

# On the representation formula of Holmes-Thompson areas

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**Abstract.** In this article, we mainly study the Holmes-Thompson areas in Minkowski space. The space of geodesics in Minkowski space has a symplectic structure which is induced by the projection from the sphere-bundle. We show that it can be also obtained from the symplectic structure on the tangent bundle of the Riemannian manifold, the tangent bundle of the Minkowski unit sphere. We give detailed descriptions and expositions on Holmes-Thompson volumes in Minkowski space by the symplectic structure and the Crofton measures for them. For Minkowski plane, a normed two-dimensional space, we express the areas explicitly in integral geometry, by constructing a measure, and this result, therefore, provides an extension of Alvarez's result to a higher dimension.

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## 1 Introductions

### 1.1 Minkowski space and geodesics

A Minkowski space is a vector space with a Minkowski norm, and a Minkowski norm is defined in [4] as

**Definition 1.1.** A function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  is a Minkowski norm if

1.  $F(x) > 0$  for any  $x \in \mathbb{R}^n \setminus \{0\}$  and  $F(0) = 0$ .
2.  $F(\lambda x) = |\lambda|F(x)$  for any  $x \in \mathbb{R}^n \setminus \{0\}$ .
3.  $F \in C^\infty(\mathbb{R}^n \setminus \{0\})$  and the symmetric bilinear form

$$(1.1) \quad g_x(u, v) := \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F^2(x + su + tv)|_{s=t=0}$$

is positively definite on  $\mathbb{R}^n$  for any  $x \in \mathbb{R}^n \setminus \{0\}$ .

We denote a Minkowski space by  $(\mathbb{R}^n, F)$ . By the way, (2) and (3) in Definition 1.1 imply the the convexity of  $F$ , on which one can refer to [4], [1], and [11].

First of all, we can infer the following theorem about geodesics in Minkowski space from Definition 1.1.

**Theorem 1.1.** *The straight line joining two points in Minkowski space is the only shortest curve joining them.*

*Proof.* For any  $p, q \in (\mathbb{R}^n, F)$ , let  $r(t)$ ,  $t \in [a, b]$  with  $F(r'(t)) = 1$ , be a curve joining  $p$  and  $q$ , which has the minimum length. Then  $r(t)$  is the minimizer of the functional  $\int_a^b F(r'(t))dt$ .

Note that  $F$  is smooth. By the fundamental lemma of calculus of variation (Let  $V(h) := \int_a^b F(r'(t) + h\delta'(t))dt$ , where  $\delta(a) = \delta(b) = 0$ . Then

$$\begin{aligned}
 V'(0) &= \frac{\partial}{\partial h} \Big|_{h=0} \int_a^b F(r'(t) + h\delta'(t))dt \\
 &= \int_a^b \frac{\partial}{\partial h} \Big|_{h=0} F(r'(t) + h\delta'(t))dt \\
 (1.2) \quad &= \int_a^b \nabla F(r'(t)) \cdot \delta'(t)dt \\
 &= \nabla F(r'(t)) \cdot \delta(t) \Big|_a^b - \int_a^b \delta(t) \cdot \frac{d}{dt} \nabla F(r'(t))dt \\
 &= - \int_a^b \delta(t) \cdot \frac{d}{dt} \nabla F(r'(t))dt.
 \end{aligned}$$

Thus we can obtain  $\frac{d}{dt} \nabla F(r'(t)) = 0$  since  $V'(0) = 0$  as  $V(0) \leq V(h)$  for any  $\delta(t)$ , or by the Euler–Lagrange equation directly, we have

$$(1.3) \quad \frac{d}{dt} \nabla F(r'(t)) = 0.$$

Using chain rule, (1.3) becomes

$$(1.4) \quad Hess(F) \frac{d^2 r(t)}{dt^2} = 0.$$

On the other hand, we have

$$(1.5) \quad \nabla F(r'(t)) \frac{d^2 r(t)}{dt^2} = 0$$

by differentiating  $F(r'(t)) = 1$ , and then by product rule, (1.4) and (1.5),

$$\begin{aligned}
 (1.6) \quad \frac{1}{2} Hess(F^2) \frac{d^2 r(t)}{dt^2} &= F(r'(t)) Hess(F) \frac{d^2 r(t)}{dt^2} + (\nabla F(r'(t)))^T \nabla F(r'(t)) \frac{d^2 r(t)}{dt^2} \\
 &= Hess(F) \frac{d^2 r(t)}{dt^2} \\
 &= 0.
 \end{aligned}$$

Hence we get  $\frac{d^2 r(t)}{dt^2} = 0$  because  $\frac{1}{2} Hess(F^2)$  is non-degenerated by (3) in Definition 1.1, and then it implies  $r(t)$ ,  $t \in [a, b]$ , is a straight line segment connecting  $p$  and  $q$ .  $\square$

Thus the space of geodesics in  $(\mathbb{R}^n, F)$  actually is the space of affine lines, denoted by  $Gr_1(\mathbb{R}^n)$ . More generally, one can define

**Definition 1.2.** The affine Grassmannian  $\overline{Gr}_k(\mathbb{R}^n)$  is the space of affine  $k$ -planes in  $(\mathbb{R}^n, F)$ .

Recently, there have been interesting works on the integral geometry in Minkowski space (see for instance, [7] and [10]). But in this paper, we will give explicit formula of the generalized areas of objects in Minkowski spaces, which are special Finsler space, in integral form. Additionally, while recently, there have been studies on complex Minkowski space (see for instance, [6] and [9]), this paper is mainly focus on real Minkowski spaces, as they have some interesting structures and properties.

## 1.2 Symplectic structures on cotangent bundle

The Minkowski space  $(\mathbb{R}^n, F)$ , as a differentiable manifold, has a canonical symplectic structure on its cotangent bundle  $T^*\mathbb{R}^n$ , from which a symplectic structure on its tangent bundle  $T\mathbb{R}^n$  can be derived as well.

