M-projective curvature tensor over cosymplectic manifolds

G. Ayar and S. K. Chaubey

Abstract. In this paper, properties of the α-cosymplectic manifolds with M-projective curvature tensor are studied. Meanwhile, we obtain some connections between different curvature tensors.

M.S.C. 2010: 53C25, 53D10, 53C21.
Key words: cosymplectic manifolds, M-projective curvature tensor, projective curvature tensor, Riemannian manifolds.

1 Introduction and preliminaries

Although M-projective curvature tensor studies are based on old times, the properties of the M-projective curvature tensor has recently come popular and has been studied by many mathematicians. In particular studies on contact manifolds with M-projective curvature tensor have contributed significantly to the literature. To mention some of these; at the beginning, in 1958, Boothby and Wong [18] introduced odd dimensional manifolds with contact and almost contact structures from topological point of view. Next, they were re-investigated by Sasaki and Hatakeyama in 1961 [19] using tensor calculus. In 1985, R.H Ojha showed some properties of M-projective curvature tensor in a Sasakian manifold [12]. An important study on this subject in 2010 by Chaubey and Ojha was about the properties of M-projective curvature tensor in Riemannian manifolds and also in Kenmotsu manifolds [17]. They obtained the relation between different curvature tensors. In addition in 2012, O. F. Zengin studied M-projectively flat spacetimes and showed that M-projectively flat Riemannian manifold is an Einstein manifold [20].

If we take an m-dimensional differentiable manifold M^m with differentiability class C^∞ on board, the elements on the manifold are as a $(1,1)$ tensor field φ, a vector field ξ, a contact form η and the associated Riemannian metric g. In 1971 on an m-dimensional Riemannian manifold, ones [10] defined a tensor field W^* as

(1.1)
$$W^*(X,Y)Z = R(X,Y)Z - \frac{1}{2(m-1)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]$$

so that

$$'W^*(X,Y,Z,U) \overset{\text{def}}{=} g(W^*(X,Y)Z,U) = 'W^*(Z,U,X,Y) \tag{1.2}$$

Differential Geometry - Dynamical Systems, Vol.21, 2019, pp. 23-33.

and

$$'W^*_{ijkl}w^{ij}w^{kl} =' W_{ijkl}w^{ij}w^{kl} \tag{1.3}$$

where $'W^*_{ijkl}$ and $'W_{ijkl}$ are components of $'W^*$and $'W$ respectively and w^{kl} is a skew-symmetric tensor [11, 15]. They called this tensor W^* as M-projective curvature tensor. In [11, 12], authors have focused on Sasakian and Kähler manifolds admitting M-projective curvature tensor. Some connections between conformal, con-harmonic, con-circular and H-projective curvature tensors have also been studied by them.

In addition to these definitions, the Weyl projective curvature tensor W, concircular curvature tensor C and conformal curvature tensor V are given as follows [13]

$$W(X,Y)Z = R(X,Y)Z - \frac{1}{m-1}\{S(Y,Z)X - S(X,Z)Y\}, \tag{1.4}$$

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{m(m-1)}\{g(Y,Z)X - g(X,Z)Y\} \tag{1.5}$$

and

$$\begin{aligned} V(X,Y)Z &= R(X,Y)Z - \tfrac{1}{(m-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &\quad -g(X,Z)QY] + \tfrac{r}{(m-1)(m-2)}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{1.6}$$

Now we will give some important theorems and corollary that we use in some of our results, without proofs.

Theorem 1.1. *[17] The M-projective and Weyl projective curvature tensors of the Riemannian manifold M are linearly dependent if and only if M is an Einstein manifold.*

Theorem 1.2. *[17] The necessary and sufficient condition for a Riemannian manifold to be an Einstein manifold is that the M-projective curvature tensor $W^*$and concircular curvature tensor C are linearly dependent.*

Theorem 1.3. *[17] A Riemannian manifold becomes an Einstein manifold. if and only if conformal and M-projective curvature tensors of the manifold are linearly dependent.*

Corollary 1.4. *[17] In an Riemannian manifold M, the following are equivalent*

i) M is an Einstein manifold.

ii) M-projective and Weyl projective curvature tensors are linearly dependent.

iii) M-projective and con-circular curvature tensors are linearly dependent.

iv) M-projective curvature and conformal curvature tensors are linearly dependent.

