Almost pseudo Ricci symmetric spacetimes

Ecem Bektaş and Füsun Özen Zengin

Abstract. The present paper is concerned with an almost pseudo Ricci symmetric spacetime. Under some conditions, we determine the properties of these spacetimes. In the first section, considering our spacetime as a perfect fluid, we prove that our spacetime reduces to an Einstein, quasi-Einstein or $\eta-$Einstein space with some assumptions. In addition, we show that a dust and a radiation fluid in an almost pseudo Ricci symmetric spacetime are vacuum.

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1 Introduction

Astronomers have struggled with basic questions about the size and age of the universe for thousand of years. At the beginning of the twentieth century, the astronomer Edwin Hubble made a critical discovery that soon led to reasonable answers to these questions. Those measurements marked the first evidence that our universe is expanding. This discovery caused a profound revolution in our in view of the universe, and understanding the source of its expansion is arguably the most dominant question in cosmology today.

In recent years, the standard cosmological model that emerges as a particular solution of the Einstein field equations has dramatically improved our understanding of the universe. It is built on several fundamental assumptions and principles; among them the so-called cosmological principle is particularly important. It states that our current universe is homogeneous and isotropic, that is, there is neither a preferred place nor direction, at least approximately on large scales. Although this mode has been successful in explaining the majority of the current observations, such as the current expansion of the universe and the spectrum of the cosmic microwave background [24] it lack of the fundamental justification. In fact, the theory of quantum mechanics [30] states that the early universe (just after the universe was born in the big bang) should have been extremely inhomogeneous and anisotropic due to quantum fluctuations that yield the creation of the fundamental particles and eventually

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to the formation of different kinds of matter distributions such as galaxies, stars and planets. Hence, fundamental questions a rise about how much primordial homogenities evolved and why they are essential absent in the present universe on large scales that any consistent cosmological model must be able to answer.

One way to understand this is by the assumption of an extremely short, but particularly violent, phase of expansion just after the big bang, called inflation. This basic idea is that during this phase initial inhomogenities are effectively smoothed out, and from the point of view of any local observer, the universe rapidly becomes essentially homogeneous and isotropic. However, there is theoretical evidence, [4], [20] that some assumptions of this model lead to significant drawbacks as important phenomena are ignored. In fact, it is conceivable that some homogeneities caused by quantum fluctuations would trigger the formation of primordial black holes [5] during the evolution of the universe. Therefore, if such phenomenon was better understood and taken into account, then we could obtain more general and realistic cosmological models.

The gravitational evolution of celestial bodies may be modeled by the Einstein field equations. These are a system of ten highly coupled partial differential equations expressing an equivalence between matter and geometry. The equations are extremely diffucult to solve in general and so simpler cases have to be treated to gain an understanding into how certain types of matter behave under the influence of the gravitational field. For example, the most studied configuration of a matter distribution is that of a static spherically symmetric perfect fluid.

This paper is concerned with certain investigations in general relativity by the coordinate free method of differential geometry. In this method of study, spacetime of general relativity is regarded as a connected four dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-,+,+,+)$. The geometry of the Lorentzian manifold begins with the study of the casual character of vectors of the manifold. It is due to this causality that the Lorentzian manifold becomes a convenient choice for the study of general relativity. The equations imply that the energy-momentum tensor is of vanishing divergence [31]. This requirement is satisfied if the energy-momentum tensor is covariant-constant [11]. In Ref [11], M.C. Chaki and S. Roy showed that a general relativistic spacetime with covariant-constant energy-momentum tensor is Ricci symmetric, that is, $\nabla S = 0$ where S is the Ricci tensor of the spacetime. Many authors have been studied spacetimes with special properties such as spacetimes with semisymmetric energy momentum tensor by De and Velimirovic [18], M-projectively flat spacetime by Zengin [40], pseudo Z symmetric spacetime by Mantica and Suh [28],[29], generalized quasi-Einstein spacetimes by Güler and Demirbag [21], a spacetime with pseudo-projective curvature tensor by Mallick, Suh and De, [27], on generalized Ricci recurrent manifolds with applications to relativity by Mallick, De and De [26], generalized Robertson-Walker spacetimes by Arslan et al,[3] and many other.