The canonical contact form  $\alpha$  on  $T^*\mathbb{R}^n$  is defined as  $\alpha_\xi(X) := \xi(\pi_{0*}X)$  for  $X \in T_\xi T^*\mathbb{R}^n$ , where  $\pi_0 : T^*\mathbb{R}^n \rightarrow \mathbb{R}^n$  is the natural projection. And then the canonical symplectic form on  $T^*\mathbb{R}^n$  is defined as  $\omega := d\alpha$ .

On the other hand, we know that the dual of Minkowski metric is defined as

$$(1.7) \quad F^*(\xi) := \sup \{ |\xi(v)| : v \in T\mathbb{R}^n, F(v) \leq 1 \}$$

for  $\xi \in T^*\mathbb{R}^n$ , and there is a natural correspondence between the sphere bundle  $S\mathbb{R}^n$  and the cosphere bundle  $S\mathbb{R}^n = \{ \xi \in T^*\mathbb{R}^n : F^*(\xi) = 1 \}$  of the Minkowski space  $(\mathbb{R}^n, F)$ .

By the convexity and the positive homogeneity of  $F$  (see for instance, [17] and [8]), we can obtain  $F^*(dF(\bar{\xi})) = 1$  and  $dF^*(dF(\bar{\xi})) = \bar{\xi}$  for any  $\bar{\xi} \in S_x\mathbb{R}^n$  and  $x \in \mathbb{R}^n$ , where  $dF$  is the gradient of  $F$  and similarly for  $dF^*$ . Thus  $dF$  is a diffeomorphism from  $S_x\mathbb{R}^n$  to  $S_x^*\mathbb{R}^n$ , which induces another diffeomorphism

$$(1.8) \quad \begin{aligned} \varphi_F : S\mathbb{R}^n &\rightarrow S^*\mathbb{R}^n \\ \varphi_F((x, \bar{\xi}_x)) &= (x, dF(\bar{\xi}_x)) \end{aligned}$$

for any  $\bar{\xi}_x \in S_x\mathbb{R}^n$ . More generally, there is another diffeomorphism  $\frac{1}{2}dF^2$  from  $T_x\mathbb{R}^n \setminus \{0\}$  to  $T_x^*\mathbb{R}^n \setminus \{0\}$  for any  $x \in (\mathbb{R}^n, F)$ , thus we obtain a diffeomorphism

$$(1.9) \quad \begin{aligned} \bar{\varphi}_F : T\mathbb{R}^n &\rightarrow T^*\mathbb{R}^n \\ \bar{\varphi}_F((x, \bar{\xi}_x)) &= (x, \frac{1}{2}dF^2(\bar{\xi}_x)) \end{aligned}$$

by ignoring the 0-sections.

The diffeomorphism (1.8) induces a 2-form  $\bar{\omega} := \varphi_F^*\omega$  on  $S\mathbb{R}^n$ . Without loss of elegance, we can express it more concretely. Since  $T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^{n*}$ , for  $(x, \xi) \in T^*\mathbb{R}^n$  the canonical symplectic form  $\omega$  on  $T^*\mathbb{R}^n$  is actually  $\omega = \text{tr}(dx \wedge d\xi)$ , here we denote  $dx \wedge d\xi := (dx_i \wedge d\xi_j)_{n \times n}$  and similarly  $dx \wedge d\bar{\xi} := (dx_i \wedge d\bar{\xi}_j)_{n \times n}$ ,  $n \times n$  matrices with 2-forms as entries, where  $\xi_j(\xi) = \xi(\frac{\partial}{\partial x_j})$ ,  $\bar{\xi}_j(\bar{\xi}) = dx_j(\bar{\xi})$  and  $F^*(\bar{\xi}) = 1$ . Then using chain rule, we can obtain

$$(1.10) \quad \bar{\omega} = \varphi_F^*(\omega|_{S^*\mathbb{R}^n}) = \text{Hess}(F) \star dx \wedge d\bar{\xi}|_{S\mathbb{R}^n},$$

where  $\star$  is the Frobenius inner product which is the sum of the entries of the entry-wise product of two matrices.

### 1.3 Gelfand transform

Gelfand transform on a double fibration as a generalization of Radon transform plays an important role in making use of the symplectic form of Section 1.2 in integral geometry of Minkowski space.

**Definition 1.3.** Let  $M \xleftarrow{\pi_1} \mathcal{F} \xrightarrow{\pi_2} \Gamma$  be double fibration where  $M$  and  $\Gamma$  are two manifolds,  $\pi_1 : \mathcal{F} \rightarrow M$  and  $\pi_2 : \mathcal{F} \rightarrow \Gamma$  are two fiber bundles, and  $\pi_1 \times \pi_2 : \mathcal{F} \rightarrow M \times \Gamma$  is an submersion. Let  $\Phi$  be a density on  $\Gamma$ , then the Gelfand transform of  $\Phi$  is defined as  $GT(\Phi) := \pi_{1*} \pi_2^* \Phi$ . In the case  $\Phi$  is a differential form and the fibers are oriented, then we also have a well-defined Gelfand transform  $GT(\Phi) := \pi_{1*} \pi_2^* \Phi$ , noting that the pushforward of a form is the integral of contracted form over the fibre.

To make it clear, let's see how the degree of a density or form changes by the transform. Suppose  $\Phi$  is a density or form of degree  $m$  on  $\Gamma$  and the dimension of fibre  $\pi_1$  is  $q$ , then  $\pi_2^* \Phi$  has degree  $m$ , and then  $GT(\Phi) = \pi_{1*} \pi_2^* \Phi = \int_{\pi_1^{-1}(x)} \pi_2^* \Phi$  for  $x \in M$  has degree  $m - q$ .

An application of Gelfand transforms in integral geometry is the following fundamental theorem [14], whose proof is quite simple.

