Generalized Aminov surfaces given by a Monge patch in the Euclidean four space

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Abstract. A depth surface of $\mathbb{E}^3$ is a range image observed from a single view can be represented by a digital graph (Monge patch) surface. That is, a depth or range value at a point (u, v) is given by a single valued function $z = f(u, v)$. In the present study we consider the surfaces in Euclidean 4-space $\mathbb{E}^4$ given with a Monge patch $z = f(u, v) = r(u)\cos(\varphi(u) + v)$, $w = g(u, v) = r(u)\sin(\varphi(u) + v))$. We investigated the curvature properties of these surfaces. We also give some special examples.

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1 Introduction

In recent years there has been a tremendous increase in computer vision research using range images (or depth maps) as sensor input data [5]. The most attractive feature of range images is the explicitness of the surface information. Many industrial and navigational robotic tasks will be more easily accomplished if such explicit depth information can be efficiently obtained and interpreted. Classical differential geometry provides a complete local description of smooth surfaces [6]. The first and second fundamental forms of surfaces provide a set of differential-geometric shape descriptors that capture domain-independent surface information. Gaussian curvature is an intrinsic surface property which refers to an isometric invariant of a surface [6]. Both Gaussian and mean curvatures have attractive characteristics of translational and rotational invariance. A depth surface is a range image observed from a single view can be represented by a digital graph (Monge patch) surface. That is, a depth or range value at a point (u, v) is given by a single-valued function $z(u, v)$. The famous theorem of Bernstein states that the only entire minimal graphs in the Euclidean space $\mathbb{R}^3$ are planes. More precisely, if $z(u, v)$ is an entire (i.e., defined over all of $\mathbb{R}^2$) smooth function whose graph is a minimal surface, then it is an affine function, and the graph is a plane [3]. One interesting class of surfaces in $\mathbb{E}^3$ is that of translation surfaces, which can be parametrized, locally, as $z(u, v) = f(u) + g(v)$, where f and g are smooth functions. From the definition, it is clear that translation surfaces are

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double curved surfaces. Therefore, translation surfaces are made up of a quadrilateral, that is, four-sided, facets. Because of this property, translation surfaces are used in architecture to design and construct free-form glass roofing structures, see [11]. Scherk's surface, obtained by H. Scherk in 1835, is the only non-flat minimal surface, that can be represented as a translation surface [17], [14]. Translation surfaces have been investigated from the various viewpoints by many differential geometers. L. Verstraelen, J. Walrave, and S. Yaprak have investigated minimal translation surfaces in n-dimensional Euclidean spaces [19].

Minimal graphs has been generalized in higher dimension and co dimension under various conditions. See [9], [10], [18] and the references therein for the codimension one case and [13], [21], [22] for the higher codimension case. Recently, in [12] the authors have obtained a Bernstein type result for entire two dimensional minimal graphs $z = f(u,v)$, $w = g(u,v)$ in $\mathbb{R}^4$, which extends a previous result due to L. Ni (see [15]). Moreover, they provide a characterization for complex analytic curves. See also [12].

In [4] we consider Aminov surfaces given with the Monge patch $f(u,v) = r(u)\cos v$, $g(u,v) = r(u)\sin v$ in $\mathbb{E}^4$. We also present some examples of these surfaces. We obtain some equations on $r(u)$, when on M the equation $K + K_N = 0$ has place. Then we obtain the condition for the case M is a Wintgen ideal surface.

This paper is organized as follows: Section 2 gives some basic concepts of the surfaces in $\mathbb{E}^4$. Section 3 tells about the surfaces given with a Monge patch in $\mathbb{E}^4$. Further this section provides some basic properties of surfaces in $\mathbb{E}^4$ and the structure of their curvatures. In the final section we consider *generalized Aminov surfaces* given with the Monge patch

$$\begin{aligned} f(u,v) &= r(u)\cos(\varphi(u)+v), \\ g(u,v) &= r(u)\sin(\varphi(u)+v), \end{aligned} \tag{1.1}$$

for some smooth functions $r(u)$ and $\varphi(u)$. We investigated the curvature properties of these surfaces. We also give some special examples of these surfaces which are first defined in [4] by the present authors.

