Seiberg-Witten-like equations on 6-manifolds without self-duality

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Abstract. In this paper, Seiberg−Witten−like equations are defined on four and six dimensional manifolds without being dependent on the self-duality concept. Then, by giving a global solution to these equations on the 4−manifolds, their similarities with the well−known Seiberg-Witten equations are shown. Finally, on the 6−manifolds not only a global solution is given, also a bound is given to the solutions of these equations.

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1 Introduction

The Seiberg−Witten equations used in topology, geometry and mathematical physics were firstly introduced by E. Witten in [12]. These equations consist of two parts. The first of these equations is known as the Dirac equation in the literature and it is sufficient for the manifold to have the Spin^c−structure to be identifiable. However, in order to be able to define the second equation called the curvature equation, it is necessary to define the concept of self−duality as well as having the Spin^c−structure of the manifold. The definition of Seiberg−Witten equations is based on the self−duality in the sense of hodge in 4− dimensional manifolds, but this is not possible in manifolds different than 4. To overcome this, mathematicians and physicists have proposed the concept of the generalized self−duality [1, 3, 2, 8]. In this paper, Seiberg−Witten−like equations are defined without using the concept of self−duality as in [5, 6]. Moreover, these equations are not only defined in 4−dimension, also a global solution is given to them on the Kähler manifolds and similarities with the classical Seiberg−Witten equations are shown. In addition, these equations are defined on the 6−dimensional manifold different than the equations costructed by Ş. Karapazar in [3]. This type of identification allows to obtain a bound on the obtained equations and to give a global solution on them.

In this paper, we begin with a section introducing some basic facts concerning *Spin*^{c}−structure and Kähler manifolds. In section 4, on 4−dimensional manifolds Seiberg−Witten−like equations are defined without using self−duality concept and the similarities with the classical Seiberg−Witten equations are indicated. In section

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5, Seiberg−Witten−like equations are defined on 6−dimensional manifolds without using self−duality concept and a bound is obtained for the solution of these equations, Fianlly, a global solution is given to them on the Kähler manifold with respect to negative and constant scalar curvature of M.

2 Some basic materials

2.1 Spin^c−structure and Dirac operator

Let M be an orientable Riemannian manifold with an open covering $\{U_\alpha\}_{\alpha\in A}$. Then, there exist transitions functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow SO(n)$ for TM. If there exists another collection of transition functions

$$\widetilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow Spin^c(n)$$

satisfied $\lambda \circ \widetilde{g}_{\alpha\beta} = g_{\alpha\beta}$ where $\lambda : Spin^c(n) \longrightarrow SO(n)$ and the cocycle condition $\widetilde{g}_{\alpha\beta}(x) \circ \widetilde{g}_{\beta\gamma}(x) = \widetilde{g}_{\alpha\gamma}(x)$ on $U_\alpha \cap U_\beta \cap U_\gamma$, then M is called $Spin^c$ manifold. On the $Spin^c$ manifold, one can construct $P_{SO(n)}, P_{Spin^c(n)}$ and P_{S^1} principal bundles by using principal bundle construction lemma [10]. Also, by using P_{S^1} principal bundle one can construct determinant line bundle

$$\mathcal{L} := P_{Spin^c(n)} \times_l \mathbb{C} = P_{S^1} \times_{U(1)} \mathbb{C} \tag{2.1}$$

where

$$l : U_\alpha \cap U_\beta \longrightarrow Spin^c(n). \tag{2.2}$$

Moreover, an associated complex vector bundle $\mathbb{S} = P_{Spin^c(n)} \times_{\kappa_n} \Delta_n$ can be constructed by considering spinor representations

$$\kappa_n : Spin^c(n) \longrightarrow Aut(\Delta_n)$$

where $\Delta_n = \mathbb{C}^{2^{\frac{n}{2}}}$. If the dimension of M even, then $\mathbb{S}$ spinor bundle splits into two pieces $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ [7]. The sections of the complex vector bundle are called spinor fields. On the complex vector bundle $\mathbb{S}$ one can define Hermitian inner product as follows:

$$\begin{aligned} \langle , \rangle : \Gamma(\mathbb{S}) \times \Gamma(\mathbb{S}) &\rightarrow \mathbb{C} \\ \big([p,\psi],[p,\phi]\big) &\longmapsto \langle \psi, \phi \rangle = \overline{\psi} \cdot \phi. \end{aligned} \tag{2.3}$$

By using Hermitian inner product defined in (2.3), one can associate each spinor Ψ to an endomorphism of $\mathbb{S}$ by the formula

$$\begin{aligned} \Psi\Psi^* : \mathbb{S} &\longrightarrow \mathbb{S} \\ (\tau) &\longmapsto \langle \Psi, \tau \rangle \Psi. \end{aligned}$$

Following bundle homomorphisms are useful while studying on spinors. Extended map of κ_n is defined by

$$\kappa : TM \rightarrow End(\mathbb{S}). \tag{2.4}$$

The Clifford multiplication with κ is defined as:

$$X \cdot \Psi := \kappa(X)(\Psi)$$

where $X \in \Gamma(TM)$ and $\Psi \in \Gamma(\mathbb{S})$.