Again in this passage to introduce an almost contact manifold, we repeat the relevant material from Blair [7] without proofs.

Let M be a $(2n+1)$-dimensional differentiable manifold equipped with a triplet (φ, ξ, η), where φ is a type of $(1,1)$ tensor field, ξ is a vector field, η is a 1-form on M such that

$$\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi, \tag{1.7}$$

which implies

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad rank(\varphi) = 2n. \tag{1.8}$$

M is said to have an almost contact metric structure (φ, ξ, η, g) when it admits a Riemannian metric g, such that

$$\begin{aligned} &g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y), \\ &\eta(X) = g(X, \xi), \end{aligned} \tag{1.9}$$

On this type of manifold, Φ is the fundamental 2-form on M^{2n+1} and defined by

$$\Phi(X,Y) = g(\varphi X, Y), \tag{1.10}$$

for vector fields X, Y on M^{2n+1}.

An almost contact manifold $(M^{2n+1}, \varphi, \xi, \eta)$ is said to be normal if the Nijenhuis torsion

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + \varphi^2[X,Y] + 2d\eta(X,Y)\xi,$$

vanishes for any vector fields X, Y on M^{2n+1}. With the normallity condition, an almost cosymplectic manifold is called a cosymplectic manifold. Equipping with almost contact metric structure, if both $\bigtriangledown\eta$ and $\bigtriangledown\Phi$ vanish then we can say that the manifold is cosymplectic. Meanwhile, an almost contact metric structure is Kenmotsu if and only if

$$(\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X. \tag{1.11}$$

A normal almost cosymplectic and almost Kenmotsu manifolds are called a cosymplectic manifold and Kenmotsu manifold, respectively.

An almost contact metric manifold M^{2n+1} is said to be almost α-Kenmotsu if $d\eta = 0$ and $d\Phi = 2\alpha\eta \wedge \Phi$, α is a non-zero real constant. Geometrical properties and examples of almost α-Kenmotsu manifolds are studied by many mathematicians in [1, 5, 6, 8]. Giving an almost Kenmotsu metric structure (φ, ξ, η, g), considering the deformed structure

$$\eta' = \frac{1}{\alpha}\eta, \quad \xi' = \alpha\xi, \quad \varphi' = \varphi, \quad g' = \frac{1}{\alpha^2}g, \quad \alpha \neq 0, \quad \alpha \in \mathbb{R},$$

where α is a non-zero real constant and we get an almost α-Kenmotsu structure $(\varphi', \xi', \eta', g')$. We can call this deformation as a homothetic deformation. It should be noted that almost α-Kenmotsu structures are associated with some of the specific local conformal deformations of almost cosympectic structures(see [8]).

If we examine these in two classes, we get a new expression about an α-cosymplectic manifold as defined by the formula

$$d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi,$$

for any real number α (see [1]). It is obvious that a normal almost α-cosymplectic manifold is an α-cosymplectic manifold. In addition, an α-cosymplectic manifold is either cosymplectic under the condition $\alpha = 0$ or α-Kenmotsu ($\alpha \neq 0$) for $\alpha \in \mathbb{R}$. Almost α-cosymplectic manifolds have been studied by several authors [2, 3, 4].

Suppose that M^{2n+1} is an α-cosymplectic manifold. Denoted by A the $(1,1)$-tensor field on M^{2n+1} defined by

$$A = -\nabla\xi, \tag{1.12}$$

and given by the following relations where $\mathcal{L}$ is the Lie derivative of g. It is obvious that, $A(\xi) = 0$ Furthermore, the tensor fields A is a symmetric operators and satisfies the following relations

$$AX = -\alpha\varphi^2 X, \tag{1.13}$$

$$(\nabla_X \eta)Y = \alpha\left[g(X,Y) - \eta(X)\eta(Y)\right], \tag{1.14}$$

$$\delta\eta = -2\alpha n,$$

$$tr(A) = -2\alpha n, \tag{1.15}$$

$$tr(\varphi A) = 0, \tag{1.16}$$

$$A\varphi + \varphi A = -2\alpha\varphi, \tag{1.17}$$

$$A\xi = 0, \tag{1.18}$$

$$(\nabla_X A)\,\xi = A^2 X, \tag{1.19}$$

for any vector fields X, Y on M^{2n+1}.

