Dynamical processes. Stability and chaos

M.-L. Bucur

Abstract. In this work we are studying some dynamical processes. The dynamical processes represent the natural generalization of the dynamical systems. A discrete time dynamical process having the generators $(f_n)_{n\in\mathbb{N}}$ is given by the difference equation $x_{n+1} = f_n(x_n)$. The discrete dynamical systems are particular cases of dynamical processes corresponding to a constant sequence of generators, i.e. $f(x_n)$.

The asymptotic behavior, the stability, the sensitivity of some dynamical processes is analysed in this paper using the pre-equilibrium points and we will calculate the Lyapunov exponents.

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1 Introduction

Most of the processes actually proceed from the discrete dynamical systems which, in their turn, proceed from the discretisation of the systems in continuous time. The approximations made by the computer to every step lead to the idea that, actually, following the discretisation of a continuous system we get rather a process than a dynamical system. It is the case of the discretisation made by variable step.

Then, if a discrete dynamical system is generating by a function $f : X \to X$, a dynamical processes is given by a functions sequence $(f_n)_{n\in\mathbb{N}}$, $f_n : X \to X$ for all $n \in \mathbb{N}$, with the recurrence

$$x_{n+1} = f_n(x_{n)}, \text{ for all } n \in \mathbb{N}.$$

In many case (for example variable step numerical methods, feed-back control schemes) it is appropriate to use dynamical processes instead dynamical systems. Many notions and results coming from the classical theory of dynamical systems can be extended for dynamical processes.

2 Study of some dynamical processes

Consider the dynamical process

$$x_{n+1} = f_n(x_n) \text{ for all } n \in \mathbb{N}, \tag{2.1}$$

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where $f_n : X \to X$ and (X, d) is a metric space.

Definition 2.1. Let $\left(X,\, (f_n)_{n\in\mathbb{N}}\right)$ be a dynamical process. We call **orbit of a point** $x_0 \in X$ the set:

$$O(x_0) = \{x_0,\ f_0(x_0),\ f_1 \circ f_0\,(x_0)\,,\ ..., f_n \circ f_{n-1} \circ ...f_0\,(x_0)\,, ...\}.$$

Definition 2.2. A point $p \in X$ is called **fixed point** for the dynamical process (2.1) iff $f_n(p) = p$ for all $n \in \mathbb{N}$.

Definition 2.3. A point $p \in X$ is called **pre-equilibrium point** for the dynamical process (2.1) iff

$$p = \lim_{n\to\infty} f_n \circ f_{n-1} \circ ...f_0\,(p)\,.$$

The **attraction basin** for a pre-equilibrium point p is the set

$$B(p) = \left\{x \in X \mid \lim_{n\to\infty} f_n \circ f_{n-1} \circ ...f_0\,(x) = p\right\}.$$

Definition 2.4. The system (2.1) is said to be **sensitive at a point** $x_0 \in X$ if there exists a constant $\delta > 0$ such that for any neighborhood $U \in V(x_0)$, there exists $y_0 \in U$ and $N \in \mathbb{N}$ such that $d(f_N \circ f_{N-1} \circ ...f_0\,(y_0)\,,\, f_N \circ f_{N-1} \circ ...f_0\,(x_0)) > \delta$.

Definition 2.5. The system (2.1) is said to be **stable at a point** $x_0 \in X$ if for any $\varepsilon > 0$ there exists a constant $\delta > 0$ such that for any $y_0 \in X$ with $d(y_0, x_0) < \delta$ and any $n \in \mathbb{N}$ we have $d(f_n \circ f_{n-1} \circ ...f_0\,(y_0)\,,\, f_n \circ f_{n-1} \circ ...f_0\,(x_0)) < \varepsilon$.

Definition 2.6. If X is a closed interval and f_n are C^1 map for any $n \in \mathbb{N}$, we define the **Lyapunov exponent** of the system (2.1) by:

$$\lambda(x_0) = \limsup_{n\to\infty} \frac{1}{n} \ln\left|(f_n \circ f_{n-1} \circ ...f_0)'\,(x_0)\right| = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln\left|f_k'\,(x_k)\right|,$$

where $(x_k)_{k=\overline{0,\infty}}$ is the orbit of system (2.1) becoming from x_0.

