \nabla du$, which is a section of the bundle $u^*(T_u N)$, where ∇ is the connection on the vector bundle $T^*M \otimes u^*(TN)$ induced by the Levi-Civita connections on M and N .

For a smooth function f on M , we write $|\nabla f|^2 = g^{ij} \nabla_i f \nabla_j f$ and Δ is the Laplace Beltrami operator, while the volume measure on M is denoted by $dM = \sqrt{|g|} dx^i$, $i = 1, 2, \dots, m$, where $|g|$ is the determinant of g .

2.2 Statement of results

The main results of this paper which concern classification and rigidity of the gradient almost $(RH)_\alpha$ solitons are stated in this section. Basic formulas which will help in proving them are dealt with in Section 3.

Theorem 2.1. *Let (M, g) be a compact Riemannian manifold. Suppose (g, u, f, σ) is a gradient almost $(RH)_\alpha$ soliton such that $\Delta \sigma \leq 0$ on M . Setting $S_{\min} := \min_M S$, $\sigma_* = \inf_M \sigma$ and $\sigma^* = \sup_M \sigma$. Then*

- (i) *if gradient almost $(RH)_\alpha$ soliton is steady, then $S_{\min} = 0$.*
- (ii) *if gradient almost $(RH)_\alpha$ soliton is shrinking with $0 < \sigma \leq \sigma^*$, then $0 \leq S_{\min} \leq m\sigma^*$. In particular $S \geq 0$.*
- (iii) *if gradient almost $(RH)_\alpha$ soliton is expanding with $\sigma_* \leq \sigma < 0$, then $m\sigma_* \leq S_{\min} \leq 0$. In particular $S \geq m\sigma_*$.*

Corollary 2.2. *With the assumptions of Theorem 2.1.*

- (i) *In cases of shrinking and expanding (i.e., cases (ii) and (iii) of Theorem 2.1), (M, g) has nonnegative scalar curvature, $R \geq \alpha |\nabla u|^2 \geq 0$.*
- (ii) *In the shrinking case, if in addition $S_{\min} = m\sigma^*$ then f is constant, u is harmonic, $S \equiv S_{\min} = m\sigma^*$ and $Sc = \sigma g$. Meaning that gradient shrinking almost $(RH)_\alpha$ soliton is trivial.*

The next result is the generalization to almost $(RH)_\alpha$ soliton of [28, Theorem 1.1] for Ricci solitons and [5, Theorem 3] for almost Ricci solitons.

Theorem 2.3. *Let (M, g) be compact with $m \geq 3$. Then gradient almost $(RH)_\alpha$ soliton is rigid in the sense of triviality if*

$$\int_M \left(Sc(\nabla f, \nabla f) + (m-2)\langle \nabla \sigma, \nabla f \rangle \right) dM \leq 0,$$

where dM is the volume measure of M .

Proposition 2.4. *Let (M, g) be compact with $m \geq 2$. Then gradient almost $(RH)_\alpha$ soliton with $Sc \geq 0$ ($Sc \leq 0$) has constant S if and only if $Sc(\nabla f, \nabla f) = 0$ and $\langle \nabla \sigma, \nabla f \rangle \leq 0$.*

If u is a constant map, we obtain a version of [5, Corollary 1] as an application of integral equations proved in the next section (Theorem 3.2).

Proposition 2.5. *Let (M, g) be compact and (g, u, f, σ) be a nontrivial gradient almost $(RH)_\alpha$ soliton. Suppose $m \geq 2$ and u is a constant map, then (M, g) is isometric to an m -Euclidean sphere, if and only if any of the following conditions holds:*

(i) (M, g) has constant scalar curvature.

$$(ii) \int_M \langle Sc, \Delta f \rangle dM = 0.$$

$$(iii) \int_M \left(Sc(\nabla f, \nabla f) + (m-2)\langle \nabla \sigma, \nabla f \rangle \right) dM \leq 0.$$

(iv) ∇f is a conformal vector field.

Here dM is the volume measure of M .

We notice also that if u is a nonconstant map, one can still deduce the last result. See the discussion in Remark 4.2.

3 Basic equations for almost Ricci-harmonic solitons

Firstly, we shall prove some basic equations which will be applied to derive some classification and rigidity results. The equations are extension to almost $(RH)_\alpha$ solitons of the classical ones derived for almost Ricci solitons. We include the proofs for completeness sake and since some new terms come in.