A spinor covariant derivative operator ∇^A is obtained by using an $A : TP_{S^1} \longrightarrow i\mathbb{R}$, $i\mathbb{R}-$valued $1-$form in the principal bundle P_{S^1} and Levi$-$Civita connection ∇ on M as follows

$$\nabla^A_X \Psi = d\Psi(X) + \frac{1}{2}\sum_{i<j} \omega_{ij}(X)\kappa(e_i) \cdot \kappa(e_j)(\Psi) + \frac{1}{2}A(X)\Psi$$

where $\Psi \in \Gamma(\mathbb{S})$ and $X \in \Gamma(TM)$. In the following Dirac operator is defined.

Definition 2.1. Let $e = \{e_1, e_2, ..., e_{2n}\}$ be any local orthonormal frame on $U \subset M$. Then the local expression of the Dirac operator $D_A : \Gamma(\mathbb{S}) \to \Gamma(\mathbb{S})$ is

$$D_A(\Psi) = \sum_{i=1}^{2n} \kappa(e_i)\nabla^A_{e_i}\Psi.$$

where $\Psi \in \Gamma(\mathbb{S})$ and $A \in \Omega(M, i\mathbb{R})$. Dirac operator decomposes into $D_A = D^+_A \oplus D^-_A$ in the case of dimension of M is even.

By using κ, another bundle map ρ, associated each $2-$form to an endomorphism of $\mathbb{S}$, can be defined on the orthonormal frame $\{e_1, e_2, ..., e_{2n}\}$ as follows

$$\begin{array}{rcl} \rho : \Lambda^2(T^*M) & \to & End(\mathbb{S}) \\ \eta = \sum_{i<j} \eta_{ij} e^i \wedge e^j & \to & \rho(\eta) = \sum_{i<j} \eta_{ij}\kappa(e_i)\kappa(e_j). \end{array}$$

Also, ρ can be extend to a complex valued $2-$forms [11], such that

$$\rho : \Lambda^2(T^*M) \otimes \mathbb{C} \to End(\mathbb{S}).$$

In addition, ρ can be defined on the half spinor bundles $\mathbb{S}^{\pm}$. The half$-$spinor bundles $\mathbb{S}^{\pm}$ are invariant under $\rho(\eta)$ for all $\eta \in \Lambda^2\big(T^*(M)\big)$. That is,

$$\begin{array}{ll} \rho(\eta)(\psi) \in \mathbb{S}^+, & \forall \psi \in \mathbb{S}^+ \\ \rho(\eta)(\psi) \in \mathbb{S}^-, & \forall \psi \in \mathbb{S}^-. \end{array}$$

Then, we obtain the following maps by restriction $\rho^+(\eta) = \rho(\eta)\Big|_{\mathbb{S}^+}$, $\rho^-(\eta) = \rho(\eta)\Big|_{\mathbb{S}^-}$. In this case

$$\rho^+ : \Lambda^2(T^*M) \otimes \mathbb{C} \to End(\mathbb{S}^+)$$

is expressed as follows:

$$\rho^+(\eta) = \rho^+\big(\sum_{i<j} \eta_{ij} e^i \wedge e^j\big) = \sum_{i<j} \eta_{ij}\kappa(e_i)\kappa(e_j).$$

Note that, the space of $i\mathbb{R}-$valued $2-$forms $\Lambda^2(M, i\mathbb{R})$ is a subbundle of $\Lambda^2(M, i\mathbb{R}) \otimes \mathbb{C}$. We consider the subbundle $W = \rho^+\Big(\Lambda^2(M, i\mathbb{R})\Big)$ of $End(\mathbb{S})$ to define curvature equation.