We will exemplify with some processes provided by the logistic map. First we consider

$$x_{n+1} = c_n x_n^2\,(1 - x_n)\,,\ \text{for all } n \in \mathbb{N} \tag{2.2}$$

so

$$f_n\,(x) = c_n x_n^2\,(1 - x_n)\,,\ \text{ for all } n \in \mathbb{N}.$$

The graphics of these functions are represented in the following figure:

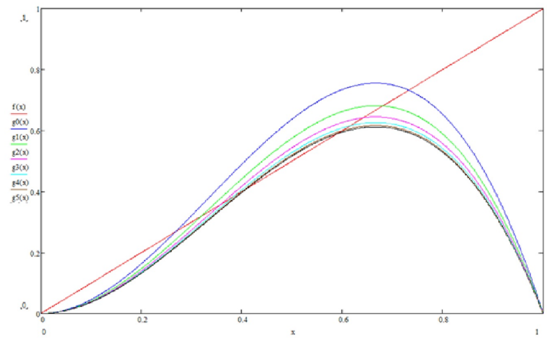
Using the Mathcad software we get the following values:

For $c_n = 4.1 + \frac{1}{2^n}$:

If $x_0 = 0.2$, $x_5 = 0.0001924$, $x_7 = 9.626 \cdot 10^{-14}$, $x_{100} = 0$.

If $x_0 = 0.3$, $x_5 = 0.22804912$, $x_{10} = 7.787 \cdot 10^{-5}$, $x_{12} = 2.535 \cdot 10^{-15}$.

If $x_0 = 0.3333$, $x_5 = 0.4567808$, $x_{10} = 0.51744303$, $x_{20} = 0.57656781$, $x_{100} = 0.57808688$.



If $x_0 = 0.5780868809$, $x_5 = 0.60937378$, $x_{10} = 0.58216971$, $x_{20} = 0.5781341$, $x_{100} = 0.57808688$.

If $x_0 = 0.91$, $x_5 = 0.46433097$, $x_{10} = 0.52854822$, $x_{20} = 0.57708157$, $x_{100} = 0.57808688$.

If $x_0 = 0.92$, $x_5 = 0.33117813$, $x_{10} = 0.7183195$, $x_{14} = 3.973 \cdot 10^{-10}$, $x_{20} = 0$.

For $c_n = 5.1 + 1/2^n$:

If $x_0 = 0.732210118$, $x_{10} = 0.73716048$, $x_{100} = 0.73221018$

If $x_0 = 0.33$,$x_{10} = 0.72587071$, $x_{100} = 0.73221018$

All these results can be summarized as follows:

Proposition 2.1. *If we analyze the process* (2.2) *where the sequence* $(c_n)_n$ *decrease to* $c \in (0, 4)$*, then, we have a unique fixed point* $a_0 = 0$ *and* $B(a_0) = [0, 1]$.

If $(c_n)_n$ *decrease to* $c \in (4, \frac{9}{2})$*, then, we have also a unique fixed point* $a_0 = 0$ *and appear a pre-equilibrium point* $p = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4}{c}}$*, which is not a fixed point for the process. In this case the fixed point keep his attractivity and* $B(a_0) = \left[0, p_0^{'}\right] \bigcup \left[q_0^{'}, 1\right]$*, where* $p_0^{'} = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4}{c_0}}$ *and* $f_0\left(q_0^{'}\right) = p_0^{'}$, $p_0^{'} < q_0^{'}$.

Proof. A graphical analysis prove that for $(c_n)_n$ decrease to $c \in (0, 4)$, $f_n(x) \leq x$, for all $x \in [0, 1]$, with equality only for $x = 0$.

$f_{n+1}(x) \leq f_n(x)$ for all $x \in [0, 1]$ and for all $n \in \mathbb{N}$.

So, for any $x_0 \in [0, 1]$, the sequence $(x_n)_n$ will be decreasing, $x_{n+1} = f_n \circ f_{n-1} \circ ...f_0(x_0)$.

But $x_n \in [0, 1]$ for all $n \in \mathbb{N}$, so $(x_n)_n$ will converge to the only fixed point $a_0 = 0$.

Suppose that the sequence $(c_n)_n$ decreases to $c \in (4, \frac{9}{2})$. We may assume that $c_n \in (4, \frac{9}{2})$.

The functions f_n both admit a maximum $\frac{2}{3}$ and three fixed points for all $n \in \mathbb{N}$: $a_0 = 0$, $p_n^{'} = \frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4}{c_n}}$ and $p_n = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4}{c_n}}$ and:

- If $x \in \left[0, p_n^{'}\right] \bigcup [p_n, 1]$ we have $f_n(x) \leq x$.
- If $x \in (p_n^{'}, p_n)$ we have $f_n(x) > x$.

Consider $y_0 = p = \frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{4}{c}}$, $y_{n+1} = f_n \circ f_{n-1} \circ ... f_0\,(y_0)$ and $p_n = \frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{4}{c_n}}$.
Because $(c_n)_n$ decrease to c, we have for all $n \in \mathbb{N}$:

$p'_n = \frac{1}{2} - \frac{1}{2}\sqrt{1-\frac{4}{c_n}} < y_0 = \frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{4}{c}} < \frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{4}{c_n}} < \frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{4}{c_0}}$

$y_0 \in (p_0^{'},\, p_0)$, so $f_0\,(y_0) > y_0$, that means $y_0 < y_1$.

The maximum value for f_n is $\frac{4c_n}{27} < \frac{2}{3}$, so $y_n < \frac{2}{3}$, for all $n \in \mathbb{N}$.

