Semiconformal curvature tensor and Spacetime of General Relativity

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Abstract. The aim of the present paper is to study the semiconformal curvature tensor for spacetime of general relativity. We establish the results on Killing and conformally Killing vectors satisfying Einstein's field equations in the case of semiconformally flat spacetimes. We extend the same case for the study of cosmological models with dust and perfect fluid.

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1 Introduction and Preliminaries

Let g_{ij} and $\tilde{g_{ij}}$ be two metric tensors of the Riemannian spaces V and $\tilde{V}$ respectively, related by the equation

$$\tilde{g_{ij}} = e^{2\beta} g_{ij}, \tag{1.1}$$

for β a real scalar function of the co-ordinates.

The Riemannian spaces V and $\tilde{V}$ are conformal spaces and the correspondence between V and $\tilde{V}$ is identified as a conformal transformation ([2]). Harmonic functions are the functions whose Laplacian is zero and generally these do not transform into harmonic functions under conformal transformation. The conharmonic transformation introduced by Ishii ([5]) form a subgroup of the conformal transformations (1.1). The condition under which the harmonic functions remain invariant is [1]

$$\beta^{i}_{;i} + \beta_{;i}\beta^{i}_{;} = 0, \tag{1.2}$$

A four ranked tensor Z^{l}_{ijk} which remains invariant under conharmonic transformations in an n-dimensional ($n \geq 4$) Riemannian manifold (M^n, g) is

$$Z^{l}_{ijk} = R^{l}_{ijk} + \frac{1}{n-2}(\delta^{l}_{j} R_{ik} - \delta^{l}_{k} R_{ij} + g_{ik} R^{l}_{j} - g_{ij} R^{l}_{k}), \tag{1.3}$$

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[1] The indices take hereafter values from $1, 2, 3......, n$. Semicolon (;) and comma (,)indicate the covariant derivative and the partial derivative, respectively.

where R^l_{ijk}, R_{ik} are the Riemann and the Ricci curvature tensors, respectively.

Here, in this paper study a curvature-like tensor introduced by J. Kim ([6]), which caries the same property of invariance like Z^l_{ijk} under conharmonic transformation. He named this new curvature-like four rank tensor S^l_{ijk} as *semiconformal curvature tensor*, which can be expressed as ([7])

$$S^l_{ijk} = -(n-2)\sigma W^l_{ijk} + [\lambda + (n-2)\sigma] Z^l_{ijk}, \tag{1.4}$$

provided that λ and σ are non-simultaneously vanishing constants.

The mixed form (of type $(1,3)$) of the conformal curvature tensor is given by

$$W^l_{ijk} = R^l_{ijk} + \frac{1}{n-2}(\delta^l_c R_{ik} - \delta^l_d R_{ij} + g_{ik} R^l_j - g_{ij} R^l_k) + \frac{R}{(n-1)(n-2)}(\delta^l_k g_{ij} - \delta^l_j g_{ik}), \tag{1.5}$$

where R_{ij} is the Ricci tensor and R is the scalar curvature tensor.

Under the remarkable substitutions $\lambda = 1$ and $\sigma = -\dfrac{1}{n-2}$, the semiconformal curvature tensor reduces to the conformal curvature tensor, while for $\lambda = 1$ and $\sigma = 0$, it reduces to the conharmonic curvature tensor. For instance, it may be noted that the semiconformal curvature tensor S_{lijk} of type $(0,4)$ satisfies the following symmetry properties

$$S_{lijk} = -S_{iljk} = -S_{likj} = S_{jkli}, \tag{1.6}$$

and

$$S_{lijk} + S_{jlik} + S_{ijlk} = 0. \tag{1.7}$$

When the semiconformal curvature tensor vanishes at every point of a manifold, then the manifold is called *semiconformally flat*. So, this tensor shows the deviation of the manifold from semiconformally flatness due to non-vanishing components. In this paper the relativistic significance of the semiconformal curvature tensor has been investigated, and it is seen that a semiconformally flat spacetime is an Einstein space and consequently a space of constant curvature under certain conditions. We have studied the Killing and conformal Killing vector fields and their existence for the spacetime of general relativity satisfying Einstein's field equations with cosmological term. Also, we consider the vanishing of the semiconformal curvature tensor, in some special cases (different models of the universe). We are encouraged to use the concept of invariance of a tensor under conharmonic transformation from the paper in [11].