2 Preliminaries

As is well known, symmetric spaces play an important role in differential geometry. The study of Riemannian symmetric metric spaces was initiated in the last twenties by Cartan [6], who, in particular, obtained a classification of those spaces.

Let $(M^n, g)(n \equiv dimM)$ be a Riemannian manifold, i.e., a manifold M with Riemannian metric g and ∇ be the Levi-Civita connection of (M^n, g). A Riemannian manifold is called locally symmetric, [6] if $\nabla R = 0$ where R is Riemannian curvature tensor of (M^n, g). This connection of local symmetric is equivalent to the fact that energy point $\rho \epsilon M$, the local geodesic symmetry $F(\rho)$ is an isometry, [31]. The class of Riemannian symmetric manifolds is very natural generalized of the class of manifolds of constant curvature.

A Riemannian manifold (M^n, g) $(n \geq 3)$ is called semi-symmetric if $R.R = 0$ holds on M. It is well-known that the class of semi-symmetric manifolds includes the set of locally symmetric manifolds ($\nabla R = 0$) as a proper subset. A fundamental study on such Riemannian manifolds was made by Szabó [35]-[37] and Kowalski [25].

During the last five decades the notion of locally symmetric manifolds have been weakened by many authors in several ways to a different extent such as conformally symmetric manifolds by Chaki and Gupta, [9], recurrent manifolds introduced by Walker [41], conformally recurrent manifolds by Adati and Miyazawa, [1], pseudo-Riemannian manifolds with recurrent concircular curvature tensor by Olszak and Olszak, [32], pseudo-symmetric manifolds introduced by Chaki, [7], weakly symmetric manifolds by Tamassy and Binh, [39], De and Bandyopadhyay [13], projectively symmetric manifolds by Soos [34] and generalized conharmonically recurrent manifold by De and Kamilya, [15], etc. In the paper [11], Chaki and Roy had shown that a general relativistic spacetime with covariant-constant energy-momentum tensor is Ricci symmetric, that is, $\nabla S = 0$ where S is the Ricci tensor of spacetime. If however $\nabla S \neq 0$, then such a spacetime may be called pseudo Ricci symmetric. It can be said that the Ricci symmetric condition is only a special case of the pseudo Ricci symmetric condition. It is, therefore, meaningful to study the properties of pseudo Ricci symmetric spacetimes in general relativity.

Let Q be the symmetric endomorphism corresponding to the Ricci tensor as indicated below

$$S(X, Y) = g(QX, Y)$$

for all vector fields X and Y.

A non-flat Riemannian manifold is called pseudo Ricci symmetric and denoted by $(PRS)_n$ if the Ricci tensor S of type $(0, 2)$ of the manifold is non-zero and satisfies the condition

$$(\nabla_Z S)(X, Y) = 2A(Z)S(X, Y) + A(X)S(Y, Z) + A(Y)S(X, Z) \tag{2.1}$$

where ∇ denotes the Levi-Civita connection and A is a non-zero $1-$form such that

$$g(X, \rho) = A(X) \tag{2.2}$$

for all X, ρ being the vector field corresponding to the associated $1-$form A. If in (2.1), the $1-$form $A = 0$, then the manifold reduces to Ricci symmetric manifold or covariantly constant

$$(\nabla_Z S)(X, Y) = 0. \tag{2.3}$$

The notion of pseudo Ricci symmetry is different from that of R. Deszcz [19].

Since pseudo Ricci symmetric manifolds have some importance in general theory of relativity, by this motivation, Chaki and Kawaguchi [10] generalized pseudo Ricci symmetric manifold and introduced the notion of almost pseudo Ricci symmetric manifold as

$$(\nabla_Z S)(X,Y) = [A(Z)+B(Z)]S(X,Y) + A(X)S(Y,Z) + A(Y)S(X,Z) \tag{2.4}$$

where A and B are two non-zero $1-$forms and ∇ denotes the operator of the covariant differentiation with respect to the metric g. Let $g(X,\rho) = A(X)$ and $g(X,Q) = B(X)$ for all X. Then ρ, Q are called basic vector fields of the manifold corresponding to the associated $1-$forms A and B, respectively. Such a manifold is denoted by $A(PRS)_n$.