2 Basic Concepts

Let M be a smooth surface in $\mathbb{E}^4$ given with the patch $X(u,v) : (u,v) \in D \subset \mathbb{E}^2$. The tangent space to M at an arbitrary point $p = X(u,v)$ of M span $\{X_u, X_v\}$. In the chart (u,v) the coefficients of the first fundamental form of M are given by

$$E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle, \tag{2.1}$$

where $\langle , \rangle$ is the Euclidean inner product. We assume that $W^2 = EG - F^2 \neq 0$, i.e. the surface patch $X(u,v)$ is regular. For each $p \in M$, consider the decomposition $T_p\mathbb{E}^4 = T_pM \oplus T_p^{\perp}M$ where $T_p^{\perp}M$ is the orthogonal component of T_pM in $\mathbb{E}^4$. Let $\widetilde{\nabla}$ be the Riemannian connection of $\mathbb{E}^4$. Given any local vector fields X_i, X_j tangent to M.

Let $\chi(M)$ and $\chi^{\perp}(M)$ be the space of the smooth vector fields tangent to M and the space of the smooth vector fields normal to M, respectively. Consider the second fundamental map $h : \chi(M) \times \chi(M) \to \chi^{\perp}(M)$;

$$h(X_i, X_j) = \widetilde{\nabla}_{X_i} X_j - \nabla_{X_i} X_j; \quad 1 \le i, j \le 2, \tag{2.2}$$

where $\widetilde{\nabla}$ is the induced connection. This map is well-defined, symmetric and bilinear.

For any arbitrary orthonormal normal frame field $\{N_1, N_2\}$ of M, recall the shape operator $A : \chi^{\perp}(M) \times \chi(M) \to \chi(M)$;

$$A_{N_i} X_i = -(\widetilde{\nabla}_{X_i} N_i)^T, \quad X_i \in \chi(M). \tag{2.3}$$

This operator is bilinear, self-adjoint and satisfies the following equation:

$$\langle A_{N_k} X_i, X_j \rangle = \langle h(X_i, X_j), N_k \rangle = c_{ij}^k, \quad 1 \le i, j, k \le 2. \tag{2.4}$$

The equation (2.2) is called Gaussian formula, and

$$h(X_i, X_j) = \sum_{k=1}^{2} c_{ij}^k N_k; \quad 1 \le i, j \le 2, \tag{2.5}$$

where c_{ij}^k are the coefficients of the second fundamental form.

Further, the *Gaussian curvature* and *Gaussian torsion* of a regular patch $X(u, v)$ are given by

$$K = \frac{1}{W^2} \sum_{k=1}^{2} (c_{11}^k c_{22}^k - (c_{12}^k)^2), \tag{2.6}$$

and

$$K_N = \frac{1}{W^2} \left(E \left(c_{12}^1 c_{22}^2 - c_{12}^2 c_{22}^1 \right) - F \left(c_{11}^1 c_{22}^1 - c_{11}^2 c_{22}^1 \right) + G \left(c_{11}^1 c_{12}^2 - c_{11}^2 c_{12}^1 \right) \right), \tag{2.7}$$

respectively.

Further, the mean curvature vector of a regular patch $X(u, v)$ is defined by

$$\overrightarrow{H} = \frac{1}{2W^2} \sum_{k=1}^{2} (c_{11}^k G + c_{22}^k E - 2c_{12}^k F) N_k. \tag{2.8}$$

Recall that a surface M is said to be *minimal* if its mean curvature vector vanishes identically [7]. The surface patch $X(u, v)$ is called *pseudo-umbilical* if the shape operator with respect to H is proportional to the identity (see, [7]).

3 Surfaces Given with the Monge Patch in $\mathbb{E}^4$

2-dimensional surfaces in $\mathbb{E}^4$ are an interesting object for investigation of geometers. Here we have some difficult problems which wait for its solutions. For example, it is unknown does there exist an isometric regular immersion of the whole Lobachevsky plane into $\mathbb{E}^4$. Hence the investigation of various classes of surfaces in $\mathbb{E}^4$ with a point of view of the influence of the principal invariants Gauss curvature K, Gauss torsion

K_N and the vector of mean curvature $\overrightarrow{H}$ on the behavior of surfaces is an actual problem.