Proposition 3.1. *Let (g, u, f, σ) be a gradient almost $(RH)_\alpha$ soliton, then we have the following equations:*

$$(3.1) \quad \operatorname{div} Sc = \frac{1}{2} \nabla S - \alpha \tau_g(u) \nabla u$$

$$(3.2) \quad \langle \nabla S, \nabla f \rangle = 2(m-1)\langle \nabla \sigma, \nabla f \rangle + 2Sc(\nabla f, \nabla f)$$

$$(3.3) \quad Sc(\nabla f, \cdot) = \frac{1}{2} \nabla S - (m-1)\nabla \sigma$$

$$(3.4) \quad \nabla(S + |\nabla f|^2) = 2(m-1)\nabla \sigma + 2\sigma \nabla f$$

$$(3.5) \quad \frac{1}{2} \Delta |\nabla f|^2 = |\operatorname{Hess} f|^2 - (m-2)\langle \nabla \sigma, \nabla f \rangle - Sc(\nabla f, \nabla f) + \alpha |\langle \nabla u, \nabla f \rangle|^2.$$

The proofs of the above equations follow from standard computation which is usually performed with the method of the moving frame in a local orthonormal frame.

Proof. The first formula is a consequence of contracted second Bianchi identity $\operatorname{div} Rc = \frac{1}{2}\nabla R$ and the identity $\operatorname{div}(\nabla u \otimes \nabla u) = \tau_g(u)\nabla u + \frac{1}{2}\nabla|\nabla u|^2$ proved in [20, Appendix]

$$\begin{aligned} \frac{1}{2}\nabla R &= \operatorname{div} Rc \\ &= \operatorname{div} Sc + \alpha \operatorname{div}(\nabla u \otimes \nabla u) \\ &= \operatorname{div} Sc + \alpha \tau_g(u)\nabla u + \frac{\alpha}{2}\nabla|\nabla u|^2, \end{aligned}$$

which implies $\frac{1}{2}\nabla(R + \alpha\nabla|\nabla u|^2) = \operatorname{div} Sc + \alpha\tau_g(u)\nabla u$, the first formula to be proved.

Note that the first equation in (1.4) implies

$$(3.6) \quad S_{ij} = \sigma g_{ij} - f_{ij}.$$

Taking covariant derivative of (3.6) we have

$$(3.7) \quad S_{ij,k} = \sigma_k g_{ij} - f_{ijk}$$

since $g_{ij,k} = 0$ (covariant derivative of g) by metric compatibility of Levi-Civita connection. Tracing with respect to j and k we have

$$(3.8) \quad S_{ik,k} = \sigma_i - f_{ikk}.$$

Recall the Ricci identity

$$(3.9) \quad f_{ikk} - f_{kki} = R_{ik}f_k$$

and the first formula in the proposition

$$(3.10) \quad S_{ij,i} = \frac{1}{2}S_j - \alpha\tau_g(u)\nabla u.$$

Now, putting (3.9) and (3.10) into (3.8), we have

$$(3.11) \quad \frac{1}{2}S_i - \alpha\tau_g(u)\nabla u = \sigma_i - f_{kki} - R_{ik}f_k.$$

Tracing (3.7) with respect to i and j , we have

$$(3.12) \quad S_i = m\sigma_i - f_{kki}.$$

Substituting (3.12) into (3.11) we have

$$(3.13) \quad S_i = 2(m-1)\sigma_i + 2R_{ik}f_k - 2\alpha\tau_g(u)\nabla u.$$

In particular, we have from (3.13)

$$\begin{aligned} \langle \nabla S, \nabla f \rangle &= 2(m-1)\langle \nabla \sigma, \nabla f \rangle + 2Rc(\nabla f, \nabla f) - 2\alpha\tau_g(u)\langle \nabla u, \nabla f \rangle \\ (3.14) \quad &= 2(m-1)\langle \nabla \sigma, \nabla f \rangle + 2Sc(\nabla f, \nabla f), \end{aligned}$$

where we have used $\tau_g(u) = \langle \nabla u, \nabla f \rangle$ and $Rc = Sc + \alpha \nabla u \otimes \nabla u$. This proves the second formula in the proposition.