**Theorem 1.2.** Suppose  $M_\gamma := \pi_1(\pi_2^{-1}(\gamma))$  are smooth submanifolds of  $M$  for  $\gamma \in \Gamma$ ,  $\overline{M} \subset M$  is an immersed submanifold, and  $\Phi$  is a top degree density on  $\Gamma$ . Then

$$(1.11) \quad \int_{\Gamma} \#(\overline{M} \cap M_\gamma) \Phi(\gamma) = \int_{\overline{M}} GT(\Phi).$$

*Proof.* Working on the transitions of measures on manifolds and the transformations of intersection numbers, we have

$$(1.12) \quad \begin{aligned} \int_{\overline{M}} GT(\Phi) &= \int_{\overline{M}} \pi_{1*} \pi_2^* \Phi = \int_{\pi_1^{-1}(\overline{M})} \pi_2^* \Phi \\ &= \int_{\Gamma} \#(\pi_2^{-1}(\gamma) \cap \pi_1^{-1}(\overline{M})) \Phi(\gamma) \\ &= \int_{\Gamma} \#(\overline{M} \cap M_\gamma) \Phi(\gamma). \end{aligned}$$

□

## 2 The symplectic structure on the space of geodesics

The symplectic structure on the space of geodesics in a Minkowski space is induced naturally from the canonical symplectic structure on its cotangent bundle.

The process of construction of symplectic form on  $\overline{Gr_1(\mathbb{R}^n)}$  in Minkowski space  $(\mathbb{R}^n, F)$  is based on the following diagram

$$(2.1) \quad \begin{array}{ccccc} S\mathbb{R}^n & \xrightarrow{\varphi_F} & S^*\mathbb{R}^n & \xrightarrow{i} & T^*\mathbb{R}^n \\ \downarrow p & & & & \\ \overline{Gr_1(\mathbb{R}^n)} & & & & \end{array}$$

where  $p$  is the projection from  $S\mathbb{R}^n$  onto  $\overline{Gr_1(\mathbb{R}^n)}$  defined by

$$(2.2) \quad p((x, \bar{\xi})) := l(x, \bar{\xi}),$$

where  $l(x, \bar{\xi})$  is the line passing through  $x$  with direction  $\bar{\xi}$ .

Consider the geodesic vector field  $\bar{\mathcal{X}}(\bar{\xi}_x) := (\bar{\xi}_x, 0)$  on  $T\mathbb{R}^n$  for any  $\bar{\xi}_x \in S\mathbb{R}^n$ ,  $\varphi_F$  in (1.8) induces another vector field  $\mathcal{X} := d\varphi_F(\bar{\mathcal{X}})$  on  $T^*\mathbb{R}^n$  with

$$(2.3) \quad \mathcal{X}(dF(\bar{\xi}_x)) = (d\varphi_F(\bar{\mathcal{X}})(\varphi_F(\bar{\xi}_x))) = (\bar{\xi}_x, 0)$$

for  $\bar{\xi}_x \in S\mathbb{R}^n$ .

We have the following vanishing property about  $\mathcal{X}$  and  $\omega$  on  $S^*\mathbb{R}^n$ .

**Lemma 2.1.**  $i_{\mathcal{X}}\omega = 0$  on  $S^*\mathbb{R}^n$ .

*Proof.* Noting that  $\omega(X, Y) = \langle X_1, Y_2 \rangle - \langle Y_1, X_2 \rangle$  for any  $X = (X_1, X_2)$  and  $Y = (Y_1, Y_2)$  in  $T_{\xi_x}S^*\mathbb{R}^n \subset T_{\xi_x}T^*\mathbb{R}^n$  because  $T^*\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^{n*}$ , where the inner product is the dual space action, by (2.3) we have

$$(2.4) \quad \omega_{\xi_x}(\mathcal{X}, Y) = \langle \bar{\xi}_x, Y_2 \rangle = \langle dF^*(\xi_x), Y_2 \rangle = 0$$

because  $Y_2 \in T_{\xi_x}S^*\mathbb{R}^n$  is “normal” to  $dF^*(\xi_x)$ , precisely, that can be obtained by differentiating  $F^*(\xi_x) = 1$  and noting  $Y_2 \in T_{\xi_x}S^*\mathbb{R}^n$ .  $\square$

Furthermore, the Lie derivative of  $\omega$  along geodesic vector field  $\mathcal{X}$  is

$$(2.5) \quad \mathcal{L}_{\mathcal{X}}\omega = di_{\mathcal{X}}\omega + i_{\mathcal{X}}d\omega = 0$$

by Lemma 2.1. Then (2.5) implies  $(\varphi_F)^*\omega|_{S^*\mathbb{R}^n}$  is invariant under  $\bar{\mathcal{X}}$ .

Based on the invariance of  $\omega$  we can construct a symplectic structure on  $\overline{Gr_1(\mathbb{R}^n)}$ . However, in order to do that, we need to give a manifold structure for  $\overline{Gr_1(\mathbb{R}^n)}$  first.

In fact, we can build a bijection  $\psi$  between  $\overline{Gr_1(\mathbb{R}^n)}$  and  $TS_F^{n-1}$ , where  $S_F^{n-1}$  is the unit sphere in  $(\mathbb{R}^n, F)$ . For any  $l(x, \bar{\xi}) \in \overline{Gr_1(\mathbb{R}^n)}$ , let  $\bar{\eta}$  be the tangent vector pointing at  $l(x, \bar{\xi}) \cap T_{\bar{\xi}}S_F^{n-1}$ , in fact,  $\bar{\eta} = x - dF(\bar{\xi})(x)\bar{\xi} \in T_{\bar{\xi}}S_F^{n-1}$ , see Figure 1 on page 80, and one can define

$$(2.6) \quad \psi(l(x, \bar{\xi})) := (\bar{\xi}, \bar{\eta}) = (\bar{\xi}, x - dF(\bar{\xi})(x)\bar{\xi}).$$

Thus we have a homeomorphism  $\psi$  from  $\overline{Gr_1(\mathbb{R}^n)}$  to  $TS_F^{n-1}$ , and then the manifold structure on  $TS_F^{n-1}$  provides one for  $\overline{Gr_1(\mathbb{R}^n)}$ .