In this paper, properties of α-cosymplectic manifolds with M-projective curvature tensor are studied. While Section 2 includes the curvature properties of α-cosymplectic manifolds, in Section 3 we show that an M-projectively flat $2n+1$-dimensional α-cosymplectic manifold M^{2n+1} is locally isometric to the hyperbolic space $H^{2n+1}(-1)$. In Section 4 the zero state of W^* is examined and interpreted. In this part we give an α-cosymplectic manifold satisfying this condition and we obtain some results. And finally in Section 5, we proved that in an α-cosymplectic manifold $M^{2n+1}$with M-projective curvature tensor is irrotational if and only if it is locally isometric to the hyperbolic space $H^{2n+1}(-\alpha^2)$.

2 Basic curvature relations

In this section, basic curvature relations will be briefly given. Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an α-cosymplectic manifold. We denote the curvature tensor g by R and Ricci tensor of g by S. Meanwhile, we define a self adjoint operator $l = R(., \xi)\xi$ (the Jacobi operator with respect to ξ). One can easily see the followings

Proposition 2.1. *[2] Let M^{2n+1} be an $\alpha-$cosymplectic manifold. Then we have*

$$R(X,Y)\xi = \alpha^2\left[\eta(X)Y - \eta(Y)X\right], \tag{2.1}$$

$$R(X,\xi)\xi = \alpha^2\varphi^2 X, \tag{2.2}$$

$$R(\xi,X)Y = \alpha^2\left[\eta(Y)X - g(X,Y)\xi\right], \tag{2.3}$$

$$R(X,\xi)\xi - \varphi R(\varphi X,\xi)\xi = 2\alpha^2\varphi^2 X, \tag{2.4}$$

$$S(X,\xi) = -\alpha^2 2n\eta(X), \tag{2.5}$$

$$\eta(R(X,Y)Z) = \alpha^2\left[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)\right], \tag{2.6}$$

$$tr\,(l) = S(\xi,\xi) = -2n\alpha^2,$$

for any vector fields X, Y on M^{2n+1}.

If a $(2n+1)$-dimensional α-cosymplectic manifold (M^{2n+1}, g) is η-Einstein, then its non-vanishing Ricci-tensor S can be written as follows.

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{2.7}$$

for arbitrary vector fields X, Y and a, b are smooth functions on (M^{2n+1}, g). When $b = 0$, then η-Einstein manifold becomes Einstein manifold. Kenmotsu [5] proved that an η-Einstein Kenmotsu manifold (M^{2n+1}, g) satisfies the relation $a + b = -2n$. With the same way, we can prove the following lemma.

Lemma 2.2. *On an almost α-cosymplectic manifold (M^{2n+1}, g), $a + b = -2n\alpha^2$.*

Proof. In view of (1.7)-(1.9) and (2.7), we have

$$QX = aX + b\eta(X)\xi, \tag{2.8}$$

such that Ricci operator Q is defined by

$$S(X,Y) \overset{\text{def}}{=} g(QX,Y). \tag{2.9}$$

Again, contracting (2.8) with respect to X and using (1.7)-(1.9), we have

$$r = (2n+1)a + b. \tag{2.10}$$

Now, putting ξ instead of X and Y in (2.7) and then using the equations in (1.7)-(1.9) and (2.5) we get

$$a + b = -2n\alpha^2. \tag{2.11}$$

Equations (2.10) and (2.11) give

$$a = \left(\frac{r}{2n} + \alpha^2\right) \;\; and \;\; b = -\left(\frac{r}{2n} + (2n+1)\alpha^2\right). \tag{2.12}$$

Equation (2.12) prove the statement of the Lemma 2.2.