In the considering work we use the representation of surfaces in the explicit form

$$X(u,v) = (u, v, f(u,v), g(u,v)), \tag{3.1}$$

where f and g are some smooth functions. The parametrization (3.1) is called *Monge patch* in $\mathbb{E}^4$.

In [4] we proved the following result.

Theorem 3.1. *[4] Let M be a smooth surface given with the Monge patch (3.1). Then the mean curvature vector of M becomes*

(3.2)
$$\begin{aligned}\overrightarrow{H} =& \frac{1}{2\sqrt{A}W^2}\{(Gf_{uu} - 2Ff_{uv} + Ef_{vv})N_1 \\ &+ \frac{1}{W}\left(G(-Bf_{uu} + Ag_{uu}) - 2F(-Bf_{uv} + Ag_{uv}) + E(-Bf_{vv} + Ag_{vv})\right)N_2\},\end{aligned}$$

where

$$\begin{aligned} A &= 1 + (f_u)^2 + (f_v)^2, \\ B &= f_u g_u + f_v g_v, \\ C &= 1 + (g_u)^2 + (g_v)^2, \end{aligned} \tag{3.3}$$

such that $EG - F^2 = AC - B^2$.

In [16], R. Osserman has constructed an example of complete minimal surface is given with the Monge patch (3.1) (i.e. a minimal two dimensional graph in $\mathbb{R}^4$).

Example 3.1. M be a smooth surface given with the Monge patch of the form

$$\begin{aligned} f(u,v) &= \frac{1}{2}\left(e^u - 3e^{-u}\right)\cos\frac{v}{2}, \\ g(u,v) &= -\frac{1}{2}\left(e^u - 3e^{-u}\right)\sin\frac{v}{2}, \end{aligned}$$

has vanishing mean curvature.

In [2] Yu. Aminov proved the following result.

Theorem 3.2. *[2] Let M be a smooth surface given with the Monge patch (3.1). Then the Gaussian curvature K and Gaussian torsion K_N of M become*

$$K = \frac{C(f_{uu}f_{vv} - f_{uv}^2) - B(f_{uu}g_{vv} + g_{uu}f_{vv} - 2f_{uv}g_{uv}) + A(g_{uu}g_{vv} - g_{uv}^2)}{W^4}, \tag{3.4}$$

and

$$K_N = \frac{E(f_{uv}g_{vv} - g_{uv}f_{vv}) - F(f_{uu}g_{vv} - g_{uu}f_{vv}) + G(f_{uu}g_{uv} - g_{uu}f_{uv})}{W^4}, \tag{3.5}$$

respectively.

Definition 3.2. A surface M is said to be *Wintgen ideal surface* in $\mathbb{E}^4$ if the equality

$$K + |K_N| = \left\| \overrightarrow{H} \right\|^2, \tag{3.6}$$

holds [20].

Proposition 3.3. *Let M be a smooth surface given with the Monge patch (3.1). If M is Wintgen ideal surface then the equality*

$$\begin{aligned} 0 &= G^2 \left(C f_{uu}^2 - 2B f_{uu} g_{uu} + A g_{uu}^2 \right) + E^2 \left(C f_{vv}^2 - 2B f_{vv} g_{vv} + A g_{vv}^2 \right) \\ &\quad + 2 \left(2F^2 - EG \right) \left(C f_{uu} f_{vv} + A g_{uu} g_{vv} \right) + 2EGB \left(g_{uu} f_{vv} + f_{uu} g_{vv} \right) \\ &\quad - 4EF \left(C f_{vv} f_{uv} + A g_{vv} g_{uv} \right) - 4FG \left(C f_{uu} f_{uv} + A g_{uu} g_{uv} \right) \\ &\quad + 4 \left(F(B - F) + EG \right) \left(G g_{uu} f_{uv} + E g_{uv} f_{vv} - F g_{uu} f_{vv} \right) \\ &\quad + 4 \left(F(B + F) - EG \right) \left(G g_{uv} f_{uu} + E g_{vv} f_{uv} - F f_{uu} g_{vv} \right), \end{aligned}$$

holds.