In order to be able to give a global solution for the Seiberg−Witten−like equation defined without self−duality on $n-$dimension, the manifold must be endowed with $SU(n)-$structure. That is guarantees the existence of a Hermitian metric compatible with the complex structure of a Hermitian manifold. On the Hermitian manifold one can construct canonical $Spin^c-$structure and by using this structure spinorial bundle can be defined with a spinorial connection. Also, Dirac operator is associated to a such connection. As a result Seiberg−Witten−like equation without self−duality is defined on such manifold and a global solution can be given to them with respect to the negative and constant scalar curvature.

In the following, before the global solution is given, a short brief of the Kähler manifolds is introduced.

2.2 Kähler Manifolds

On the $2n-$manifolds endowed with $SU(n)-$structure, there exist an almost complex structure J defined by,

$$J : TM \longrightarrow TM,\ J^2 = -I_d.$$

A smooth manifold endoved with an almost complex structure is called an almost complex manifold and donated by (M, J).

The almost complex structure J acts on the space of $1-$forms as follows:

$$\begin{array}{rcl} J : T^*M & \longrightarrow & T^*M \\ \omega & \longmapsto & J\big(\omega\big)(X) := \omega(JX) \end{array}$$

where $\omega \in \Gamma(T^*M)$ and $X \in \Gamma(TM)$. Moreover, J acts on the complexification of the cotangent bundle of M as:

$$\begin{array}{rcl} J : T^*M \otimes_{\mathbb{R}} \mathbb{C} & \longrightarrow & T^*M \otimes_{\mathbb{R}} \mathbb{C} \\ \omega \otimes z & \longmapsto & \big(J\omega\big) \otimes z. \end{array}$$

Since $J^2 = -I_d$, $\pm i$ are eigenvalues of J. Then $T^*M \otimes_{\mathbb{R}} \mathbb{C}$ is the direct sum of

$$T^*M \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^{1,0}(M) \oplus \Lambda^{0,1}(M),$$

where

$$\begin{array}{rcl} \Lambda^{1,0}(M) & = & \{Z \in T^*M \otimes_{\mathbb{R}} \mathbb{C} \big| JZ = iZ\}, \\ \Lambda^{0,1}(M) & = & \{Z \in T^*M \otimes_{\mathbb{R}} \mathbb{C} \big| JZ = -iZ\}. \end{array}$$

The space of $r-$forms is given as:

$$\Lambda^r(M) \quad = \quad \sum_{a+b=r} \Lambda^{a,b}(M)$$

where $\Lambda^{p,q}(M) = span\{x \wedge y | x \in \Lambda^a\big(\Lambda^{1,0}(M)\big), y \in \Lambda^b\big(\Lambda^{0,1}(M)\big)\}$ is the space of (a, b) type complex forms. Finally, Kähler manifold is defined as follows.

Definition 2.2. Let (M, J) be an almost complex manifold. Then, a Riemannian metric g is called Hermitian metric if it is compatible with the almost complex structure J :

$$g(JX, JY) = g(X, Y)$$

where $X, Y \in \Gamma(TM)$.

The associated smooth 2−form Φ defined by

$$\Phi(X,Y) = g(X,JY)$$

is called the Kähler 2−form and satisfies $\Phi(JX,JY) = \Phi(X,Y)$. If Φ is closed then M is called Kähler Manifold and the metric on M is called a Kähler metric.

3 Dirac Operator on the Kähler Manifolds

Let M be n−dimensional Kähler manifold. Since the structure group of any Kähler manifold of dimension n is $U(n)$, it admits a canonical $Spin^c$−structure given by:

$$P_{Spin^c(n)} \quad = \quad P_{U(n)} \times_F Spin^c(n)$$

where $F : U(n) \longrightarrow Spin^c(2n)$ is the lifting map [7]. The associated canonical spinor bundle then has the form:

$$\mathbb{S} \cong \Omega^{0,*}(M).$$

where $\Omega^{0,*}(M)$ is the direct sum of $\Omega(M)^{0,1} \oplus \Omega(M)^{0,2} \oplus ... \oplus \Omega(M)^{0,i}$, $i \in \mathbb{N}$.

There are two ways to include a spinorial Levi−Civita connection on $\mathbb{S}$

The first of these is obtained by the extension of the connection to forms and the latter is obtained via $Spin^c$−structure. In this work, we mainly focused on the canonical $Spin^c$−structure with the following isomorphism:

$$\mathbb{S} \cong \Omega^{0,*}(M).$$

On this bundle, we described Dirac operator defined on $\mathbb{S}$ and we give the relation with the Dirac−type operator defined on $\Omega^{0,*}(M)$.