Definition 1.1. Einstein-like Spacetime. A Riemannian space $V_n (n > 2)$ is said to be an *Einstein space* if its Ricci tensor has the form

$$R_{ij} = \varphi g_{ij} \tag{1.8}$$

where $\varphi = \dfrac{R}{n}$ is a scalar and R is the scalar curvature. If the Ricci tensor is simply proportional to the metric tensor, we call this space as *Einstein-like space*.

Definition 1.2. Einstein field equations with cosmological constant. The interaction of matter and gravitation through Einstein's famous field equations with cosmological term is given by

$$R_{ij} - \frac{1}{2}Rg_{ij} + \wedge g_{ij} = kT_{ij}, \tag{1.9}$$

where k is a constant, T_{ij} is the energy-momentum tensor and $\wedge$ is the cosmological constant.

Definition 1.3. Spaces of constant curvature. In cosmology, the spaces of constant curvature or the space forms are of great importance. The highlights of spaces of constant curvature are *no preferred point/direction* which by default follow the principle of cosmology, i.e., the universe is isotropic and homogeneous. These spaces of constant curvature are also called as *maximally symmetric spaces* and have $\frac{1}{2}n(n+1)$ independent Killing vectors in the Riemannian spaces of dimension n. So, in spacetimes (of constant curvature), i.e., spaces whose curvature depends on the time, the independent Killing vectors are ten and these spacetimes are de-Sitter models. The three dimensional space-like surfaces of constant curvature are Robertson-Walker metrics which are acquired by the de-Sitter cosmological model [8].

Definition 1.4. Symmetries of Spacetime. In general theory of relativity, the curvature tensor characterizes the gravitational field into two parts, viz., the matter part and the free gravitational part. The interaction between the matter and the gravitational parts is described through the Bianchi identities. In gravitational physics, the main objective of all investigations is the construction of the gravitational potential satisfying the Einstein field equations. For a given distribution of matter, this is achieved by imposing symmetries on the geometry compatible with the dynamics of the chosen distribution of matter. The geometrical symmetries of the spacetime are representable through the following equation [12]

$$\pounds_\xi D - 2\mho D = 0, \tag{1.10}$$

where $\pounds_\xi$ represents the Lie derivative with respect to the vector field ξ^i (this vector may be time-like, space-like or null), D denotes a geometrical/physical quantity and $\mho$ is a scalar.

A simple example can be provided here as the metric inheritance symmetry for $D = g_{ij}$ in equation (1.10) and if $\mho$ is zero, then ξ^i is a Killing vector ([9],[10]). Until this date, more than 30 geometric symmetries have been found in literature. The semiconformal curvature tensor is used throughout in the present paper and the symmetry of spacetime corresponding to this tensor is given in [3]. (For a comprehensive analysis of symmetry inheritance and related results, see [1]).

2 Semiconformally flatness in spacetime of general relativity

The semiconformal curvature tensor from equation (1.4) in V_4 is

$$S^l_{ijk} = -2\sigma W^l_{ijk} + [\lambda + 2\sigma]Z^l_{ijk}, \tag{2.1}$$

where W^l_{ijk} and Z^l_{ijk} are the conformal and the conharmonic curvature tensors, respectively. In V_4, from equations (1.3) and (1.5), these tensors may be written in the following form

$$Z^l_{ijk} = R^l_{ijk} + \frac{1}{2}(\delta^l_c R_{ik} - \delta^l_k R_{ij} + g_{ik}R^l_j - g_{ij}R^l_k), \tag{2.2}$$

and

$$W^l_{ijk} = R^l_{ijk} + \frac{1}{2}(\delta^l_j R_{ik} - \delta^l_k R_{ij} + g_{ik}R^l_j - g_{ij}R^l_k) + \frac{R}{6}(\delta^l_k g_{ij} - \delta^l_j g_{ij}). \tag{2.3}$$