Proof. By the use of (3.4) and (3.5) with mean curvature of (3.2) and substituting them on the (3.6) we get the result. □

Definition 3.3. The surface given with the parametrization (3.1) by the parametrization

$$f(u,v) = f_3(u) + g_3(v), \quad g(u,v) = f_4(u) + g_4(v), \tag{3.7}$$

is called *translation surface* in Euclidean 4-space $\mathbb{E}^4$ [8].

In the case (3.7) we obtain simple expressions for K, K_N and $\overrightarrow{H}$. As a consequence of Theorem 3.1 and Theorem 3.2 we get the following results.

Corollary 3.4. *[4] Let M be a translation surface given with the Monge patch (3.7). Then the Gaussian curvature K and Gaussian torsion K_N of M becomes*

$$K = \frac{f_3''(u)g_3''(v)C - (f_3''(u)g_4''(v) + f_4''(u)g_3''(v))B + f_4''(u)g_4''(v)A}{W^4},$$

and

$$K_N = \frac{F(f_4''(u)g_3''(v) - f_3''(u)g_4''(v))}{W^4},$$

respectively, where

$$\begin{aligned} E &= 1 + (f_3'(u))^2 + (f_4'(u))^2, \\ F &= f_3'(u)g_3'(v) + f_4'(u)g_4'(v), \\ G &= 1 + (g_3'(v))^2 + (g_4'(v))^2, \end{aligned}$$

and

$$\begin{aligned} A &= 1 + (f_3'(u))^2 + (g_3'(v))^2, \\ B &= f_3'(u)f_4'(u) + g_3'(v)g_4'(v), \\ C &= 1 + (f_4'(u))^2 + (g_4'(v))^2. \end{aligned}$$

Corollary 3.5. *[4] Let M be a translation surface given with the Monge patch (3.7). Then the mean curvature vector of M becomes*

$$\overrightarrow{H} = \frac{f_3''(u)G + g_3''(v)E}{2\sqrt{A}W^2}N_1 + \frac{G(f_4''(u)A - f_3''(u)B) + E(g_4''(v)A - g_3''(v)B)}{2\sqrt{A}W^3}N_2.$$

Example 3.4. The translation surface given with the surface patch of

$$X(u,v) = (u, v, u^2 + v^2, u^2 - v^2),$$

has vanishing Gaussian curvature and Gaussian torsion [2].

Theorem 3.6. *[8] Let M be a translation surface in $\mathbb{E}^4$. Then M is minimal if and only if either M is a plane or*

$$\begin{aligned} f_k(u) &= \frac{c_k}{\sum c_k^2}\left(\log\left|\cos(\sqrt{a}u)\right| + cu\right) + e_k u, \\ g_k(v) &= \frac{c_k}{\sum c_k^2}\left(-\log\left|\cos(\sqrt{b}v)\right| + dv\right) + p_k v, \quad k = 3, 4, \end{aligned}$$

where c_k, e_k, p_k, $a > 0$, $b > 0$, c, d are real constants.

Corollary 3.7. *Let M be a translation surface given with the surface patch (3.7). If M is Wintgen ideal surface then the equality*

$$\begin{aligned} 0 &= 4W^2A\left(Cf_3''(u)g_3''(u) + Af_4''(u)g_4''(u) + (F - B)f_4''(u)g_3''(u) - (F + B)f_3''(u)g_4''(u)\right) \\ &\quad -W^2\left(f_3''(u)G + g_3''(v)E\right)^2 - \left(G(f_4''(u)A - f_3''(u)B) + E(g_4''(v)A - g_3''(v)B)\right)^2, \end{aligned}$$

holds.

4 Generalized Aminov Surfaces in $\mathbb{E}^4$

In the present section we consider the surfaces M given with

(4.1) $$f(u,v) = r(u)\cos(\varphi(u) + v), \quad g(u,v) = r(u)\sin(\varphi(u) + v).$$

We call such surfaces *generalized Aminov surfaces* in Euclidean 4-space $\mathbb{E}^4$.

For the case $\varphi(u) = 0$ the surface patch (4.1) becomes

(4.2) $$f(u,v) = r(u)\cos v, \quad g(u,v) = r(u)\sin v,$$

which earlier were been considering in the work [1]. We call such surfaces Aminov surfaces in Euclidean 4-space $\mathbb{E}^4$ [4].