The third equation in the proposition can be seen as a direct consequence of (3.13) or (3.14). But we prove it this way: Taking divergence of the first equation in (1.4) yields

$$(3.15) \quad \operatorname{div} Rc - \alpha \operatorname{div}(\nabla u \otimes \nabla u) + \operatorname{div} \operatorname{Hess} f = \operatorname{div}(\sigma g).$$

Routine computation gives $\operatorname{div}(\nabla u \otimes \nabla u) = \tau_g(u) \nabla u + \frac{1}{2} \nabla |\nabla u|^2$ and $\operatorname{div} \operatorname{Hess} f = \nabla \Delta f + Rc(\nabla f, \cdot) = m \nabla \sigma - \nabla S + Rc(\nabla f, \cdot)$. Then (3.15) implies

$$(3.16) \quad \operatorname{div} Rc + (m-1) \nabla \sigma + Rc(\nabla f, \cdot) - \alpha \tau_g(u) \nabla u - \nabla S - \frac{\alpha}{2} \nabla |\nabla u|^2 = 0.$$

Using $\operatorname{div} Rc = \frac{1}{2} \nabla R$ in (3.16) we have

$$(3.17) \quad \frac{1}{2} \nabla (R - \alpha |\nabla u|^2) - \nabla S + (m-1) \nabla \sigma + (Rc - \alpha \nabla u \otimes \nabla u)(\nabla f, \cdot) = 0.$$

Then

$$(3.18) \quad -\frac{1}{2} \nabla S + (m-1) \nabla \sigma + Sc(\nabla f, \cdot) = 0,$$

which is the third equation of the proposition.

The fourth equation is another consequence of (3.13). It has been previously derived in [5, 18] by different methods. Equation (3.13) implies

$$\begin{aligned} \nabla S &= 2(m-1) \nabla \sigma + 2Sc(\nabla f, \cdot) \\ &= 2(m-1) \nabla \sigma + 2(\sigma g - \operatorname{Hess} f) \nabla f \\ &= 2(m-1) \nabla \sigma + 2\sigma \nabla f - 2\operatorname{Hess} f(\nabla f) \\ &= 2(m-1) \nabla \sigma + 2\sigma \nabla f - \nabla |\nabla f|^2, \end{aligned}$$

which implies

$$(3.19) \quad \nabla (S + |\nabla f|^2) - 2(m-1) \nabla \sigma - 2\sigma \nabla f = 0.$$

The fourth equation is proved.

Inserting $\Delta f = m\sigma - S$ into the well known Bochner formula we have

$$\begin{aligned} \frac{1}{2} \Delta |\nabla f|^2 &= |\operatorname{Hess} f|^2 + \langle \nabla \Delta f, \nabla f \rangle + Rc(\nabla f, \nabla f) \\ &= |\operatorname{Hess} f|^2 + m \langle \nabla \sigma, \nabla f \rangle - \langle \nabla S, \nabla f \rangle + Rc(\nabla f, \nabla f). \end{aligned}$$

Using the second equation of the proposition yields

$$\begin{aligned} \frac{1}{2} \Delta |\nabla f|^2 &= |\operatorname{Hess} f|^2 + m \langle \nabla \sigma, \nabla f \rangle - 2(m-1) \langle \nabla \sigma, \nabla f \rangle - 2Sc(\nabla f, \nabla f) \\ &\quad + Rc(\nabla f, \nabla f) \\ &= |\operatorname{Hess} f|^2 - (m-2) \langle \nabla \sigma, \nabla f \rangle - Sc(\nabla f, \nabla f) + \alpha \nabla u \otimes \nabla u(\nabla f, \nabla f), \end{aligned}$$

which implies the fifth equation. Therefore the proof of the proposition is complete. \square

Lastly, we derive some integral equations which will be used to derive an application which generalizes results obtained in [5] for almost Ricci solitons.

Theorem 3.2. *Let (M, g) be compact and (g, u, f, σ) be gradient almost $(RH)_\alpha$ soliton. Then we have*

$$(3.20) \quad \int_M \left| \text{Hess}f - \frac{1}{m}g\Delta f \right|^2 dM = \frac{m-2}{2m} \int_M \langle \nabla S, \nabla f \rangle dM - \alpha \int_M |\langle \nabla u, \nabla f \rangle|^2 dM$$

and

$$(3.21) \quad \int_M \left| \text{Hess}f - \frac{1}{m}g\Delta f \right|^2 dM = \frac{m-2}{m} \int_M \left(\text{Sc}(\nabla f, \nabla f) + (m-1)\langle \nabla \sigma, \nabla f \rangle \right) dM - \alpha \int_M |\langle \nabla u, \nabla f \rangle|^2 dM,$$

where dM is the volume measure of M .