The use of equations (2.2) and (2.3) in equation (2.1) yields

$$S^l_{ijk} = \lambda[R^l_{ijk} + \frac{1}{2}(\delta^l_j R_{ik} - \delta^l_k R_{ij} + g_{ik}R^l_j - g_{ij}R^l_k)] - \frac{\sigma R}{3}(\delta^l_k g_{ij} - \delta^l_j g_{ik}). \tag{2.4}$$

The contraction of equation (2.4) provides

$$S_{ij} = -\left(\frac{\lambda + 2\sigma}{2}\right)Rg_{ij}, \tag{2.5}$$

and under the condition (1.2), the equation (2.5) is invariant.

If the semiconformal curvature tensor vanishes, then (2.4) leads to

$$\lambda R^l_{ijk} = \frac{\lambda}{2}\left(\delta^l_k R_{ij} - \delta^l_j R_{ik} + g_{ij}R^l_k - g_{ik}R^l_j\right) + \frac{\sigma R}{3}\left(\delta^l_k g_{ij} - \delta^l_j g_{ik}\right), \tag{2.6}$$

or,

$$-\lambda R^l_{ijk} = \frac{\lambda}{2}\left(\delta^l_j R_{ik} - \delta^l_k R_{ij} + g_{ik}R^l_j - g_{ij}R^l_k\right) + \frac{\sigma R}{3}\left(\delta^l_j g_{ik} - \delta^l_k g_{ij}\right).$$

The contractions over l and j give rise to

$$R_{ik} = -\left(\frac{\lambda + 2\sigma}{\lambda}\right)\frac{1}{4}Rg_{ik}, \tag{2.7}$$

where R is the scalar curvature.

Thus, we may state the following result:

Theorem 2.1. *A semiconformally-flat spacetime is an Einstein-like space.*

Corollary 2.2. *A semiconformally-flat spacetime is an Einstein space for $\sigma = 0$ and λ non-zero constant.*

Moreover, from equation (2.6), we have

$$\lambda R_{tijk} = \lambda g_{tl}R^l_{ijk} = \frac{\lambda}{2}\left(g_{tk}R_{ij} - g_{tj}R_{ik} + g_{ij}R_{tk} - g_{ik}R_{tj}\right) + \frac{\sigma R}{3}\left(g_{tk}g_{ij} - g_{tj}g_{ij}\right). \tag{2.8}$$

In view of equation (2.7), the above equation reduces to

$$R_{tijk} = \left(\frac{3\lambda + 2\sigma}{12\lambda}\right)R(g_{tj}g_{ik} - g_{tk}g_{ij}), \tag{2.9}$$

Thus, we have the following theorem

Theorem 2.3. *A semiconformally-flat spacetime is of constant curvature.*

The semiconformal curvature tensor stands for the deviation of the manifold V_4 from the spacetime having the constant curvature. We use the following operator for the measurement of the deviation from semiconfromal flatness

$$S = sup_D \frac{|S_{tijk} D^{ti} D^{jk}|}{D_{ij} D^{ij}}$$

where $D^{tj} = Y^t X^j - Y^j X^t$; X and Y are mutually orthogonal unit vectors. Also,

$$D_{ij} = -D_{ji}.$$

The covariant derivative of equation (2.7) gives

$$R_{ik;j} = -\left(\frac{\lambda + 2\sigma}{\lambda}\right)\frac{1}{4} g_{ik} R_{,j}. \tag{2.10}$$

The product of g^{kl} in the equation (2.10), leads to

$$R^l_{i;j} = -\left(\frac{\lambda + 2\sigma}{\lambda}\right)\frac{1}{4} \delta^l_i R_{,j}; \tag{2.11}$$

contracting over l and i, we get

$$R_{,j} = 0, \qquad \text{since } 2(\lambda + \sigma) \neq 0. \tag{2.12}$$

This equation implies that the spacetime is of constant scalar curvature.