In the case of Kähler manifold endowed with a canonical $Spin^c$−structure, there is a spinorial connection ∇^A on the associated spinor bundle $\mathbb{S}$ induced by an unitary connection 1−form A on the determinant line bundle $\mathcal{L}$ together with the spinorial Levi−Civita connection ∇. Also, on the associated spinor bundle one can describe Dirac operator as follows:

Let $\{e_i\}$ $i = 1, ..., 2n$ be a local orthonormal frame on M. Then the Dirac operator D^A is given by:

$$D_A = \sum_{i=1}^{2n} e_i \cdot \nabla^A_{e_i}. \tag{3.1}$$

Moreover, by considering Kähler manifolds with $\Omega^{0,*}(M)$ associated spinor bundle the Dirac type operator is defined as follows

Let

$$\overline{\partial} : \Omega^{0,r}(M) \longrightarrow \Omega^{0,r+1}(M), \ \overline{\partial}^* : \Omega^{0,r}(M) \longrightarrow \Omega^{0,r-1}(M) \tag{3.2}$$

respectively given by:

$$\overline{\partial_0} = \sum_{i=1}^{n} \overline{Z}_i^* \wedge \nabla_{\overline{Z_i}}, \ \overline{\partial_2}^* = -\sum_{i=1}^{n} \iota(\overline{Z_i})^* \wedge \nabla_{\overline{Z_i}}$$

where ∇ is the extension of the Levi−Civita connection to $\Omega^{0,*}(M)$ and ι is the contraction operator. Since $\mathbb{S} \cong \Omega^{0,*}(M)$, one has

$$D_{A_0} = \sqrt{2}\big(\overline{\partial_0} + \overline{\partial_2}^*\big) \tag{3.3}$$

where A_0 is the Levi−Civita connection of the line bundle $L = \Omega^2(M)$ of the canonical Spin^c−structure. Also, the curvature of the connection 1−form A_0 is given by

$$F_{A_0} = i\rho_{ric} \tag{3.4}$$

where $\rho_{ric}(X,Y) = g\big(X, J \circ Ric(Y)\big)$ and $Ric : TM \to TM$ denotes the Ricci tensor.

4 Seiberg-Witten−Like Equations :

Definition 4.1. Let (M,g) be a n−dimensional $Spin^c$ manifold. Then Seiberg−Witten like equations for the pair (A,Ψ) is given by

$$\begin{aligned} D_A\Psi &= 0, \quad \text{Dirac Equation} \\ \rho^+\big(F_A\big) &= \frac{1}{2}\big(\Psi\Psi^*\big)^+, \quad \text{Curvature Equation} \end{aligned} \tag{4.1}$$

where F_A is the curvature of A and $\big(\Psi\Psi^*\big)^+$ is the orthogonal projection of $\Psi\Psi^*$ onto $W = \rho^+(\Omega^2(M, i\mathbb{R}))$. In the local orthonormal frame $\{e_1, ..., e_n\}$,

$$\begin{aligned} \big(\Psi\Psi^*\big)^+ &= Proj_W\big(\Psi\Psi^*\big) \\ &= \sum_{i<j} \frac{\big\langle \rho^+(e^i \wedge e^j), \Psi\Psi^* \big\rangle}{\big\langle \rho^+(e^i \wedge e^j), \rho^+(e^i \wedge e^j)\big\rangle} \rho^+(e^i \wedge e^j). \end{aligned}$$

4.1 Seiberg−Witten−Like Equations on $\mathbb{R}^4$

In $\mathbb{R}^4$, since $\omega_{ij} = g(\nabla e_i, e_j)$ is vanished, $\nabla^A\Psi$ described as follows:

$$\nabla^A\Psi = d\Psi + \tfrac{1}{2}A\Psi.$$

By considering the following Spin$^c(4)$−structure

$$\kappa(v) = \begin{bmatrix} 0 & \gamma(v) \\ -\gamma(v)^* & 0 \end{bmatrix}$$

where $\gamma : \mathbb{R}^4 \to End(\mathbb{C}^2)$ is defined on generators $\{e_1, e_2, e_3, e_4\}$ by the followings:

$$\gamma(e_1) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \gamma(e_2) = \begin{bmatrix} \mathfrak{i} & 0 \\ 0 & -\mathfrak{i} \end{bmatrix}, \gamma(e_3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \gamma(e_4) = \begin{bmatrix} 0 & \mathfrak{i} \\ \mathfrak{i} & 0 \end{bmatrix},$$

one has the explicit form of the Dirac operator as follows:

$$\begin{aligned} \frac{\partial\psi_1}{\partial x_1} + A_1\psi_1 &= i\Big(\frac{\partial\psi_1}{\partial x_2} + A_2\psi_1\Big) + \frac{\partial\psi_2}{\partial x_3} + A_3\psi_2 + i\Big(\frac{\partial\psi_2}{\partial x_4} + A_4\psi_2\Big), \\ \frac{\partial\psi_2}{\partial x_1} + A_1\psi_2 &= -i\Big(\frac{\partial\psi_2}{\partial x_2} + A_2\psi_2\Big) - \frac{\partial\psi_1}{\partial x_3} + A_3\psi_1 + i\Big(\frac{\partial\psi_1}{\partial x_4} + A_4\psi_1\Big). \end{aligned} \tag{4.2}$$

Also, the curvature 2−form of A is obtained as follows

$$\begin{bmatrix} iF_{12}+iF_{34} & F_{13}-F_{24}+iF_{14}+iF_{23} \\ -F_{13}+F_{24}+iF_{14}+iF_{23} & -iF_{12}-iF_{34}. \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(|\psi_1|^2-|\psi_2|^2) & \psi_1\overline{\psi_2} \\ \psi_2\overline{\psi_2} & \frac{1}{2}(-|\psi_1|^2+|\psi_2|^2) \end{bmatrix},$$

where $F_{ij} = \left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right),\ i<j,\ i,j=1,...,4.$ By analyzing these two matrix, one gets:

$$\begin{aligned} F_{12}+F_{34} &= -\tfrac{i}{2}\left(|\psi_1|^2-|\psi_2|^2\right) \\ F_{14}+F_{23} &= -\tfrac{i}{2}\left(\psi_1\overline{\psi_2}+\psi_2\overline{\psi_1}\right) \\ F_{13}-F_{24} &= \tfrac{1}{2}\left(\psi_1\overline{\psi_2}-\psi_2\overline{\psi_1}\right). \end{aligned}$$

Notice that these equation sets are the same as in the classical equation sets [11].

4.1.1 A Global Solutions to the Seiberg−Witten−Like Equations on the 2− Dimensional Kahler Manifold

In this section, a global solution to the Seiberg−Witten−like equations without self−duality is given on the 4−dimensional Kähler manifolds with respect to negative and constant scalar curvature of M. Let $\Phi(X,Y)=g(X,JY)$ be the Kähler form on the 2−dimensional Kähler manifold endowed with a canonical $Spin^c$−structure and $\{e_1, e_2=J(e_1), e_3, e_4=J(e_3)\}$ be a local orthonormal frame with the dual basis $\{e^1,e^2,e^3,e^4\}$. Then the explicit form of the Kähler 2−form is

$$\Phi = e^1\wedge e^2 + e^3\wedge e^4.$$

Here Φ acts as an endomorphism in the spinor bundle $\Phi:\mathbb{S}^+\to\mathbb{S}^+$ and has the eigenvalues $\pm 2i$. Then, the spinor bundle $\mathbb{S}^+$ splits into $\mathbb{S}^+ = S^+(2i)\oplus\mathbb{S}^+(-2i)\cong\Omega^{0,2}(M)\oplus\Omega^{0,0}(M)$ where $\mathbb{S}^+(k)=\{\Psi\in\mathbb{S}:\Phi\Psi=k\Psi\}$, $(k=\pm 2i)$ are the corresponding subspaces. Let Ψ_0 be a spinor in $S^+(-2i)\cong\Omega^{0,0}(M)$ corresponding to a constant function 1, in the chosen coordinates $\Psi_0=\begin{bmatrix}0\\1\end{bmatrix}$. By using Ψ_0, one has

$$\frac{(\Psi_0\Psi_0^*)^+}{2} = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

Theorem 4.1. *Let (M,g,J) be 4−dimensional Kähler manifold. Then for a given negative and constant scalar curvature s, $(A_0, \Psi=\sqrt{-s}\Psi_0)$ is the solution of the Seiberg− Witten−lile equations without self−duality.*

Proof. Since $\Psi=\sqrt{-s}\Psi_0\in\Omega^{0,0}(M)$ and Ψ_0 is the spinor field corresponding to the constant function 1, by using (3.3), one gets $D_{A_0}\equiv 0$. Then, it is remaining that satisfying the curvature equation. According to the local coordinates, matrix form of the almost komplex structure J and the Ricci tensor can be given respectively by:

$$J=\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, Ric=\begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{bmatrix}.$$

Since $J \circ Ric = Ric \circ J$, the reduced form of the Ric is obtained in the following way:

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & R_{24} \\ R_{31} & R_{32} & R_{33} & R_{34} \\ R_{41} & R_{42} & R_{43} & R_{44} \end{bmatrix} = \begin{bmatrix} 0 & -A & D & -C \\ A & 0 & C & D \\ -D & -C & 0 & -B \\ C & -D & B & 0 \end{bmatrix}$$

where $R_{11} = R_{22} = A,\ R_{14} = -R_{23} = D,\ R_{24} = R_{13} = C,\ R_{33} = R_{44} = B$. Then, one can obtain the explicit form of ρ_{ric} as follows:

$$\rho_{ric} = -Ae_1 \wedge e_2 + D(e_1 \wedge e_3 + e_2 \wedge e_4) + C(e_2 \wedge e_3 - e_1 \wedge e_4) - Be_3 \wedge e_4.$$

By using Ric in (3.4), one gets $\rho^+(F_{A_0}) = i\rho^+(\rho_{ric})$. The explicit form of $\rho^+(F_{A_0}) = i\rho^+(\rho_{ric})$ is:

$$\begin{bmatrix} A+B & 0 \\ 0 & -A-B \end{bmatrix} .$$

Since $s = tr(Ric) = (R_{11} + R_{22} + R_{33} + R_{44}) = (2A + 2B) = s$, $i\rho^+(\rho_{ric}) = \frac{1}{2}(\Psi\Psi^*)^+$ is satisfied. This means $\rho^+(F_{A_0}) = \frac{1}{2}(\Psi\Psi^*)^+$.

Remark 4.2. $(A_0, \Psi = \sqrt{-s}\Psi_0)$ is the solution of both the Seiberg−Witten equations without self−duality and classical Seiberg−Witten equations.

□

5 Seiberg−Witten−Like Equations on 6−Manifolds

Let $\kappa : \mathbb{R}^6 \to End(\mathbb{C}^4)$ be the $Spin^c$−structure given as in [3]:

$$\kappa(e_1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \kappa(e_2) = \begin{bmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}, \quad \kappa(e_3) = \begin{bmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{bmatrix}$$

$$\kappa(e_4) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \kappa(e_5) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad \kappa(e_6) = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$$

where $\kappa(e_i) : \mathbb{R}^6 \to End(\mathbb{C}^4)$ is defined on generators $\{e_1, e_2, e_3, ..., e_6\}$. In the following, a bound to the solution of the Seiberg−Witten−like equations is obtained.

Lemma 5.1. *Let (M, g) be a compact oriented smooth $6-$dimensional Riemannian manifold endowed with the $Spin^c-$structure given in [3]. Then, following equalities are hold*

1. $\langle \sigma(\Psi)\Psi,\ \Psi \rangle = 3|\Psi|^4$
2. $\langle \sigma(\Psi),\ \sigma(\Psi) \rangle = 3|\Psi|^4$

where $\Psi \in \Gamma(\mathbb{S}^+)$ and $\sigma(\Psi) \in \Omega^2(M, i\mathbb{R})$.

Proof.

1. Let $\Psi = \begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{bmatrix} \in \Gamma(\mathbb{S}^+)$. Taking Hermitian inner product of $\big\langle \sigma(\Psi)\Psi,\ \Psi\big\rangle$ is obtained as:

$$\begin{aligned}\big\langle \sigma(\Psi)\Psi,\ \Psi\big\rangle &= 3\big(|\psi_1|^4+|\psi_2|^4+|\psi_3|^4+|\psi_4|^4+2|\psi_1|^2|\psi_2|^2+2|\psi_1|^2|\psi_3|^2\\ &\quad +2|\psi_1|^2|\psi_4|^2+2|\psi_2|^2|\psi_3|^2+2|\psi_2|^2|\psi_4|^2+2|\psi_3|^2|\psi_4|^2\big)\\ &= 3|\Psi|^4.\end{aligned}$$

2. A straightforward computation gives

$$\big\langle \sigma(\Psi),\ \sigma(\Psi)\big\rangle = 3|\Psi|^4.$$

□

Theorem 5.2. *Let (A,Ψ) be a solution of $D_A^+\Psi = 0,\ \rho^+(F_A) = \frac{1}{2}\big(\Psi\Psi^*\big)^+$ over a compact oriented smooth 6−dimensional Riemannian manifold M with a negative constant scalar curvature s. Then, at each point, the following inequality is satisfied:*

$$\tfrac{\sqrt{3}}{2}|\Psi(x)|^2 \leq -s_{min},\ s_{min} = min\{s(m) : m \in M\}.$$