Proof. Taking divergence of the fourth equation in Proposition 3.1 yields

$$(3.22) \quad \Delta S + \Delta |\nabla f|^2 - 2(m-1)\Delta \sigma - 2\langle \nabla \sigma, \nabla f \rangle - 2\sigma \Delta f = 0.$$

Using second and fifth equations in Proposition 3.1 into (3.22) we have

$$(3.23) \quad \Delta S + 2|\text{Hess}f|^2 - \langle \nabla S, \nabla f \rangle - 2(m-1)\Delta \sigma - 2\sigma \Delta f + 2\alpha |\langle \nabla u, \nabla f \rangle|^2 = 0.$$

By using the identity

$$\left| \text{Hess}f - \frac{1}{m}g\Delta f \right|^2 = |\text{Hess}f|^2 - \frac{1}{m}(\Delta f)^2$$

(3.23) can be then written as

$$\begin{aligned} \Delta S + 2 \left| \text{Hess}f - \frac{1}{m}g\Delta f \right|^2 - \langle \nabla S, \nabla f \rangle - 2(m-1)\Delta \sigma \\ - 2\Delta f \left(\sigma - \frac{1}{m}\Delta f \right) + 2\alpha |\langle \nabla u, \nabla f \rangle|^2 = 0. \end{aligned}$$

Note that $S = \sigma - \frac{1}{m}\Delta f$ (from $\Delta f = m\sigma - S$), then

$$\begin{aligned} \Delta S + 2 \left| \text{Hess}f - \frac{1}{m}g\Delta f \right|^2 - \langle \nabla S, \nabla f \rangle - 2(m-1)\Delta \sigma \\ - 2S\Delta f + 2\alpha |\langle \nabla u, \nabla f \rangle|^2 = 0. \end{aligned}$$

Integrating over M and using compactness of M , we have

$$(3.24) \quad 2 \int_M \left| \text{Hess}f - \frac{1}{m}g\Delta f \right|^2 dM = \frac{m-2}{m} \int_M \langle \nabla S, \nabla f \rangle - 2\alpha \int_M |\langle \nabla u, \nabla f \rangle|^2 dM$$

from where (3.20) follows. Now using the second equation of Proposition 3.1 in (3.24) we obtain (3.21) at once. The proof is complete. \square

4 Proofs of main results

The aim of this section is to use some of the equations obtained in the last section to classify gradient almost $(RH)_\alpha$ solitons and determine when they are rigid as stated in Section 2. The first result to prove is Theorem 2.1.

4.1 Proof of Theorem 2.1

To start with, taking divergence of the fourth equation in Proposition 3.1 gives

$$(4.1) \quad \Delta S + \Delta|\nabla f|^2 - 2(m-1)\Delta\sigma - 2\langle\nabla\sigma, \nabla f\rangle - 2\sigma\Delta f = 0.$$

Using fifth and second equations in Proposition 3.1, identity $\Delta f = m\sigma - S$, the condition $\Delta\sigma \leq 0$ and the inequality $|Sc|^2 \geq \frac{2}{m}S^2$ into (4.1) we have

$$\begin{aligned} 0 &= \Delta S + 2|\text{Hess}f|^2 - \langle\nabla S, \nabla f\rangle - 2(m-1)\Delta\sigma - 2\sigma\Delta f + 2\alpha|\langle\nabla u, \nabla f\rangle|^2 \\ &= \Delta S + 2|\sigma g - Sc|^2 - \langle\nabla S, \nabla f\rangle - 2(m-1)\Delta\sigma - 2\sigma(m\sigma - S) + 2\alpha|\langle\nabla u, \nabla f\rangle|^2 \\ &= \Delta S + 2|Sc|^2 - 2\sigma S - \langle\nabla S, \nabla f\rangle - 2(m-1)\Delta\sigma + 2\alpha|\langle\nabla u, \nabla f\rangle|^2 \\ &\geq \Delta S + \frac{2}{m}S^2 - 2\sigma S - \nabla S(\nabla f). \end{aligned}$$