If $B = A$, then the equation (2.4) reduces to (2.1), that is, $A(PRS)_n$ reduces to a pseudo Ricci symmetric manifold [8]. In 1993, Tamassy and Binh [38] introduced the notion of weakly Ricci symmetric manifold which is the generalization of pseudo Ricci symmetric manifold in the sense of Chaki. It may be mentioned that an $A(PRS)_n$ is not a particular case of a weakly Ricci symmetric manifold introduced by Tamassy and Binh [38].

Almost pseudo Ricci symmetric manifolds on some structures have been studied by many authors such as De and Gazi [14], Shaikh, Hui and Bagewadi [33], De, Özgür and De [12], Hui and Özen Zengin [22], De and Mallick [16], De and Pal [17], Kırık and Özen Zengin [23], and many others.

The present paper deals with almost pseudo Ricci symmetric spacetimes. Our aim is to study the properties of this spacetime assuming this spacetime to be a perfect fluid, a radiation fluid or a pressureless fluid (a dust).

3 Perfect fluid spacetimes

It is know that Einstein field equation with cosmological constant can be written as

$$S(X,Y) - \frac{1}{2}rg(X,Y) = \kappa T(X,Y), \tag{3.1}$$

where κ is the gravitational constant, T is the energy momentum tensor for perfect fluid given by

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y) \tag{3.2}$$

with σ is the energy density and p is the isotropic pressure of the fluid, respectively and μ is given by $g(X,\rho)$=$A(X)$ for all X, ρ is the flow vector field of the fluid.

Using (3.2), we can express (3.1) as follows

$$S(X,Y) = \kappa(\sigma + p)A(X)A(Y) + (\kappa p + \frac{r}{2})g(X,Y). \tag{3.3}$$

Contracting (3.3) over X and Y, then we have

$$r = \kappa(\sigma - 3p). \tag{3.4}$$

From (3.3) and (3.4),

$$S(X,Y) = \kappa(\sigma + p)A(X)A(Y) + \frac{\kappa}{2}(\sigma - p)g(X,Y). \tag{3.5}$$

4 Almost pseudo Ricci symmetric spacetimes

If we differentiate the equation (3.5), we get

$$\begin{aligned}(\nabla_Z S)(X,Y) =&\kappa(d\sigma + dp)A(X)A(Y) + \kappa(\sigma + p)[(\nabla_Z A)(X)A(Y) \\ &+ A(X)(\nabla_Z A)(Y)] + \frac{\kappa}{2}(d\sigma - dp)g(X,Y).\end{aligned} \tag{4.1}$$

Comparing (2.4) with (4.1) and using (3.5), then we get for $\kappa \neq 0$

$$\begin{aligned}&[A(Z) + B(Z)][(\sigma + p)A(X)A(Y) + \frac{1}{2}(\sigma - p)g(X,Y)] \\ &+ 2(\sigma + p)A(X)A(Y)A(Z) + \frac{1}{2}(\sigma - p)[A(X)g(Y,Z) \\ &+ A(Y)g(X,Z)] = (d\sigma + dp)A(X)A(Y) \\ &+ (\sigma + p)[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)] \\ &+ \frac{1}{2}(d\sigma - dp)g(X,Y).\end{aligned} \tag{4.2}$$

Contracting (4.2) over X and Y, we find

$$-6pA(Z) + (\sigma - 3p)B(Z) = d\sigma - 3dp. \tag{4.3}$$

Again if we contract (4.3) over Z, it can be obtained that

$$6p = d\sigma(\rho) - 3dp(\rho). \tag{4.4}$$

On the other hand, similarly, contracting (4.2) over X and Y,

$$\frac{3}{2}A(Z)(\sigma + 3p) + \frac{1}{2}B(Z)(\sigma + 3p) = \frac{1}{2}(d\sigma + 3dp). \tag{4.5}$$

And, contracting (4.5) over Z, we get

$$-3(\sigma + 3p) = d\sigma(\rho) + 3dp(\rho) \tag{4.6}$$

Finally, from (4.4) and (4.6), we find

$$d\sigma(\rho) = -\frac{3}{2}(\sigma + p). \tag{4.7}$$

If the derivative of σ is orthogonal to the vector field generated by the 1−form A then we have the following theorem:

Theorem 4.1. *In a perfect fluid $A(PRS)_4$ spacetime satisfying Einstein field equation without cosmological constant, a necessary and sufficient condition the vector field generated by the energy density be orthogonal to the vector field generated by the 1-form A is that it must be $\sigma + p = 0$.*

Assume that $\sigma + p = 0$ in (4.17), (3.5) reduces to

$$S(X, Y) = \kappa\sigma g(X, Y). \tag{4.8}$$

With the help of (3.4) and (4.8), we find $S(X, Y) = \frac{r}{4} g(X, Y)$.
Hence, we can state the following theorem:

Theorem 4.2. *In a perfect fluid $A(PRS)_4$ spacetime, if the vector field generated by the energy density is orthogonal to the vector field generated by the $1-$form A then this spacetime reduces to an Einstein space.*

In this case, considering the equations (4.4) and (4.7), we obtain

$$dp(\rho) = -\frac{1}{2}(\sigma + 5p). \tag{4.9}$$

Thus, we have the following theorem:

Theorem 4.3. *In a perfect fluid $A(PRS)_4$ spacetime satisfying Einstein field equation without cosmological constant, a necessary and sufficient condition the vector field generated by the isotropic pressure be orthogonal to the vector field generated by the $1-$form A is that it must be $\sigma = -5p$.*

With the help of (3.5) and (4.9) we find

$$S(X, Y) = \frac{4\kappa\sigma}{5} A(X)A(Y) + \frac{3\kappa\sigma}{5} g(X, Y). \tag{4.10}$$

This shows that our spacetime is an quasi-Einstein.
Hence, we have the following theorem:

Theorem 4.4. *In a perfect fluid $A(PRS)_4$ spacetime satisfying Einstein field equation without cosmological constant, if the vector field generated by the isotropic pressure is orthogonal to the vector field generated by the $1-$form A then this spacetime reduces to a quasi-Einstein as in the form (4.10).*

Now, contracting (3.5) over X, then we obtain

$$A(QY) = -\frac{\kappa}{2}(\sigma + 3p)A(Y). \tag{4.11}$$

If we take the covariant derivative of (4.11), we find

$$(\nabla_Z A)(QY) = -\frac{\kappa}{2}(d\sigma + 3dp)A(Y) - -\frac{\kappa}{2}(\sigma + 3p)(\nabla_Z A). \tag{4.12}$$

Using (2.4), (3.5) and (4.11) in (4.12), it can be found that

$$-2(\sigma+2p)A(Y)A(Z)-\frac{1}{2}(\sigma+3p)A(Y)B(Z)-\frac{1}{2}(\sigma-p)g(Y,Z) + (\sigma+p)(\nabla_Z A)(Y)+\frac{1}{2}(\sigma+3p)A(Y)=0. \tag{4.13}$$

Then contracting over Y and Z in (4.13), we find

$$6p+(\sigma+p)divA+\frac{1}{2}[d\sigma(\rho)+3d(\rho)]=0. \tag{4.14}$$

From (4.6) and (4.14), it can be easily seen that

$$divA=\frac{3(\sigma-p)}{2(\sigma+p)}. \tag{4.15}$$

Thus, we have the following theorem :

Theorem 4.5. *In a perfect fluid $A(PRS)_4$ spacetime satisfying Einstein field equation without cosmological constant, the divergence of the vector field generated by the $1-$form A is related by the energy density and the isotropic pressure as in (4.15).*

If the vector field generated by the $1-$form A is divergence$-$free then by the aid of (4.15), it must be $\sigma=p$. In this case, we have from (3.5),

$$S(X,Y)=2\kappa\sigma A(X)A(Y).$$

In this case, we can say that our spacetime is $\eta-$Einstein spacetime where $\eta=2\kappa\sigma$.

In global transcription, a vectorfield ψ in a Riemannian manifold M is called torse-forming if it satisfies

$$\nabla_X\psi=\varrho X+\alpha(X)\psi \tag{4.16}$$

where $X\in TM$, α is a linear form and ϱ is a scalar function. In local transcription, this reads

$$\psi^h{}_{,i}=\varrho\delta^h{}_i+\psi^h\alpha_i \tag{4.17}$$

where ψ^h and α_i are the components of the vector fields generated by ψ and α and $\delta^h{}_i$ is Kronecker symbol [1].