In view of (2.7), equation (1.9) yields

$$\left\{ \wedge - \left(\frac{3\lambda + 2\sigma}{\lambda}\right)\frac{1}{4} R \right\} g_{ij} = kT_{ij}, \tag{2.13}$$

since R is due to (2.12); then, operating the Lie derivative on equation (2.13), we have

$$\left\{ \wedge - \left(\frac{3\lambda + 2\sigma}{\lambda}\right)\frac{1}{4} R \right\} \pounds_\xi g_{ij} = k \pounds_\xi T_{ij}. \tag{2.14}$$

For ξ as Killing vector field, (2.14), gives

$$\pounds_\xi g_{ij} = 0. \tag{2.15}$$

Using equations (2.14) and (2.15), we infer

$$\pounds_\xi T_{ij} = 0. \tag{2.16}$$

Conversely, if equation (2.16) holds and since k, $\wedge$, $\left(\frac{3\lambda + 2\sigma}{\lambda}\right)$ and R are constants, then the use of equation (2.16), leads to

$$\pounds_\xi g_{ij} = 0.$$

Hence ξ is a Killing vector field. Thus, we may state

Theorem 2.4. *In a semiconformally-flat spacetime admitting Einstein field equations with a cosmological term, for the existence of Killing vector ξ, the necessary and sufficient condition is that the Lie derivative of energy-momentum tensor vanishes along ξ.*

Under a conformal transformation, the angle between two arbitrary vectors of the Riemannian manifold is preserved (cf., equation (1.1)). For having an one-parameter group of transformations spanned by the vector field ξ as group of conformal transformations, the necessary and sufficient condition is that ξ satisfies the following equation (cf., [12]),

$$\pounds_\xi g_{ij} = 2\mho g_{ij}, \tag{2.17}$$

where $\mho$ is a scalar function. In (2.17) the vector field ξ is called *conformal Killing vector field* (CKV). The invariance of the semiconformal curvature tensor under conharmonic transformations, which is a special case of conformal transformations provides an idea to investigate the role of semiconformally-flat spacetimes.

Making use of (2.17) in (2.14), we get

$$\left\{ \wedge - \left(\frac{3\lambda + 2\sigma}{\lambda} \right) \frac{1}{4} R \right\} 2\mho g_{ij} = k \pounds_\xi T_{ij}. \tag{2.18}$$

From (2.13) and (2.18), we get

$$\pounds_\xi T_{ij} = 2\mho T_{ij}, \tag{2.19}$$

which implies that the symmetry inheritance property is admitted by the energy-momentum tensor. Conversely, we can show that (2.17) will hold, if the equation (2.19) holds.

So, we deduce the next result

Theorem 2.5. *The necessary and sufficient condition for a semiconformally-flat spacetime satisfying Einstein's field equations with cosmological term to admit a conformal Killing vector field, is that it admits the symmetry inheritance property for the energy-momentum tensor.*

3 Cosmological model with vanishing semiconformal curvature tensor

For setting some cosmological models of the universe, we use Einstein's equations. Einstein considered the global case, i.e., the universe on large scale follows cosmological principles. The presence of matter doesn't violate the isotropy and homogeneity conditions of universe. The matter contents of the universe, i.e., Stars, Galaxies, Nebulas etc., were considered to be a perfect fluid.

Let us consider a perfect fluid spacetime with vanishing semiconformal curvature tensor. The energy-momentum tensor T_{ij} of a perfect fluid is given by

$$T_{ij} = (\mu + \rho) u_i u_j + \rho g_{ij}, \tag{3.1}$$

where μ is the energy density, ρ is the isotropic pressure and u_i is the fluid four velocity vector, such that $u_i u^i = -1$.