Proposition 4.1. *Let M be a generalized Aminov surface given with the Monge patch (4.1). Then the Gaussian curvature K and Gaussian torsion K_N of M become*

(4.3) $$K = -\frac{rr''\left(1 + r^2(1 + (\varphi')^2)\right) + r'\left(r'(1 + (r')^2) - r^3\varphi'\varphi''\right)}{\left((1 + (r')^2 + r^2(\varphi')^2)(1 + r^2) - r^4(\varphi')^2\right)^2},$$

and

(4.4) $$K_N = \frac{r'r''\left(1 + r^2\right) + r\left(r'\left(1 + (r')^2\right) + \varphi'\left(r\varphi'\right)'\right)}{\left((1 + (r')^2 + r^2(\varphi')^2)(1 + r^2) - r^4(\varphi')^2\right)^2},$$

respectively.

Proof. Differentiating the functions on (4.1) with respect to u and v and substituting them on the (3.4) and (3.5) we get the result. □

Proposition 4.2. *[2] Let M be a smooth surface given with the Monge patch of the form*

$$\begin{aligned} f(u,v) &= \phi_u(u,v), \\ g(u,v) &= \phi_v(u,v), \end{aligned} \tag{4.5}$$

then the Gaussian curvature K coincides with the Gaussian torsion K_N of M, where $\phi(u,v)$ is a smooth function of two variables.

As a consequence of Proposition 4.2 we get the following result.

Corollary 4.3. *Let M be a smooth generalized Aminov surface given with the Monge patch (4.1) of the form*

$$\begin{aligned} r(u) &= \sqrt{\lambda^2 e^{2\lambda u} + \mu^2 e^{2\mu u}}, \\ \varphi(u) &= \frac{\mu}{\lambda} \arctan(e^{(\mu-\lambda)}), \end{aligned} \tag{4.6}$$

then the Gaussian curvature K coincides with the Gaussian torsion K_N of M.

Proof. Suppose M is given with the parametrization (4.6) then an easy calculation gives

$$\begin{aligned} r(u)\cos\varphi(u) &= \lambda e^{\lambda u}, \\ r(u)\sin\varphi(u) &= \mu e^{\mu u}. \end{aligned} \tag{4.7}$$

So the surface patch becomes

$$\begin{aligned} f(u,v) &= \phi_u(u,v) = \lambda e^{\lambda u}\cos v - \mu e^{\mu u}\sin v. \\ g(u,v) &= \phi_v(u,v) = -e^{\lambda u}\sin v - e^{\mu u}\cos v, \end{aligned}$$

where

$$\phi(u,v) = e^{\lambda u}\cos v - e^{\mu u}\sin v.$$

So by Proposition 4.2 one can say that the Gaussian curvature K coincides with the Gaussian torsion K_N of M. □

Proposition 4.4. *Let M be a generalized Aminov surface given with the Monge patch (4.1). If the Gaussian curvature K coincides with the Gaussian torsion K_N of M, then the equality*

$$\begin{aligned} 0 &= r(1+(\varphi')^2)\left(r' + r^2 r''\right) + (r')^2(1 + rr' + (r')^2) \\ &\quad + r''\left(r + r'(1+r^2)\right) + r^2\varphi'\varphi''(1 - rr'), \end{aligned} \tag{4.8}$$

holds.

Proof. By the use of the equalities (4.3) and (4.4) we obtain the (4.8). □

In [4] we proved the following results;

Proposition 4.5. *[4] Let M be an Aminov surface (i.e., $\varphi(u) = 0$) given with the Monge patch (4.2). Then the Gaussian curvature K and Gaussian torsion K_N of M becomes*

$$K = -\frac{r(u)r''(u)(1+r^2(u)) + (r'(u))^2(1+(r'(u))^2)}{(1+r^2(u))^2(1+(r'(u))^2)^2}, \tag{4.9}$$

and

$$K_N = \frac{r'(u)r''(u)(1+r^2(u)) + r(u)r'(u)(1+(r'(u))^2)}{(1+r^2(u))^2(1+(r'(u))^2)^2}, \tag{4.10}$$

respectively.

Corollary 4.6. *[4] Let M be an Aminov surface given with the Monge patch (4.2). If $K + K_N = 0$, then the equality*

$$(r(u) - r'(u))\left\{r'(u)\left(1+(r'(u))^2\right) - r''(u)\left(1+r^2(u)\right)\right\} = 0, \tag{4.11}$$

holds.