Proof.

$$\begin{aligned}0 \leq \Delta|\Psi|^2 &= 2\big\langle(\nabla^A)^*\nabla^A\Psi,\Psi\big\rangle - 2\big\langle\nabla^A\Psi,\nabla^A\Psi\big\rangle\\ &\leq 2\big\langle(\nabla^A)^*\nabla^A\Psi,\Psi\big\rangle\\ &= 2\big\langle\Delta_A\Psi,\Psi\big\rangle,\ (D_A^2\Psi = \Delta_A\Psi + \tfrac{s}{4}\Psi + \tfrac{1}{2}dA\Psi)\\ &= 2\big\langle D_A^2\Psi - \tfrac{s}{4}\Psi - \tfrac{1}{2}dA\Psi,\ \Psi\big\rangle,\\ &= \big\langle -\tfrac{s}{2}\Psi - dA\Psi,\ \Psi\big\rangle,\\ &= -\tfrac{s}{2}|\Psi|^2 - \big\langle dA\Psi,\ \Psi\big\rangle,\ \ (dA\Psi = \rho^+(F_A)\Psi)\\ &= -\tfrac{s}{2}|\Psi|^2 - \big\langle\rho^+(F_A)\Psi,\Psi\big\rangle,\ (\big\langle\rho^+(F_A)\Psi,\Psi\big\rangle = \tfrac{1}{8}\big\langle\sigma(\Psi)\Psi,\ \Psi\big\rangle)\\ &= -\tfrac{s}{2}|\Psi|^2 - \tfrac{3}{8}|\Psi|^4.\end{aligned}$$

Since $0 \leq -\frac{s}{2}|\Psi|^2 - \frac{3}{8}|\Psi|^4,\ \frac{\sqrt{3}}{2}|\Psi|^2 \leq -s.$ □

Theorem 5.3. *Let (A,Ψ) be a solution of $D_A\Psi = 0,\ \rho^+(F_A) = \frac{1}{2}\big(\Psi\Psi^*\big)^+$ over a compact oriented smooth 6−dimensional Riemannian manifold M. If $\frac{\sqrt{3}}{2}|\Psi|^2 \leq -s$, then, $|F_A| \leq \frac{1}{4}|s|$.*

Proof.

$$\begin{aligned}|F_A|^2 &= \big\langle\rho^+(F_A),\rho^+(F_A)\big\rangle\\ &= \tfrac{1}{4}\big\langle(\Psi\Psi^*)^+,(\Psi\Psi^*)^+\big\rangle\\ &= \tfrac{1}{64}\big\langle\sigma(\psi),\sigma(\psi)\big\rangle\\ &= \tfrac{3}{64}|\Psi|^4\\ &\Rightarrow |F_A| = \tfrac{\sqrt{3}}{8}|\Psi|^2 \leq \tfrac{|s|}{4}.\end{aligned}$$

□

5.1 A Global Solution to the Seiberg−Witten−Like Equations on 6−Manifolds

In this section, a global solution to the Seiberg−Witten−like equations without self−duality is given on the $SU(3)$−manifolds.

The existence of the $SU(3)$−structure guarantees the existence of the Φ standard symplectic form, the ϕ standard complex volume form and the complex structure J. Φ standard symplectic form is denoted by

$$\Phi = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6 \tag{5.1}$$

where $\{e^1, ..., e^6\}$ is the dual basis of the standard basis of $\{e_1, ..., e_6\}$. Also, standard complex form is given

$$\phi = (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6) \tag{5.2}$$

and the complex structure J is given as

$$J(e_1) = e_2,\ J(e_3) = e_4,\ J(e_5) = e_6. \tag{5.3}$$

Standart 2−form Φ acts as an endomorphism in the spinor bundle $\mathbb{S}$, as follows

$$\Phi : \mathbb{S} \to \mathbb{S}. \tag{5.4}$$

This endomorphism has the eigenvalues $\{\pm 3i, \pm i\}$. According to these eigenvalues the spinor bundle $\mathbb{S}$ splits into $\mathbb{S} = \mathbb{S}(3i) \oplus \mathbb{S}(i) \oplus \mathbb{S}(-i) \oplus \mathbb{S}(-3i)$ where the corresponding subspaces given by $\mathbb{S}(k) = \{\Psi \in \mathbb{S} : \Phi\Psi = k\Psi\}$, $(k = 3i, i, -i, -3i)$. More explicitly the subbundles $\mathbb{S}^+$ and $\mathbb{S}^-$ are given in [3] as:

$$\begin{aligned} \mathbb{S}^+ &= \mathbb{S}(i) \oplus \mathbb{S}(-3i) \cong \Omega^{0,2}(M) \oplus \Omega^{0,0}(M), \\ \mathbb{S}^- &= \mathbb{S}(-i) \oplus \mathbb{S}(3i) \cong \Omega^{0,1}(M) \oplus \Omega^{0,3}(M). \end{aligned}$$