3 M-projectively flat α-cosymplectic manifolds

In view of $W^* = 0$, (1.1) becomes

$$R(X,Y)Z = \frac{1}{4n}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \tag{3.1}$$

Substituting $Z = \xi$ in (3.1) and then using (1.7)-(1.9) and (2.5), we obtain

$$2\alpha^2 n(\eta(Y)X - \eta(X)Y) + 4nR(X,Y)\xi = \eta(Y)QX - \eta(X)QY. \tag{3.2}$$

Again putting $Y = \xi$ in the above relation and using (1.7), (1.8), (1.9) and (2.5), we have

$$QX = -2\alpha^2 nX \Longleftrightarrow S(X,Y) = -2\alpha^2 ng(X,Y) \tag{3.3}$$

and

$$r = -2\alpha^2 n(2n+1). \tag{3.4}$$

In consequence of (3.3), the equation (3.1) becomes

$$R(X,Y)Z = -\alpha^2\{g(Y,Z)X - g(X,Z)Y\}. \tag{3.5}$$

A space form is said to be hyperbolic if and only if the sectional curvature tensor is negative [12, 14]. Thus, we can express the following theorem.

Theorem 3.1. *An almost α-cosymplectic manifold M^{2n+1} is M-projectively flat if and only if it is locally isometric to the hyperbolic space $H^{2n+1}(-\alpha^2)$.*

In view of the equations (3.3) and (3.4), and the Theorem 3.1, we can give the following corollaries.

Corollary 3.2. *Every M-projectively flat almost α-cosymplectic manifold M^{2n+1} is Ricci symmetric.*

Corollary 3.3. *An M-projectively flat almost α-cosymplectic manifold M^{2n+1} possesses a constant scalar curvature.*

A triplet (g, V, λ) defined on M^{2n+1} is said to a Ricci soliton if it satisfies

$$(\mathfrak{L}_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0 \tag{3.6}$$

for vector fields X and Y on M^{2n+1}, where $\mathfrak{L}_V g$ denotes the Lie derivative of the Riemannian metric g along the complete vector field V on M^{2n+1} and λ is a real constant. The Ricci soliton (g, V, λ) is said to be shrinking, expanding or steady if λ is < 0, > 0 or $= 0$, respectively.

If possible, we suppose that the M-projectively flat almost α-cosymplectic manifolds admit a Ricci soliton (g, V, λ). Replacing X, Y and V by the structure vector field ξ in (3.6) and keeping in mind that $\nabla_\xi \xi = 0$ on M^{2n+1} and the equation (3.3), we find

$$\lambda = -S(\xi,\xi) = 2n\alpha^2$$

and

$$(\mathfrak{L}_V g)(X,Y) = 0.$$

This shows that the complete vector field V is Killing, that is $\mathfrak{L}_V g = 0$, on M^{2n+1}. By the way we can specify the following theorem.

Theorem 3.4. *If an M-projectively flat almost α-cosymplectic manifold M^{2n+1} admits a Ricci soliton (g,V,λ), then the vector field V is Killing and the Ricci soliton (g,V,λ) to be expanding.*

4 M-projectively semi-symmetric α-cosymplectic manifolds

In view of (1.7)-(1.9), (2.3), (2.7), (2.8) and (2.12), the equation (1.1) becomes

$$W^*(\xi,X)Y=\left\{\alpha^2+\frac{1}{4n}\left(\frac{r}{2n}-\alpha^2\left(2n-1\right)\right)\right\}\{\eta(Y)X-g(X,Y)\xi\}. \tag{4.1}$$

Now, we have

$$\begin{aligned}(W^*(\xi,X)\cdot R)(Y,Z)U&=W^*(\xi,X)R(Y,Z)U-R(W^*(\xi,X)Y,Z)U\\&-R(Y,W^*(\xi,X)Z)U-R(Y,Z)W^*(\xi,X)U.\end{aligned} \tag{4.2}$$

Let us suppose that an α-cosymplectic manifold M^{2n+1} is M-projectively semi-symmetric, that is, $W^*(X,Y)\cdot R=0$ for all vector fields X and Y on M^{2n+1}. In consequence of $W^*(\xi,X)\cdot R=0$, the equation (4.2) becomes

$$\begin{aligned}&W^*(\xi,X)R(Y,Z)U-R(W^*(\xi,X)Y,Z)U-R(Y,W^*(\xi,X)Z)U\\&-R(Y,Z)W^*(\xi,X)U=0.\end{aligned}$$