As a consequence of Corollary 4.6 we can give the following example.

Example 4.1. [4] The Aminov surface given with the surface patch of

$$X(u,v) = (u, v, \lambda e^u \cos v, \lambda e^u \sin v), \tag{4.12}$$

satisfies the relation $K + K_N = 0$.

As a consequence of Theorem 4.1 we get the following results.

Proposition 4.7. *Let M be a smooth generalized Aminov surface given with the Monge patch (4.1). Then the mean curvature vector of M becomes*

$$\overrightarrow{H} = \frac{1}{2}(H_1N_1 + H_2N_2), \tag{4.13}$$

where

$$\begin{aligned} H_1 &= \frac{1}{W^2\sqrt{A}}\left(P\cos(\varphi(u)+v) - R\sin(\varphi(u)+v)\right), \\ H_2 &= \frac{1}{W^3\sqrt{A}}\{(LR-QP)\cos(\varphi(u)+v) + (SP-QR)\sin(\varphi(u)+v)\}, \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} P &= r''(1+r^2) - r(1+(r')^2+(\varphi')^2), \\ Q &= rr'\varphi', \quad R = 2r'\varphi' + r\varphi''(1+r^2), \\ L &= 1+(r')^2, \quad S = 1+r^2+r^2(\varphi')^2. \end{aligned} \tag{4.15}$$

Corollary 4.8. *Let M be a smooth generalized Aminov surface given with the Monge patch (4.1). Then the mean curvature of M becomes*

$$\begin{aligned} \|H\|^2 &= \frac{1}{4AW^6}\left(P^2S + R^2L - 2PQR\right)(L\cos^2(\varphi(u)+v) \\ &\quad + S\sin^2(\varphi(u)+v) - 2Q\cos(\varphi(u)+v)\sin(\varphi(u)+v)). \end{aligned} \tag{4.16}$$

Proposition 4.9. *Let M be a smooth generalized Aminov surface given with the Monge patch (4.1). If M is Wintgen ideal surface then the equation*

$$\begin{aligned} & \left(P^2S + R^2L - 2PQR\right)\left(L\cos^2(\varphi(u)+v) + S\sin^2(\varphi(u)+v)\right. \\ & -2Q\cos(\varphi(u)+v)\sin(\varphi(u)+v)) \\ = \; & 4AW^6\{r(1+(\varphi')^2)(r' - r''r^2) + (r')^2\left(rr' - 1 - (r')^2\right) \\ & +r''(r' + r'r^2 - r) + r^2\varphi'\varphi''(1+rr')\}, \end{aligned} \tag{4.17}$$

holds.

As a consequence of Corollary 4.8 and Proposition 4.9 we can give the following example.

Example 4.2. The surface given with the surface patch

$$X(u,v) = (u, v, \lambda e^u \cos v - \mu e^u \sin v, \lambda e^u \sin v + \mu e^u \cos v), \tag{4.18}$$

is minimal and also satisfies the Wintgen ideal equality.

In [4] we get the following results:

Proposition 4.10. *Let M be an Aminov surface given with the Monge patch (4.2). Then the mean curvature vector of M becomes*

$$\overrightarrow{H} = \frac{(Gr''(u) - Er(u))}{2W^2\sqrt{A}}\left\{\cos vN_1 + \frac{(1+r^2(u))}{W}\sin vN_2\right\}, \tag{4.19}$$

where

$$A = 1 + (r'(u))^2\cos^2 v + r^2(u)\sin^2 v,$$

and

$$\begin{aligned} E &= 1 + (r'(u))^2, \\ F &= 0, \\ G &= 1 + r^2(u), \end{aligned}$$

such that $EG - F^2 = AC - B^2$.

Corollary 4.11. *Let M be an Aminov surface given with the Monge patch (4.2). Then the mean curvature of M becomes*

$$H = \frac{r''(u)(1+r^2(u)) - r(u)(1+(r'(u))^2)}{2(1+r^2(u))(1+(r'(u))^2)^{3/2}}. \tag{4.20}$$

Corollary 4.12. *Let M be an Aminov surface given with the Monge patch (4.2). If M is minimal then*

$$r(u) = \frac{1}{2a}\left(a^2 e^{\pm\frac{2(u+b)}{a}} + a^2 - 1\right)e^{\pm\frac{(u+b)}{a}}, \tag{4.21}$$

where, a and b are real constants.