Let us again consider the projection (2.2) with the manifold structure on  $\overline{Gr_1(\mathbb{R}^n)}$ , and then we can obtain the following lemma

**Lemma 2.2.**  $\bar{\mathcal{X}}$  is in the kernel of  $dp$ , in other words,  $p_*(\bar{\mathcal{X}}) = 0$ .

*Proof.* Using the basic equality

$$(2.7) \quad dF(\bar{\xi})(\bar{\xi}) = F(\bar{\xi}) = 1$$

obtained by the positive homogeneity of  $F$  for any  $\bar{\xi} \in S_x\mathbb{R}^n$ , we have

$$(2.8) \quad \begin{aligned} p_*(\bar{\mathcal{X}}) &= dp((\bar{\xi}, 0)) = d(\bar{\xi}, x - dF(\bar{\xi})(x)\bar{\xi})((\bar{\xi}, 0)) \\ &= \bar{\xi} - dF(\bar{\xi})(\bar{\xi})\bar{\xi} \\ &= (1 - dF(\bar{\xi})(\bar{\xi}))\bar{\xi} \\ &= 0. \end{aligned}$$

$\square$

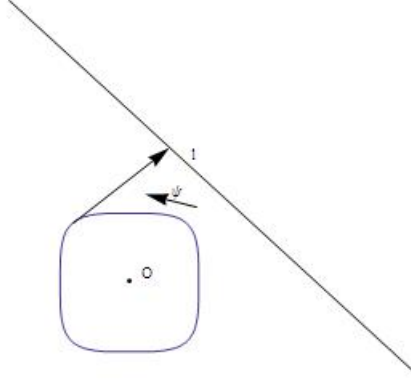


Figure 1:  $\overline{Gr_1(\mathbb{R}^n)}$  Diffeomorphic to  $TS_F^{n-1}$

One can compute the rank of the Jacobian of  $p$  which is  $2n - 2$ , that implies  $\dim(dp|_{\bar{\xi}_x}) = 1$  and then

$$(2.9) \quad \ker(dp|_{\bar{\xi}_x}) = \text{span}(\bar{\mathcal{X}}(\bar{\xi}_x))$$

by Lemma 2.2.

Now we can obtain the following theorem

**Theorem 2.3.** *There exists a symplectic form  $\omega_0$  on  $\overline{Gr_1(\mathbb{R}^n)}$ , such that  $p^*\omega_0 = \bar{\omega} = (\varphi_F)^*\omega|_{S^*\mathbb{R}^n}$ .*

*Proof.* By (2.5), (2.9) and Lemma 2.1, we know that  $\bar{\omega}_{\xi_x}(X, Y)$  is independent of the choices of preimages under the pushforward induced by projection  $p$ . Thus we have a well-defined two form  $\omega_0$  on  $\overline{Gr_1(\mathbb{R}^n)}$ ,

$$(2.10) \quad \omega_{0_{p(\xi_x)}}(\tilde{X}, \tilde{Y}) := \bar{\omega}_{\xi_x}(X, Y),$$

where  $(p_*)_{\xi_x}(X) = \tilde{X}$  and  $(p_*)_{\xi_x}(Y) = \tilde{Y}$ , such that

$$(2.11) \quad p^*\omega_0 = \bar{\omega} = (\varphi_F)^*i^*\omega.$$

□

That finishes the construction of symplectic structure on the space of geodesics in Minkowski space.

On the other hand, since  $T^*S_F^{n-1}$  as a cotangent bundle on Riemannian manifold  $S_F^{n-1}$  has a canonical symplectic structure denoted as  $\tilde{\omega}$ , and we have a canonical diffeomorphism

$$(2.12) \quad \begin{aligned} \tilde{\varphi}_F : T^*S_F^{n-1} &\rightarrow T^*S_F^{n-1} \\ \tilde{\varphi}_F(\tilde{\eta}_{\bar{\xi}}) &= \langle \tilde{\eta}_{\bar{\xi}}, \cdot \rangle_{g_F}, \end{aligned}$$

in which  $g_F$  is the Riemannian metric on  $S_F^{n-1}$ , which is actually the bilinear form

$$(2.13) \quad \langle \bar{u}, \bar{v} \rangle_{g_F} := \frac{\partial^2}{\partial s \partial t} F(\bar{\xi} + s\bar{u} + t\bar{v})|_{s=t=0}$$

for any  $\bar{u}, \bar{v} \in T_{\bar{\xi}}S_F^{n-1}$ , see [4], and then  $\tilde{\varphi}_F^*\tilde{\omega}$  is the the symplectic form induced on  $T^*S_F^{n-1} \cong \overline{Gr_1(\mathbb{R}^n)}$ . Also, we have another symplectic form  $\omega_0$  on  $\overline{Gr_1(\mathbb{R}^n)}$  from Theorem 2.3. A natural question is whether the two symplectic structures on  $\overline{Gr_1(\mathbb{R}^n)}$  are the same, the answer is yes, see the following theorem

**Theorem 2.4.**  $\omega_0 = \tilde{\varphi}_F^*\tilde{\omega}$ .

Let us first draw a diagram for this theorem by combining (2.1)

$$(2.14) \quad \begin{array}{ccccc} S\mathbb{R}^n & \xrightarrow{\varphi_F} & S^*\mathbb{R}^n & \xrightarrow{i} & T^*\mathbb{R}^n \\ \downarrow p & & & & \\ \overline{Gr_1(\mathbb{R}^n)} & \xrightarrow{\psi} & T^*S_F^{n-1} & \xrightarrow{\tilde{\varphi}_F} & T^*S_F^{n-1}. \end{array}$$

*Proof.* First, differentiating (2.7) and using chain rule, one can get

$$(2.15) \quad Hess(F) \star \bar{\xi} d\bar{\xi}|_{S\mathbb{R}^n} = 0$$

in which  $\bar{\xi} d\bar{\xi} := (\bar{\xi}_i d\bar{\xi}_j)_{n \times n}$  is a matrix and  $\star$  is the Frobenius inner product of matrices.