In view of (4.1), the last equation takes the form

$$\begin{aligned}&\left\{\alpha^2+\tfrac{1}{4n}\left(\tfrac{r}{2n}-\alpha^2\left(2n-1\right)\right)\right\}[\eta(R(Y,Z)U)X-{}'R(Y,Z,U,X)\xi-\eta(Y)R(X,Z)U\\&+g(X,Y)R(\xi,Z)U-\eta(Z)R(Y,X)U+g(X,Z)R(Y,\xi)U-\eta(U)R(Y,Z)X\\&+g(X,U)R(Y,Z)\xi]=0,\end{aligned}$$

where

$${}'R(X,Y,Z,U)\overset{\text{def}}{=}g(R(X,Y)Z,U).$$

Taking inner-product of the above equation according to the Riemannian metric g and then using (1.7)-(1.9) and (2.6), we have

$$\begin{aligned}&\left\{\alpha^2+\tfrac{1}{4n}\left(\tfrac{r}{2n}-\alpha^2(2n-1)\right)\right\}[{}'R(Y,Z,U,X)+\alpha^2\left\{g(X,Y)g(Z,U)\right.\\&\left.-g(X,Z)g(Y,U)\right\}]=0,\end{aligned} \tag{4.3}$$

which implies that either $r=-2n(2n+1)\alpha^2$, that is, the scalar curvature of M^{2n+1} is constant or

$${}'R(Y,Z,U,X)=\alpha^2\left\{g(X,Z)g(Y,U)-g(X,Y)g(Z,U)\right\},$$

equivalent to

$$R(Y,Z)U=\alpha^2\left\{g(Y,U)Z-g(Z,U)Y\right\}. \tag{4.4}$$

This reflects that the M-projectively semi-symmetric α-cosymplectic manifold M^{2n+1} is a space form. Contracting (4.4) according to the vector field Y, we find

$$S(Z,U)=-2\alpha^2ng(Z,U), \tag{4.5}$$

which gives

$$QZ = -2\alpha^2 nZ \tag{4.6}$$

and

$$r = -2\alpha^2 n(2n+1). \tag{4.7}$$

Conversely, the equations (1.7)-(1.9), (4.1), (4.4)-(4.6) and (4.2) give $W^*(\xi, X)\cdot R = 0$. Thus, consequently we state:

Theorem 4.1. *Every α-cosymplectic manifold M^{2n+1} is M-projectively semi-symmetric if and only if either M^{2n+1} is locally isometric to the hyperbolic space $H^{2n+1}(-\alpha^2)$ or M^{2n+1} has constant scalar curvature $-2\alpha^2\, n(2n+1)$.*

In presence of the Theorem 3.4 and the Theorem 4.1, we can state the following corollary.

Corollary 4.2. *If an M-projectively semi-symmetric α-cosymplectic manifold M^{2n+1} confesses a Ricci soliton (g, V, λ), then the vector field V is a Killing vector field and (g, V, λ) is expanding.*

Also, in the light of the Corollary 1.4, Theorem 3.1 and the Theorem (4.1), we have

Corollary 4.3. *A $(2n+1)$-dimensional α-cosymplectic manifold M^{2n+1} is M-projectively semi-symmetric if and only if it is conformally flat.*

5 η-Einstein α-cosymplectic manifolds with the irrotational M-projective curvature tensor

In this section, we study the properties of the irrotational M-projective curvature tensor on an η-Einstein α-cosymplectic manifold M^{2n+1}. Before going to prove our results, we give the following definition.

Definition 5.1. Let ∇ be a Riemannian connection with respect to the Riemannian metric g, then the rotation (Curl) of M-projective curvature tensor W^* on a $(2n+1)$-dimensional *alpha*-cosymplectic manifold M^{2n+1} is defined as

$$\begin{aligned} Rot\, W^* &= (\nabla_U W^*)(X,Y)Z + (\nabla_X W^*)(U,Y)Z + (\nabla_Y W^*)(X,U)Z \\ &\quad -(\nabla_Z W^*)(X,Y)U \end{aligned} \tag{5.1}$$

for all vector fields X, Y, Z and U on M^{2n+1} (see [17]). If $Rot\, W^* = 0$ on M^{2n+1}, then the M-projective curvature tensor is irrotational.