Using (3.1) and (2.13), we get

$$\left\{ \wedge - \left(\frac{3\lambda + 2\sigma}{\lambda} \right) \frac{1}{4} R - k\rho \right\} g_{ij} = k(\mu + \rho) u_i u_j. \tag{3.2}$$

Multiplication by g^{ti} of (3.2) yields

$$\left\{ \wedge - \left(\frac{3\lambda + 2\sigma}{\lambda} \right) \frac{1}{4} R - k\rho \right\} \delta^t_j = k(\mu + \rho) u^t u_j, \tag{3.3}$$

and contracting over t and j, we infer

$$\left(\frac{3\lambda + 2\sigma}{\lambda} \right) R = 4 \wedge + k(\mu - 3\rho). \tag{3.4}$$

Furthermore, multiplying by u^i and using $u^i u_i = -1$ in (3.2), we get

$$\left(\frac{3\lambda + 2\sigma}{\lambda} \right) R = 4 \wedge + 4k\mu. \tag{3.5}$$

By combining equations (3.4) and (3.5), we get

$$\mu + \rho = 0, \tag{3.6}$$

which opposes our assumption of perfect fluid.

Thus, we may state the following

Theorem 3.1. *A spacetime with vanishing semiconformal curvature tensor and satisfying the Einstein field equations with cosmological term is a perfect fluid space if* $\mu + \rho \neq 0$

In case of the dust model, the energy-momentum tensor is given by

$$T_{ij} = \mu u_i u_j. \tag{3.7}$$

Equation (2.13) with the help of (3.7) may be written as

$$\left\{ \wedge - \left(\frac{3\lambda + 2\sigma}{\lambda} \right) \frac{1}{4} R \right\} g_{ij} = k\mu u_i u_j. \tag{3.8}$$

Multiplying g^{ti} and then contracting over t and j with $u_i u^i = -1$ leads to

$$\left(\frac{3\lambda + 2\sigma}{\lambda} \right) R = 4 \wedge + k\mu. \tag{3.9}$$

Now, by operating u^i on both sides in equation (3.8), we get

$$\left(\frac{3\lambda + 2\sigma}{\lambda} \right) R = 4(\wedge + k\mu). \tag{3.10}$$

From (3.9) and (3.10), we have

$$\mu = 0,$$

which is against of our assumption.

Thus, we may state the following

Corollary 3.2. *A spacetime satisfying the Einstein field equations and having a vanishing semiconformal curvature tensor represents a dust cosmological model if the energy-density does not vanish.*

For a spacetime with radiative perfect fluid ($\mu = 3\rho$), the resulting universe must be isotropic and homogeneous ([4]). So, by using this condition $\mu = 3\rho$ in equation (3.2), we get

$$\left\{ \wedge - \left(\frac{3\lambda + 2\sigma}{4\lambda} \right) R - \frac{k\mu}{3} \right\} g_{ij} = \frac{4}{3} k\mu u_i u_j. \tag{3.11}$$

Further, by operating g^{ti} on both sides and by contracting over t and j with $u_i u^i = -1$, (3.11) yields

$$\left(\frac{3\lambda + 2\sigma}{\lambda} \right) R = 4 \wedge . \tag{3.12}$$

Now, by multiplying u^b on equation (3.11), we get

$$\left(\frac{3\lambda + 2\sigma}{\lambda} \right) R = 4 \wedge + 4k\mu. \tag{3.13}$$

Using equations (3.12) and (3.13), we have

$$\mu = 0,$$

which is not possible by our assumption.

Thus, we may state the following

Corollary 3.3. *A spacetime with vanishing semiconformal curvature tensor and satisfying the Einstein field equations with a cosmological term is an isotropic and homogeneous spacetime if the energy density of the fluid does not vansih.*

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