Next, the canonical symplectic form  $\tilde{\omega}$  on  $T^*S_F^{n-1}$ ,  $\tilde{\omega} = \omega|_{T^*S_F^{n-1}}$  in which  $\omega$  is the canonical symplectic form on the cotangent bundle  $T^*\mathbb{R}^n$ . Thus, from (2.12) and (2.13), one can obtain that

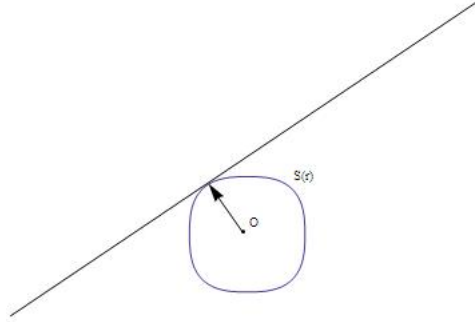
$$(2.16) \quad \tilde{\varphi}_F^*\tilde{\omega} = Hess(F) \star d\bar{\eta} \wedge d\bar{\xi}|_{T^*S_F^{n-1}},$$

here  $d\bar{\xi} \wedge d\bar{\eta}$  is a matrix with 2-form entries and  $\star$  is the Frobenius inner product of matrices.

Therefore, by plugging (2.6) into (2.16) and using (2.15), we obtain

$$(2.17) \quad \begin{aligned} p^*\tilde{\varphi}_F^*\tilde{\omega} &= Hess(F) \star d\bar{\eta} \wedge d\bar{\xi}|_{T^*S_F^{n-1}} \\ &= Hess(F) \star d(x - dF(\bar{\xi})(x)\bar{\xi}) \wedge d\bar{\xi}|_{S\mathbb{R}^n} \\ &= Hess(F) \star dx \wedge d\bar{\xi}|_{S\mathbb{R}^n} - d(dF(\bar{\xi})(x)) \wedge Hess(F) \star \bar{\xi} d\bar{\xi}|_{S\mathbb{R}^n} \\ &= Hess(F) \star dx \wedge d\bar{\xi}|_{S\mathbb{R}^n} \\ &= \tilde{\omega}, \end{aligned}$$

which by Theorem 2.3 implies the claim.  $\square$

Figure 2: “ $p$ -norm distance”  $r$ 

At the end to this section, we make a remark on the symplectic structure on  $\overline{Gr_1(\mathbb{R}^n)}$ .

**Remark 2.1.** From (1.10) we see the symplectic structure  $\bar{\omega}$  on  $T\mathbb{R}^n$  relies on the Minkowski metric  $F$ , then we know, by the above construction, the symplectic structure on  $\overline{Gr_1(\mathbb{R}^n)}$  depends on the Minkowski metric  $F$  as well. Let us see the following example of Minkowski plane with  $p$ -norm as a Minkowski metric.

**Example 2.2.** Given a Minkowski plane by  $(\mathbb{R}^2, \|\cdot\|_p)$ ,  $1 < p < \infty$ , where  $\|(\alpha, \beta)\|_p = (|\alpha|^p + |\beta|^p)^{1/p}$  and the dual norm is  $\|\cdot\|_{\frac{p}{p-1}}$  we can obtain the symplectic form  $\omega$  on  $\overline{Gr_1(\mathbb{R}^2)}$ , the space of affine lines in  $(\mathbb{R}^2, \|\cdot\|_p)$ , by following the general construction above.

By (1.10) and Theorem 2.3, we have

$$(2.18) \quad p^* \omega_0 = (p-1)\alpha^{p-2} dx \wedge d\alpha + (p-1)\beta^{p-2} dy \wedge d\beta,$$

for  $((x, y), (\alpha, \beta)) \in S\mathbb{R}^2$ .

Since  $\overline{Gr_1(\mathbb{R}^2)}$  is a 2-dimensional manifold, we can parametrize affine lines in  $\overline{Gr_1(\mathbb{R}^2)}$  with two variables in a natural way. For any straight line  $l$  passing through  $(x, y)$  with direction  $(\alpha, \beta)$  of unit  $p$ -norm, let  $(-\Theta, \Omega)$  be the unit vector in  $p$ -norm such that  $l$  is tangent to the Minkowski sphere  $S(r)$  of radius  $r$  at  $(-r\Theta, r\Omega)$ , here we can call  $r$  the “ $p$ -norm distance” of  $l$  to the origin, see Figure 2 on page 82. Thus we can denote the line by  $l(r, \Theta)$ .

We have the following theorem about the symplectic structure on  $\overline{Gr_1(\mathbb{R}^2)}$  by the above parametrization.