Since M is compact, there exists a point $q \in M$, where S attains its minimum. By the maximum principle, at this point $\nabla S(q) = 0$ and $\Delta S(q) \geq 0$. Then the last inequality implies

$$(4.2) \quad S_{min}(S_{min} - m\sigma) \leq 0.$$

The first case is when $\sigma = 0$, that is, the case of steady gradient almost $(RH)_\alpha$ soliton, we have $S_{min} = 0$ by (4.2). In the case of shrinking gradient almost $(RH)_\alpha$ soliton $\sigma > 0$ with $0 < \sigma \leq \sigma^*$, by (4.2) we have $0 \leq S_{min} \leq m\sigma^*$. In the case of expanding gradient almost $(RH)_\alpha$ soliton $\sigma < 0$ with $\sigma_* \leq \sigma < 0$, we have $m\sigma_* \leq S_{min} < 0$ and $S \geq S_{min} \geq m\sigma_*$. \square

Remark 4.1. If one follows the steps in the proof of Theorem 0.4 of [29], it can be further shown that under a pairwise control on σ , the curvature function S of gradient almost $(RH)_\alpha$ soliton is bounded from below and the lower bound can be estimated from both below and above.

We note that when S attains its minimum, by the maximum principle the inequality

$$(4.3) \quad \Delta S_{min} + \frac{2}{m}S_{min}^2 - 2\sigma S_{min} - \nabla S_{min}(\nabla f) \leq 0$$

implies that if S is nonconstant, then it must be everywhere positive and then σ is positive, which says that gradient almost $(RH)_\alpha$ soliton is shrinking. This implies the following proposition.

Proposition 4.1. *Every steady or expanding gradient almost $(RH)_\alpha$ soliton has constant S .*

Proof. By the weak maximum principle inequality (4.3) implies

$$(4.4) \quad \frac{1}{2}\Delta S_{min} \leq \frac{1}{m}S_{min}(m\sigma_* - S_{min}).$$

This implies that $\Delta S_{min} \leq 0$ since $S_{min} = 0$ (for the steady case) and $S_{min} \geq m\sigma_*$ (for the expanding case). Then $S \geq S_{min} \geq 0$ is nonnegative subharmonic and it is therefore constant. \square

The above proposition forces $\Delta f = m\sigma - S$ to be $\Delta f \geq 0$ since $S = m\sigma_* \leq m\sigma$. Thus f is superharmonic and the compactness of M implies that f is constant and the soliton is trivial. Hence, the following corollary.

Corollary 4.2. *Every steady or expanding gradient almost $(RH)_\alpha$ soliton ($S = m\sigma_*$) is trivial on compact M .*

4.2 Proof of Theorem 2.3

Integrating both sides of the fifth equation in Proposition 3.1, we have

$$\begin{aligned} \frac{1}{2} \int_M \Delta |\nabla f|^2 dM &= \int_M |\text{Hess} f|^2 dM - \int_M Sc(\nabla f, \nabla f) dM \\ &\quad - (m-2) \int_M \langle \nabla \sigma, \nabla f \rangle dM + \alpha \int_M |\langle \nabla u, \nabla f \rangle|^2 dM, \end{aligned}$$

where dM is the volume measure of M . Since M is compact we have

$$(4.5) \quad \int_M |\text{Hess} f|^2 dM = \int_M \left(Sc(\nabla f, \nabla f) + (m-2) \langle \nabla \sigma, \nabla f \rangle \right) dM$$

$$(4.6) \quad - \alpha \int_M |\langle \nabla u, \nabla f \rangle|^2 dM.$$

Note that the second on the right hand side of (4.5) is nonnegative and by assumption the first term is nonpositive. Then we obtain $\text{Hess} f = 0$ which implies ∇f is a killing vector field and that f is a constant by compactness of M . \square

4.3 Proof of Proposition 2.4

The proof of Proposition 2.4 can be established using any of the first three equations in Proposition 3.1. In what follows, we mimick the proof of Proposition 3 in [28].