Now, we assume that the vector field generated by the $1-$form $A(X)$ is a torse-forming vector field, from (4.16)

$$(\nabla_Z A)(X)=\lambda(Z)A(X)+\beta g(X,Z). \tag{4.18}$$

where λ is a linear form and β is a scalar function.

If we contract (4.18) over X, we get

$$\lambda(Z) = \beta A(Z). \tag{4.19}$$

Comparing (4.1) and (4.19), it is concluded that

$$\begin{aligned}(\nabla_Z S)(X,Y) = \kappa(d\sigma + dp)A(X)A(Y) &+ 2\kappa\beta(\sigma+p)A(X)A(Y)A(Z)\\ &+ \frac{\kappa}{2}(d\sigma - dp)g(X,Y)\\ &+ \kappa\beta(\sigma+p)[A(Y)g(X,Z) + A(X)g(Y,Z)].\end{aligned} \tag{4.20}$$

By putting (4.20) in (2.4), we find

$$\begin{aligned}\kappa(d\sigma + dp)A(X)A(Y) + 2\kappa\beta(\sigma+p)A(X)A(Y)A(Z) + \frac{\kappa}{2}(d\sigma - dp)g(X,Y)\\ +\kappa\beta(\sigma+p)[A(Y)g(X,Z) + A(X)g(Y,Z)] = [A(Z) + B(Z)]S(X,Y)\\ +A(X)S(Y,Z) + A(Y)S(X,Z).\end{aligned} \tag{4.21}$$

Since $\kappa \neq 0$, from (3.5) and (4.21),

$$\begin{aligned}(d\sigma + dp)A(X)A(Y) + \frac{1}{2}(d\sigma - dp)g(X,Y) = (\sigma+p)[A(Z) + B(Z)]A(X)A(Y)\\ +\frac{1}{2}(\sigma - p)[A(Z) + B(Z)]g(X,Y) + \frac{1}{2}(\sigma - p)[A(Y)g(X,Z) + A(Z)g(X,Y)]+\\ \kappa\beta(\sigma+p)[A(Y)g(X,Z) + A(X)g(Y,Z)].\end{aligned} \tag{4.22}$$

Now, contracting (4.22) over X and Y,

$$d\sigma - 3dp = -6pA(Z) + (\sigma - 3p)B(Z). \tag{4.23}$$

Again, contracting (4.23) over Z,

$$d\sigma(\rho) - 3dp(\rho) = 6p. \tag{4.24}$$

On the other hand, contracting (4.22) over Z, we get

$$d\sigma(\rho) + 3dp(\rho) = -3(\sigma + 3p). \tag{4.25}$$

In this case, from (4.24) and (4.25), we obtain that

$$d\sigma(\rho) = -\frac{3}{2}(\sigma + p)$$

If we assume that the vector field generated by σ is orthogonal to the torse-forming vector field generated by A then it must be $\sigma + p = 0$.

Thus, we have the following theorem:

Theorem 4.6. *Let the vector field generated by A of a perfect fluid $A(PRS)_4$ spacetime satisfying Einstein field equation without cosmological constant be a torse-forming vector field. If the vector field generated by the energy density is orthogonal to the torse-forming vector field then this spacetime reduces to an Einstein space.*

5 A Pressureless Fluid $A(PRS)_4$ Spacetimes

Assuming that our spacetime is a pressureless fluid spacetime (a dust), the energy momentum tensor is the form

$$T(X,Y) = \sigma A(X)A(Y). \tag{5.1}$$

In this case, from (3.3) and (5.1), we obtain

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa\sigma A(X)A(Y). \tag{5.2}$$

Now, contracting (5.2) over X and Y, it can be found that

$$r = \kappa\sigma. \tag{5.3}$$

In this case, from (5.2) and (5.3) , we get

$$S(X,Y) = \kappa\sigma[A(X)A(Y) + \frac{1}{2}g(X,Y)]. \tag{5.4}$$

Taking the covariant derivative of (5.4), one can easily bring out the following

$$(\nabla_Z S)(X,Y) = \kappa d\sigma[A(X)A(Y) + \frac{1}{2}g(X,Y)] + \kappa\sigma[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)]. \tag{5.5}$$