**Theorem 2.5.** *The symplectic structure on  $\overline{Gr_1(\mathbb{R}^2)}$  is*

$$(2.19) \quad \omega_0 = \frac{(p-1)^2 \Theta^{p(p-2)} \Omega^{p^2-3p+1}}{\|(\Theta, \Omega)\|_{\frac{p}{p-1}}^{(p-1)(2p-1)}} dr \wedge d\Theta.$$



*Proof.* For a line  $l$  passing through  $(x, y)$  with direction  $(\alpha, \beta)$  of unit  $p$ -norm, the “ $p$ -norm distance”

$$(2.20) \quad r = -\Theta^{p-1}x + \Omega^{p-1}y$$

and

$$(2.21) \quad (-\Theta, \Omega) = \left( -\frac{\beta^{\frac{1}{p-1}}}{(\alpha^{\frac{p}{p-1}} + \beta^{\frac{p}{p-1}})^{\frac{1}{p}}}, \frac{\alpha^{\frac{1}{p-1}}}{(\alpha^{\frac{p}{p-1}} + \beta^{\frac{p}{p-1}})^{\frac{1}{p}}} \right).$$

In order to express  $\omega_0$  in terms of  $r$  and  $\Theta$ , at first we use (2.20) and (2.21) to compute

$$(2.22) \quad \begin{aligned} dr \wedge d\Theta &= (-\Theta^{p-1}dx + \Omega^{p-1}dy) \wedge d\Theta \\ &= -\Theta^{p-1}dx \wedge d\left(\frac{\beta^{\frac{1}{p-1}}}{(\alpha^{\frac{p}{p-1}} + \beta^{\frac{p}{p-1}})^{\frac{1}{p}}}\right) + \Omega^{p-1}dy \wedge d\left(\frac{\alpha^{\frac{1}{p-1}}}{(\alpha^{\frac{p}{p-1}} + \beta^{\frac{p}{p-1}})^{\frac{1}{p}}}\right) \\ &= -\Theta^{p-1}dx \wedge d\left(\frac{1}{\left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{\frac{1}{p}}}\right) + \Omega^{p-1}dy \wedge d\left(\frac{1}{\left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{\frac{1}{p}}}\right) \\ &= -\Theta^{p-1}dx \wedge \left(-\frac{1}{p}\right)\left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{-\frac{p+1}{p}} \frac{p}{p-1} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \frac{\beta d\alpha - \alpha d\beta}{\beta^2} \\ &\quad + \Omega^{p-1}dy \wedge \left(-\frac{1}{p}\right)\left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{-\frac{p+1}{p}} \frac{p}{p-1} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \frac{\beta d\alpha - \alpha d\beta}{\beta^2} \\ &= \left(\frac{1}{p-1}\right)\left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{-\frac{p+1}{p}} (\Theta^{p-1}dx \wedge \left(\frac{1}{\beta}d\alpha - \frac{\alpha}{\beta^2}d\beta\right) \\ &\quad - \Omega^{p-1}dy \wedge \left(\frac{1}{\beta}d\alpha - \frac{\alpha}{\beta^2}d\beta\right)) \\ &= \left(\frac{1}{p-1}\right)\left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{-\frac{p+1}{p}} (\Theta^{p-1}\left(\frac{1}{\beta} + \frac{\alpha^p}{\beta^{p+1}}\right)dx \wedge d\alpha \\ &\quad - \Omega^{p-1}\left(-\frac{1}{\beta} \frac{\beta^{p-1}}{\alpha^{p-1}} - \frac{\alpha}{\beta^2}\right)dy \wedge d\beta) \\ &= \left(\frac{1}{p-1}\right)\left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{-\frac{p+1}{p}} \left(\frac{\Theta^{p-1}}{\beta^{p+1}}dx \wedge d\alpha + \frac{\Omega^{p-1}}{\alpha^{p-1}\beta^2}dy \wedge d\beta\right) \\ &= \frac{1}{(p-1)^2} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{-\frac{p+1}{p}} \left(\frac{\Theta^{p-1}}{\beta^{p+1}\alpha^{p-2}}(p-1)\alpha^{p-2}dx \wedge d\alpha \right. \\ &\quad \left. + \frac{\Omega^{p-1}}{\alpha^{p-1}\beta^p}(p-1)\beta^{p-2}dy \wedge d\beta\right). \end{aligned}$$

Indeed,

$$(2.23) \quad \frac{\Theta^{p-1}}{\beta^{p+1}\alpha^{p-2}} = \frac{\Omega^{p-1}}{\alpha^{p-1}\beta^p}$$

since

$$(2.24) \quad \frac{\Theta^{p-1}}{\Omega^{p-1}} = \frac{\beta}{\alpha}$$

by (2.21).

Therefore, using (2.23), (2.24) and  $\|(\Theta, \Omega)\|_p = 1$ , we have

$$\begin{aligned}
dr \wedge d\Theta &= \frac{1}{(p-1)^2} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{-\frac{p+1}{p}} \frac{\Theta^{p-1}}{\beta^{p+1}\alpha^{p-2}} \\
&\quad \left((p-1)\alpha^{p-2}dx \wedge d\alpha + (p-1)\beta^{p-2}dy \wedge d\beta\right) \\
&= \frac{1}{(p-1)^2} \left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{-\frac{p+1}{p}} \frac{\Theta^{p-1}}{\beta^{p+1}\alpha^{p-2}} \omega_0 \\
&= \frac{1}{(p-1)^2} \frac{\left(\frac{\alpha}{\beta}\right)^{\frac{1}{p-1}} \Theta^{p-1}}{\left(\left(\frac{\alpha}{\beta}\right)^{\frac{p}{p-1}} + 1\right)^{\frac{p+1}{p}} \beta^{p+1}\alpha^{p-2}} \omega_0 \\
(2.25) \quad &= \frac{1}{(p-1)^2} \frac{\frac{\Omega}{\Theta} \Theta^{p-1}}{\left(\left(\frac{\Omega}{\Theta}\right)^p + 1\right)^{\frac{p+1}{p}} \left(\frac{\Theta^{p-1}}{\|(\Theta^{p-1}, \Omega^{p-1})\|_p}\right)^{p+1} \left(\frac{\Omega^{p-1}}{\|(\Theta^{p-1}, \Omega^{p-1})\|_p}\right)^{p-2}} \omega_0 \\
&= \frac{1}{(p-1)^2} \frac{\Omega \Theta^{2p-1}}{\left(\frac{\Theta^{p-1}}{\|(\Theta^{p-1}, \Omega^{p-1})\|_p}\right)^{p+1} \left(\frac{\Omega^{p-1}}{\|(\Theta^{p-1}, \Omega^{p-1})\|_p}\right)^{p-2}} \omega_0 \\
&= \frac{1}{(p-1)^2} \frac{\Omega \Theta^{2p-1} \|(\Theta^{p-1}, \Omega^{p-1})\|_p^{2p-1}}{\Theta^{(p-1)(p+1)} \Omega^{(p-1)(p-2)}} \omega_0 \\
&= \frac{1}{(p-1)^2} \frac{\|(\Theta^{p-1}, \Omega^{p-1})\|_p^{2p-1}}{\Theta^{p(p-2)} \Omega^{p^2-3p+1}} \omega_0 \\
&= \frac{\|(\Theta, \Omega)\|_{p(p-1)}^{p(p-1)}}{(p-1)^2 \Theta^{p(p-2)} \Omega^{p^2-3p+1}} \omega_0,
\end{aligned}$$