Proof. Consider a nonnegative (nonpositive) definite, self-adjoint operator T , we know that $\langle Tv, v \rangle = 0$ implies $Tv = 0$. Taking T to be $(0, 2)$ -tensor, the proposition follows from the second or third equation in Proposition 3.1. That is,

$$\frac{1}{2} \nabla S(\nabla f) = (m-1) \langle \nabla \sigma, \nabla f \rangle + Sc(\nabla f, \nabla f)$$

which implies $\nabla S = 0$ by the assumption. \square

4.4 Proof of Proposition 2.5

Proof. Considering any of the conditions in the proposition, we have from either (3.20) or (3.21) that

$$(4.7) \quad \int_M \left| \text{Hess}f - \frac{1}{m}g\Delta f \right|^2 dM = 0.$$

Indeed, supposing condition (ii), we have

$$\begin{aligned} 0 &= \int_M \langle Sc, \Delta f \rangle dM = \int_M \langle \text{div}Sc, \nabla f \rangle dM \\ &= -\frac{1}{2} \int_M \langle \nabla S, \nabla f \rangle dM + \alpha \int_M |\langle \nabla u, \nabla f \rangle| dM, \end{aligned}$$

which holds if and only if S is scalar, since u is constant and ∇f is not killing (nontrivial soliton). Note that (4.9) implies $Sc - \frac{1}{m}Sg = 0$ and consequently, $Rc = \frac{1}{m}Rg$, thus, (M, g) is Einstein. Therefore (1.3) yields $\text{Hess}f = (\sigma - \frac{1}{m}R)g$, saying that ∇f is a nontrivial conformal vector field.

If on the other hand, we suppose ∇f is a conformal vector field, then (3.20) will yield $\int_M |\langle \nabla S, \nabla f \rangle| dM \geq 0$. The fact that u is constant and (4.9) give equality $\int_M |\langle \nabla S, \nabla f \rangle| dM = 0$, meaning that (M, g) has constant curvature.

The rest of the proof is now similar to that of [5, Corollary 1], where a theorem of Tashiro [32] together with Theorem [5, Theorem 2] was applied. The rest argument is therefore omitted here. Tashiro's theorem says that a complete Riemannian manifold M^m , $m \geq 2$ is conformally equivalent to the Euclidean sphere of the same dimension if it admits a nontrivial solution of the equation $\text{Hess}\rho - \frac{1}{m}g\Delta\rho = 0$, where ρ is a smooth function on M . [5, Theorem 2] says that when $m \geq 3$, (M, g) is isometric to Euclidean sphere \mathbb{S}^m if u is a constant and ∇f is a nontrivial conformal field. \square

Remark 4.2. If u is a nonconstant map, one can still deduce the conclusion of Proposition 2.5. Noticing that (3.20) and (3.21) under Theorem 3.2 become

$$(4.8) \quad \int_M \left| \text{Hess}f - \frac{1}{m}g\Delta f \right|^2 dM \leq \frac{m-2}{2m} \int_M \langle \nabla S, \nabla f \rangle dM$$

and

$$(4.9) \quad \int_M \left| \text{Hess}f - \frac{1}{m}g\Delta f \right|^2 dM \leq \frac{m-2}{m} \int_M \left(Sc(\nabla f, \nabla f) + (m-1)\langle \nabla\sigma, \nabla f \rangle \right) dM.$$

One can then invoke any of the conditions (i) – (iv) of Proposition 2.5 and conclude that

$$\int_M \left| Sc - \frac{1}{m}Sg \right|^2 dM \leq 0$$

since

$$(4.10) \quad \text{Hess}f - \frac{1}{m}g\Delta f = \text{Hess}f - \left(\sigma - \frac{1}{m}S \right) g = -Sc + \frac{1}{m}Sg.$$

Therefore

$$0 = Sc - \frac{1}{m}Sg = Rc - \frac{1}{m}S(R + \alpha(m-1)|\nabla u|^2g)$$

and we conclude that (M, g) is Einstein, since every Einstein manifold belongs to the class of Riemannian manifolds (M, g) realising the relation $R_{ij} = ag_{ij} + b\nabla_i v \nabla_j v$, where a and b are certain nonzero constants (see [10]).

Acknowledgements

The author wishes to thank the anonymous referee and the editor for their useful comments.

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Authors' addresses:

Abimbola Abolarinwa
Department of Physical Sciences,
College of Pure and Applied Sciences
Landmark University
P.M.B. 1001, Omu-Aran, Kwara State, Nigeria
E-mail: a.abolarinwa1@gmail.com, abolarinwa.abimbola@lmu.edu.ng