In consequence of (2.4) it follows from (5.5) that

$$[A(Z)+B(Z)]S(X,Y) + A(X)S(Y,Z) + A(Y)S(X,Z) = \kappa d\sigma[A(X)A(Y) + \frac{1}{2}g(X,Y)] + \kappa\sigma[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)]. \tag{5.6}$$

From (5.4) and (5.6), we find

$$\kappa\sigma[A(Z)+B(Z)][A(X)A(Y) + \frac{1}{2}g(X,Y)] + \kappa\sigma A(X)[A(Y)A(Z) + \frac{1}{2}g(Y,Z)] + \kappa\sigma A(Y)[A(X)A(Z) + \frac{1}{2}g(X,Z)] = \kappa(d\sigma)[A(X)A(Y) + \frac{1}{2}g(X,Y)] + \kappa\sigma[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)]. \tag{5.7}$$

After that, contracting (5.7) over X and Y,

$$\sigma B(Z) = d\sigma \tag{5.8}$$

which yields by virtue of (5.8) that

$$d\sigma(\rho) = 0. \tag{5.9}$$

If the vector field generated by σ is orthogonal to the vector field associated by the $1-$form A then from (4.6) and (5.9),

$$\sigma = 0. \tag{5.10}$$

Thus, with the help of (5.1) and (5.10), we conclude that

$$T(X, Y) = 0.$$

In this case, the spacetime is devoid of the matter. Hence, we can get the following theorem:

Theorem 5.1. *An $A(PRS)_4$ dust fluid spacetime satisfying Einstein field equation without cosmological constant is vacuum.*

6 A Radiation Fluid $A(PRS)_4$ Spacetimes

Now, we assume that our spacetime is a radiation fluid, then we have

$$T(X, Y) = p[4A(X)A(Y) + g(X, Y)]. \tag{6.1}$$

In this case, from (3.1) and (6.1), we find

$$S(X, Y) - \frac{r}{2} = \kappa p[4A(X)A(Y) + g(X, Y)]. \tag{6.2}$$

Now, contracting (6.2) over X and Y, we get

$$r = 0. \tag{6.3}$$

Thus, by the aid of (6.2) and (6.3), we obtain

$$S(X, Y) = \kappa p[4A(X)A(Y) + g(X, Y)]. \tag{6.4}$$

If we take the covariant derivative of (6.4), then we find

$$\begin{aligned}(\nabla_Z S)(X, Y) = \kappa dp[4A(X)A(Y) + g(X, Y)] + 4\kappa p[(\nabla_Z A)(X)A(Y) \\ + A(X)(\nabla_Z A)(Y)].\end{aligned} \tag{6.5}$$

In consequence of (2.4) it follows from (6.5) that

$$\begin{aligned}[A(Z) + B(Z)]S(X, Y) + A(X)S(Y, Z) + A(Y)S(X, Z) = \\ \kappa[dp(4A(X)A(Y) + g(X, Y)) \\ + 4p((\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y))].\end{aligned} \tag{6.6}$$

Contracting (6.6) over X and Y, it can be easily seen that

$$[A(Z) + B(Z)]r + 2A(QZ) = 0. \tag{6.7}$$

By the aid of (6.3) and (6.4) in (6.7), we obtain

$$-6\kappa p A(Z) = 0 \tag{6.8}$$

which leads to $p = 0$.

In this case, we get from, (6.1) $T(X, Y) = 0$. Thus we can state the following theorem:

Theorem 6.1. *An $A(PRS)_4$ radiation fluid spacetime satisfying Einstein field equation without cosmological constant is vacuum.*

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Authors' addresses:

Ecem Bektaş
Department of Mathematics, Faculty of Science and Letters,
Istanbul Technical University, Istanbul, Turkey.
E-mail: bektasecem@gmail.com

Füsun Özen Zengin
Department of Mathematics, Faculty of Science and Letters,
Istanbul Technical University, Istanbul, Turkey.
E-mail: fozen@tu.edu.tr