Thus we have shown

$$(2.26) \quad dr \wedge d\Theta = \frac{\|(\Theta, \Omega)\|_{p(p-1)}^{(p-1)(2p-1)}}{(p-1)^2 \Theta^{p(p-2)} \Omega^{p^2-3p+1}} \omega_0,$$

which implies (2.19) in the claim.  $\square$

So from (2.18) and (2.19) we see the symplectic structure on  $\overline{Gr_1(\mathbb{R}^2)}$  is determined by the Minkowski metric  $\|\cdot\|_p$  on  $\mathbb{R}^2$ .

### 3 Integral Geometry on length in Minkowski space

The length of a straight line segment in  $(\mathbb{R}^2, F)$  can be obtained by integrating the canonical contact form  $\alpha$  introduced in Section 1.2. For any  $x, y \in \mathbb{R}^2$ , let  $\overrightarrow{xy}$  be the vector from  $x$  to  $y$ , and

$$(3.1) \quad c(t) := \left(x + \frac{t}{F(\overrightarrow{xy})}(y-x), dF\left(\frac{\overrightarrow{xy}}{F(\overrightarrow{xy})}\right)\right), t \in [0, F(\overrightarrow{xy})]$$

be a straight line segment in  $T^*\mathbb{R}^2$ . By the positive homogeneity of  $F$ , one can get the useful fact that

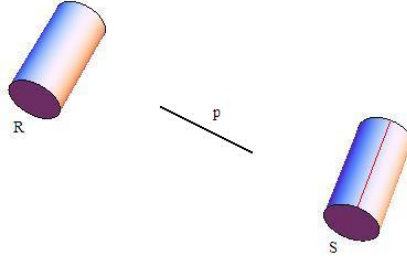
$$(3.2) \quad dF\left(\frac{\overrightarrow{xy}}{F(\overrightarrow{xy})}\right)\left(\frac{\overrightarrow{xy}}{F(\overrightarrow{xy})}\right) = F\left(\frac{\overrightarrow{xy}}{F(\overrightarrow{xy})}\right) = 1.$$

Therefore,

$$(3.3) \quad \int_c \alpha = \int_0^{F(\overrightarrow{xy})} dF\left(\frac{\overrightarrow{xy}}{F(\overrightarrow{xy})}\right)\left(\frac{\overrightarrow{xy}}{F(\overrightarrow{xy})}\right) dt = F(\overrightarrow{xy}) = \mathbf{L}(\overrightarrow{xy}),$$

where  $\mathbf{L}(\overrightarrow{xy})$  is the length of  $\overrightarrow{xy}$ .

Here let us introduce a general definition in integral geometry first.

Figure 3: From  $S\mathbb{R}^2$  to  $\overline{Gr_1(\mathbb{R}^2)}$ 

**Definition 3.1.** A Crofton measure  $\phi$  for a degree  $k$  measure  $\Phi$  on  $(\mathbb{R}^n, F)$  is a measure on  $\overline{Gr_{n-k}(\mathbb{R}^n)}$  (Definition 1.2), such that it satisfies the Crofton-type formula

$$(3.4) \quad \Phi(M) = \int_{P \in \overline{Gr_{n-k}(\mathbb{R}^n)}} \#(M \cap P) \Phi(P)$$

for any compact convex subset  $M$   $(\mathbb{R}^n, F)$ .

Furthermore, we have the following

**Proposition 3.1.** *The Crofton measure on  $\overline{Gr_1(\mathbb{R}^2)}$  for the length is  $|\omega_0|$ .*

Our treatment of applying Stokes' theorem here is primarily based on [12].

*Proof.* From Section 2, we know  $\overline{Gr_1(\mathbb{R}^2)} \xrightarrow{\psi} TS_F$  which is a cylinder, and it has a symplectic form  $\omega_0$  as  $TS_F$  embedded in  $T^*\mathbb{R}^2$ .

Let  $S := \left\{ l \in \overline{Gr_1(\mathbb{R}^2)} : l \cap \overset{\circ}{xy} \neq \emptyset \right\}$ ,  $C_x$  and  $C_y$  be the family of oriented lines passing through  $x$  and  $y$  respectively, then  $C_x \cap C_y = \{l_{xy}^+, l_{xy}^-\}$  that are the two oriented lines connecting  $x$  and  $y$ , and  $\partial S = C_x \cup C_y$ .

Let  $R := \{ \xi \in S\mathbb{R}^2 \subset T\mathbb{R}^2 : l((x + t(y-x), dF(\xi)) \cap \overset{\circ}{xy}) \neq \emptyset \}$ , where  $l((x + t(y-x), dF(\xi))$  is the line passing through  $x + t(y-x)$  with direction  $\xi$ , then  $p(R) = S$ , where  $p$  is the natural projection from  $S\mathbb{R}^2$  to  $\overline{Gr_1(\mathbb{R}^2)}$ .