In consequence of the Bianchi's second identity for the Riemannian connection ∇, the equation (5.1) becomes

$$Rot\, W^* = -(\nabla_Z W^*)(X,Y)U. \tag{5.2}$$

If the M-projective curvature tensor is irrotational, then curl $W^* = 0$ and therefore $(\nabla_Z W^*)(X,Y)U = 0$, which gives

$$\nabla_Z(W^*(X,Y)U) = W^*(\nabla_Z X,Y)U + W^*(X,\nabla_Z Y)U + W^*(X,Y)\nabla_Z U. \tag{5.3}$$

Replacing $U = \xi$ in (5.3) , we have

$$\nabla_Z(W^*(X,Y)\xi) = W^*(\nabla_Z X,Y)\xi + W^*(X,\nabla_Z Y)\xi + W^*(X,Y)\nabla_Z \xi. \tag{5.4}$$

Let us suppose that M^{2n+1} is a $(2n+1)$-dimensional η-Einstein α-cosymplectic manifold. Then substituting $Z = \xi$ in (1.1) and then using (1.7)-(1.9), (2.1), (2.5) and (2.12), we obtain

$$W^*(X,Y)\xi = k[\eta(X)Y - \eta(Y)X], \tag{5.5}$$

where

$$k = \left\{\frac{\alpha^2}{2} + \frac{1}{4n}\left(\frac{r}{2n} + \alpha^2\right)\right\}. \tag{5.6}$$

Using (1.7)-(1.9), (1.14), (5.5) and (5.6) in (5.4), we obtain

$$\alpha\, W^*(X,Y)Z = \frac{dr(Z)}{8n^2}\{\eta(X)Y - \eta(Y)X\} + k\alpha[g(X,Z)Y - g(Y,Z)X], \tag{5.7}$$

where d is an exterior derivative. Contacting (5.7) along the vector field X and then using the equations (1.1), (1.7), (2.5) and (5.6) together with $Y = \xi$, we have

$$dr(Z) = 0, \tag{5.8}$$

which shows that the scalar curvature of M^{2n+1} is constant. In consequence of (5.8), the equation (5.7) shows that either $\alpha = 0$ or

$$W^*(X,Y)Z = k[g(X,Z)Y - g(Y,Z)X], \;\; \alpha \neq 0. \tag{5.9}$$

An η-Einstein α-cosymplectic manifold M^{2n+1} is cosymplectic if $\alpha = 0$. If possible, we suppose that $\alpha \neq 0$, then the equations (1.1) and (5.9) give

$$\begin{aligned} k\,[g(X,Z)Y - g(Y,Z)X] = R(X,Y)Z \\ -\tfrac{1}{4n}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY]. \end{aligned}$$

Contracting the above equation with respect to the vector field X and then using (5.6), we find

$$S(Y,Z) = -2n\alpha^2 g(Y,Z) \Longleftrightarrow QY = -2n\alpha^2 Y, \tag{5.10}$$

which gives

$$r = -2n(2n+1)\alpha^2, \tag{5.11}$$

provided $\alpha \neq 0$. In consequence of (1.1), (5.6), (5.7), (5.10) and (5.11), we can find

$$R(X,Y)Z = -\alpha^2[g(Y,Z)X - g(X,Z)Y]. \tag{5.12}$$

Thus, we can state:

Theorem 5.1. *The M-projective curvature tensor in an η-Einstein $\alpha-$cosymplectic manifold M^{2n+1} is irrotational if and only if it is locally isometric to the hyperbolic space $H^{2n+1}(-\alpha^2)$ or locally Riemannian product of an almost Kaehler manifold with the real line.*

Theorem 4.1 together with the Theorem 5.1 lead to the following corollaries.