We obtain the following result.

Theorem 4.13. *Let M be an Aminov surface given with the Monge patch (4.2). If M is Wintgen ideal surface then the equality*

$$2r''(1+r^2)(1+(r')^2)(2r'-r)+(1+(r')^2)^2(4rr'-4(r')^2-r^2)-(r'')^2(1+r^2)^2=0, \tag{4.22}$$

holds.

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References

[1] Yu. Aminov, *Surfaces in $\mathbb{E}^4$ with a Gaussian curvature coinciding with a Gaussian torsion up to the sign*, Mathematical Notes 56 (1994), 1211-1215.

[2] Yu. Aminov, *The Geometry of Submanifolds*, Gordon and Breach Science Publishers, Singapore 2001.

[3] S. Bernstein, *Sur une theoreme de geometrie et ses applications aux equations derivees partielles du type elliptique*, Comm. Soc. Math. Kharkov 15 (1915 - 1917), 38-45.

[4] B. Bulca and K. Arslan, *Surfaces given with the Monge patch in $\mathbb{E}^4$*, Journal of Mathematical Physics, Analysis, Geometry 9 (2013), 435-447.

[5] P. J. Besl and R. C. Jain, *Invariant surface characteristics for 3D object recognition in range images*, Computer Vision, Graphics and Image Processing 33 (1986), 33-80.

[6] M.P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1976.

[7] B.Y. Chen, *Geometry of Submanifolds*, Marcel Dekker Inc., New York, 1973.

[8] F. Dillen, L. Verstraelen, L. Vrancken and G. Zafindratafa, *Classification of polynomial translation hypersurfaces of finite type*, Results in Math. 27 (1995), 244-249.

[9] K. Ecker and G. Huisken, *A Bernstein result for minimal graphs of controlled growth*, J. Differential Geom. 31 (1990), 397-400.

[10] D. Fischer-Colbrie, *Some rigidity theorems for minimal submanifolds of the sphere*, Acta Math. 145 (1980), 29-46.

[11] J. Glymph, D. Schelden, C. Ceccato, J. Mussel and H. Schober, *A parametric strategy for free-from glass structures using quadrilateral planar facets*, Automation in Construction 13 (2004), 187-202.

[12] Th. Hasanis, A. Savas-Halilaj and Th. Vlachos, *Minimal graphs in $\mathbb{R}^4$ with bounded Jacobians*, Proc. Amer. Math. Soc. 137 (2009), 3463-3471.

[13] J. Jost and Y. L. Xin, *Bernstein type theorems for higher codimension*, Calc. Var. Partial Differential Equations 9 (1999), 277-296.

[14] H. Liu, *Translation surfaces with constant mean curvature in 3-dimensinal spaces*, J. Geom. 64 (1999), 141-149.

[15] L. Ni, *A Bernstein type theorem for minimal volume preserving maps*, Proc. Amer. Math. Soc. 130 (2002), 1207-1210.

[16] R. Osserman, *A Survey of Minimal Surfaces*, Van Nostrand-Reinhold, New York, 1969.

[17] H. F. Scherk, *Bemerkungen ber die kleinste Flche innerhalb gegebener Grenzen*, J. R. Angew. Math. 13 (1835), 185-208.

[18] J. Simons, *Minimal varieties in Riemannian manifolds*, Ann. of Math. 88 (1968), 62-105.

[19] L. Verstraelen, J. Walrave and S. Yaprak, *The minimal translation surface in Euclidean Space*, Soochow J. Math. 20 (1994), 77-82.

[20] P. Wintgen, *Sur l'inégalité de Chen-Willmore*, C.R. Acad. Sci. Paris 288 (1979), 993-995.

[21] M. T. Wang, *On graphic Bernstein type results in higher codimension*, Trans. Amer. Math. Soc. 355 (2003), 265-271.

[22] Y. Yuan, *A Bernstein problem for special Lagrangian equations*, Invent. Math. 150 (2002), 117-125.

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