Let Ψ_0 be a spinor in $S^+(-3i) \cong \Omega^{0,0}(M)$ corresponding to constant function 1, in the chosen coordinates

$$\Psi_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{5.5}$$

By using Ψ_0, one gets

$$\frac{(\Psi_0\Psi_0^*)^+}{2} = \begin{bmatrix} -\frac{1}{8} & 0 & 0 & 0 \\ 0 & -\frac{1}{8} & 0 & 0 \\ 0 & 0 & -\frac{1}{8} & 0 \\ 0 & 0 & 0 & \frac{3}{8} \end{bmatrix}.$$

Theorem 5.4. *Let (M, g, J) be an 6−dimensional manifold endowed with $SU(3)$−structure. Then for a given negative and constant scalar curvature s, $(A_0, \Psi = 2\sqrt{-s}\Psi_0)$ is the solution of the Seiberg− Witten−like equations without self−duality.*

Proof. Since $\Psi = 2\sqrt{-s}\Psi_0 \in \Omega^{0,0}(M)$ and Ψ_0 is the spinor field corresponding to the constant function 1, by using (3.3), one gets $D_{A_0} \equiv 0$. Then, it is remaining that satisfying the curvature equation. Since $J \circ Ric = Ric \circ J$, one gets

$$
\begin{bmatrix} R_{11} & 0 & R_{13} & R_{14} & R_{15} & R_{16} \\ 0 & R_{11} & -R_{14} & -R_{13} & -R_{16} & R_{15} \\ R_{13} & -R_{14} & R_{33} & 0 & R_{35} & R_{36} \\ R_{14} & -R_{13} & 0 & R_{33} & -R_{36} & R_{35} \\ R_{15} & -R_{16} & R_{35} & -R_{36} & R_{55} & 0 \\ R_{16} & R_{15} & R_{36} & R_{35} & 0 & R_{55} \end{bmatrix}. \tag{5.6}
$$

To achieve this, *Ric* must be taken as follows:

$$
Ric = \begin{bmatrix} \frac{1}{2}s & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}s & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}s & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}s & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2}s & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{2}s \end{bmatrix}.
$$

Using $\rho_{ric}(X, Y) = g(X, J \circ RicY)$, one has

$$
\begin{aligned}
\rho_{ric} = & -R_{11}e_1 \wedge e_2 - R_{33}e_3 \wedge e_4 - R_{13}(e_1 \wedge e_4 - e_2 \wedge e_3) \\
& -R_{15}(e_1 \wedge e_6 - e_2 \wedge e_5) + R_{14}(e_1 \wedge e_3 + e_2 \wedge e_4) \\
& +R_{16}(e_1 \wedge e_5 + e_2 \wedge e_6) + R_{36}(e_3 \wedge e_5 + e_4 \wedge e_6) \\
& -R_{35}(e_3 \wedge e_6 - e_4 \wedge e_5).
\end{aligned}
$$

By using *Ric* in (3.4) with (5.6), one gets $\rho^+(F_{A_0}) = i\rho^+(\rho_{ric})$ as follows:

$$
\begin{bmatrix} R_{11} - R_{33} - R_{55} & -2iR_{13} - 2R_{14} & 0 & -2R_{15} + 2iR_{16} \\ 2iR_{13} - 2R_{14} & -R_{11} + R_{33} - R_{55} & 0 & -2iR_{35} - 2R_{36} \\ 0 & 0 & R_{11} + R_{33} + R_{55} & 0 \\ -2(R_{15} + iR_{16}) & 2iR_{35} - 2R_{36} & 0 & -R_{11} - R_{33} + R_{55} \end{bmatrix} = \begin{bmatrix} \frac{s}{2} & 0 & 0 & 0 \\ 0 & \frac{s}{2} & 0 & 0 \\ 0 & 0 & \frac{s}{2} & 0 \\ 0 & 0 & 0 & -\frac{3s}{2} \end{bmatrix}.
$$

This means $\rho^+(F_{A_0}) = \frac{(\Psi\Psi^*)^+}{2}$. □

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