Corollary 5.2. *A $2n+1$-dimensional η-Einstein α-cosymplectic manifold $M^{(2n+1)}$ to be M-projectively semi-symmetric if and only if the M-projective curvature tensor of $M^{(2n+1)}$ is irrotational.*

Corollary 5.3. *The M-projective curvature tensor in an η-Einstein α-cosymplectic manifold M^{2n+1} is irrotational if and only if the manifold is conformally flat.*

Corollary 5.4. *If a $(2n+1)$-dimensional η-Einstein α-cosymplectic manifold M^{2n+1} is irrotational, then M^{2n+1} possesses a constant scalar curvature.*

Corollary 5.5. *If the M-projective curvature tensor of an η-Einstein α-cosymplectic manifold M^{2n+1} is irrotational, then M^{2n+1} is either cosymplectic or α-Kenmotsu.*

References

[1] T. W. Kim and H. K. Pak, *Canonical foliations of certain classes of almost contact metric structures*, Acta Math. Sinica, Eng. Ser. Aug., 21, 4 (2005), 841–846.

[2] H. Ozturk, C. Murathan, N. Aktan and A. T. Vanlı , *Almost α-cosymplectic f-manifolds*, Analele Stiintifice ale Universitatii Al I Cuza din Iasi - Matematica 60(1) (2014).

[3] H. Ozturk, N. Aktan and C. Murathan, Almost α -Cosymplectic (κ, μ, ν)-Spaces, arXiv:1007.0527v1 [math.DG] (2010), 1-24.

[4] M. Yıldırım, N. Aktan and C. Murathan, *Almost f -Cosymplectic Manifolds*, Mediterr. J. Math. 11 (2014), 775-787.

[5] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku Math. J. 24 (1972), 93-103.

[6] Z. Olszak, *Locally conformal almost cosymplectic manifolds*, Coll. Math. 57 (1989), 73–87.

[7] D.E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics 203. Birkhuser Boston, Inc., Boston, MA, (2002).

[8] I. Vaisman, *Conformal changes of almost contact metric manifolds*, Lecture Notes in Math. Berlin-Heidelberg-New York, 792 (1980), 435-443.

[9] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure Mathematics, 3. World Scientific Publishing Co. Singapore (1984).

[10] G.P. Pokhariya and R.S. Mishra, *Curvature tensor and their relativistic significance II*, Yokohama Mathematical Journal 19 (1971), 97-103.

[11] R.H. Ojha, *A note on the M-projective curvature tensor*, Indian J. Pure Appl. Math. 8 12 (1975), 1531-1534.

[12] R.H. Ojha, *M-projectively flat Sasakian manifolds*, Indian J. Pure Appl. Math. 17 4 (1986), 481-484.

[13] R.S. Mishra, *Structures on a Differentiable manifold and their applications*, Chandrama Prakashan, 50-A Bairampur House Allahabad 1984.

[14] B.Y. Chen, Geometry of Submanifolds, Marcel Dekker, Inc. New York (1973).

[15] S. Tanno, *Curvature tensors and non-existence of killing vectors*, Tensor N.S. 22 (1971), 387-394.

[16] A. Ghosh, T. Koufogiorgos and R. Sharma, *Conformally flat contact metric manifolds*, J. Geom. 70(2001), 66-76.

[17] S.K. Chaubey and R.H. Ojha, *On the m-projective curvature tensor of a Kenmotsu manifold*, Differential Geometry-Dynamical Systems 12 (2010), 52-60.

[18] M. Boothby and R.C. Wong, *On contact manifolds*, Ann. Math. 68 (1958), 421-450.

[19] S. Sasaki and Y. Hatakeyama, *On differentiable manifolds with certain structures which are closely related to almost contact structure*, Tohoku Math. J. 13 (1961), 281-294.

[20] F.O. Zengin, *M-projectively flat Spacetimes*, Math. Reports 14(64) 4(2012), 363-370.

Authors' addresses:

Gulhan Ayar
Department of Mathematics, Kamil Ozdag Science Faculty,
Karamanoglu Mehmetbey University,
Karaman, Turkey.
E-mail: gulhanayar@gmail.com

Sudhakar Kumar Chaubey
Section of Mathematics,
Department of Information Technology,
Shinas College of Technology,
Al Aqar, Shinas, Oman.
E-mail: sudhakar.chaubey@shct.edu.om , sk22_math@yahoo.co.in