We can prove that $y_n > p_n^{'}$ for all $n \in \mathbb{N}$:

$y_0 > p_n^{'}$, so $f_0(y_0) > f_0(p_n^{'})$ because all functions f_n are increasing on the interval $(0, \frac{2}{3})$ and $p_n^{'} < y_0 < \frac{2}{3}$. Because $p_n^{'} \in (p_0^{'},\, p_0)$ we have $f_0(p_n^{'}) > p_n^{'}$, i.e., $y_1 > p_n^{'}$.

In a similar way, $y_2 > p_n^{'}, \ldots y_n > p_n^{'}$.

1. y$_n \in (p_n^{'},\, p_n)$ for all $n \in \mathbb{N}$, so $f_n(y_n) > y_n$, which means $y_n < y_{n+1}$ for all $n \in \mathbb{N}$. So, $(y_n)_n$ is increasing and $y_n \in [0, 1]$. It is also bounded, hence it has a limit y.

   Because $y_{n+1} = f_n(y_n)$ we obtain $y = f(y)$, where $f(x) = cx^2(1-x)$ (Dini's Lemma).

   So $y \in \left\{0,\ \frac{1}{2} - \frac{1}{2}\sqrt{1-\frac{4}{c}},\ \frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{4}{c}}\right\}$ and $(y_n)_n$ is increasing and positive, i.e.,

$$\lim_{n\to\infty} y_n = \frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{4}{c}}.$$

   Because $\frac{1}{2} - \frac{1}{2}\sqrt{1-\frac{4}{c}} < \frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{4}{c}} = y_0 < ... < y_n < ...$

2. There is N natural such that $y_N \in \left[0,\ p_N^{'}\right] \bigcup\, [p_N,\ 1]$. But $y_N > p_N^{'}$ so $y_N > p_N$.

   $f_N\,(y_N) > f_N\,(p_N) = p_N$, so $y_{N+1} > p_N > p_{N+1}$.

   So $y_n > p_n$ for all $n \in \mathbb{N}$.

   We get $f_n(y_n) \le y_n$, so $y_{n+1} \le y_n$.

   Therefore $(y_n)_n$ will be decreasing and it is also bounded, so it admits a limit y.

   Also in this case we obtain $\lim\limits_{n\to\infty} y_n = \frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{4}{c}}$, because $y_n > p_n > \frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{4}{c}}$ for all $n \in \mathbb{N}$.

   To prove the last part, we consider $y_0 \in [q'_0, 1]$. But f_0 is decreasing on this interval, so

   $f_0(y_0) \le f_0(q'_0) = p'_0 < p'_1$

   Hence $y_1 \le p'_1$, that means $f_n(y_1) \le y_1$, $y_2 \le y_1$.

   Similarly, $y_{n+1} \le y_n \le p'_0$, so $(y_n)_n$ is decreasing and bounded, so has it admits a limit $y \in [0, p'_0]$. So, $y = 0$.

□

For this process we can find the bf Lyapunov exponent:

$$\lambda(x_0) = \limsup_{n\to\infty} \frac{1}{n} \ln \left| (f_n \circ f_{n-1} \circ ...f_0)' (x_0) \right| = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln \left| f_k' (x_k) \right|,$$

where $(x_k)_{k=0}^{\infty}$ is the orbit of system (2.2) starting with x_0.

Let $y_0 = p = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4}{c}}$ be the pre-equilibrium point.

$$\lambda(y_0) = \limsup_{n\to\infty} \frac{1}{n} \ln \left| (f_n \circ f_{n-1} \circ ...f_0)' (y_0) \right| = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln \left| f_k' (y_k) \right|,$$

where $(y_k)_{k=0}^{\infty}$ is the orbit of system (2.2) starting with y_0.

We already proved that $(y_k)_{k=0}^{\infty}$ converges to $p = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4}{c}}$, so

$$\limsup_{n\to\infty} \frac{1}{n} \ln \left| (f_n \circ f_{n-1} \circ ...f_0)' (y_0) \right| = \limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln \left| f_k' (y_k) \right| =$$
$$\ln(p \cdot c \cdot (2p - 3p^2)) = \ln 3.$$

Using Theorem 3.1 from [2], since $\lambda(y_0) = ln3 > 0$, the process will be sensitive at y_0.

Remark 2.7.

1. We can also prove that for $c \in (\frac{9}{2}, 6)$ the point $p = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4}{c}}$ is still a pre-equilibrium point.
2. All of the above prove that 4 is a bifurcation point because when c become grater than 4, appear a pre-equilibrium point.

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Author's address:

Maria-Liliana Bucur
University of Craiova, Department of Applied Mathematics,
13 "Al. I. Cuza" Str., RO-200585 Craiova, Romania.
E-mail: lilianabucur@yahoo.com