Additionally, let  $C'_x = \{ \xi_x : \xi_x \in S_x\mathbb{R}^2 \}$ ,  $C'_y = \{ \xi_y : \xi_y \in S_y\mathbb{R}^2 \}$ ,  $l'^+_{xy} = \frac{\overset{\circ}{xy}}{F(\overset{\circ}{xy})} \in S_x\mathbb{R}^2$ , and  $l'^-_{xy} = -\frac{\overset{\circ}{xy}}{F(\overset{\circ}{xy})} \in S_y\mathbb{R}^2$ , then  $p$  maps  $C'_x$ ,  $C'_y$ ,  $l'^+_{xy}$  and  $l'^-_{xy}$  to  $C_x$ ,  $C_y$ ,  $l^+_{xy}$  and  $l^-_{xy}$  respectively, see Figure 3 on page 85.

Applying Stokes' theorem to the two regions individually, using the fact that  $\int_{C'} \alpha = 0$  because of the fixed base points for any  $C' \subset C'_x$  or  $C'_y$ , and combining with (3.3), we obtain

$$(3.5) \quad \begin{aligned} \int_S |\omega_0| &= \int_{p(R)} |\omega_0| = \int_R |p^* \omega_0| \\ &= \int_R |\omega| \\ &= 2 \int_{l'^+_{xy} \cup l'^-_{xy}} \alpha \\ &= 4 \mathbf{L}(\overset{\circ}{xy}). \end{aligned}$$

Therefore, for any rectifiable curve  $\gamma$  in  $(\mathbb{R}^2, F)$ , the length of  $\gamma$ ,

$$(3.6) \quad \mathbf{L}(\gamma) = \frac{1}{4} \int_{l \in \overline{Gr_1(\mathbb{R}^2)}} \#(\gamma \cap l) |\omega_0|,$$

which is the desired claim.  $\square$

**Remark 3.2.** The proof above can be applied to  $\mathbb{R}^2$  with projective Finsler metric, in which geodesics are straight lines. Furthermore, for  $\mathbb{R}^n$  with a projective Finsler metric  $F$ , we choose a plane  $P \subset \mathbb{R}^n$  containing  $\overline{xy}$  for any  $x, y \in \mathbb{R}^n$ , then

$$(3.7) \quad \mathbf{L}(\overline{xy}) = \frac{1}{4} \int_{l \in \overline{Gr_1(P)}} \#(\overline{xy} \cap l) |\omega_0|.$$

## 4 Volume of hypersurfaces

A standard definition of Holmes-Thompson volume in Minkowski space  $(\mathbb{R}^n, F)$  is given and its importance in Finsler geometry and integral geometry is illustrated in [2].

The Holmes-Thompson volumes are defined as follows.

**Definition 4.1.** Let  $N$  be a  $k$ -dimensional manifold and

$$(4.1) \quad D^*N := \{\xi_x \in T^*N : F^*(\xi_x) \leq 1\},$$

where  $F^*$  is the dual norm in (1.7), be the codisc bundle of  $N$ , then the  $k$ -th Holmes-Thompson volume is defined as

$$(4.2) \quad vol_k(N) := \frac{1}{\epsilon_k} \int_{D^*N} |\omega^k|,$$

where  $\epsilon_k$  is the Euclidean volume of  $k$ -dimensional Euclidean ball and  $\omega$  is the canonical symplectic form on the cotangent bundle of  $N$ .

Let  $\Lambda \in \overline{Gr_k(\mathbb{R}^n)}$  for some  $k \leq n$ ,  $\omega_0$  and  $\hat{\omega}_0$  are the natural symplectic forms on  $\overline{Gr_1(\mathbb{R}^n)}$  and  $\overline{Gr_1(\Lambda)}$  constructed in the way described in Section 2. The relation between  $\omega_0$  and  $\hat{\omega}_0$  is shown in the following

**Lemma 4.1.**  $i^*\omega_0 = \omega$  for  $i : \overline{Gr_1(\Lambda)} \hookrightarrow \overline{Gr_1(\mathbb{R}^n)}$ .

*Proof.* First consider the diagram

$$(4.3) \quad S^*\Lambda \xrightarrow{\varphi_{F^*}} S\Lambda \xrightarrow{\hat{i}} S\mathbb{R}^n \xrightarrow{\varphi_F} S^*\mathbb{R}^n.$$

We have a canonical contact form  $\hat{\alpha}_\xi(X) := \xi(\hat{\pi}_0^*X)$  for  $X \in T_\xi S^*\Lambda$  on  $S^*\Lambda$  in diagram (4.3), where  $\hat{\pi}_0 : S^*\Lambda \rightarrow \Lambda$  is the natural projection, and define  $\hat{\omega} := d\hat{\alpha}$  on  $S^*\Lambda$ .

Let  $j = \varphi_F \circ \hat{i} \circ \varphi_{F^*}$ , then for any  $X \in T_\xi S^*\Lambda$ ,

$$(4.4) \quad (j^*\alpha)_\xi(X) = \alpha_{j(\xi)}(j_*X) = j(\xi)(\pi_*j_*X) = \xi(\hat{\pi}_0^*X) = \hat{\alpha}_\xi(X)$$

in which  $\alpha$  and  $\omega$  on  $S^*\mathbb{R}^n$  are introduced in Section 1.2, then (4.4) implies

$$(4.5) \quad j^*\alpha = \hat{\alpha},$$

and furthermore we have

$$(4.6) \quad j^*\omega = \hat{\omega}$$

by differentiating (4.5).

Next, let  $\hat{p}$  be the projection taking  $\bar{\xi}_x \in S\Lambda$  to the line passing  $x$  with the direction  $\bar{\xi}_x$ , and similarly for  $p$  which is described in (2.1). Consider the diagram

$$(4.7) \quad \begin{array}{ccccccc} S^*\Lambda & \xrightarrow{\varphi_{F^*}} & S\Lambda & \xrightarrow{\hat{i}} & S\mathbb{R}^n & \xrightarrow{\varphi_F} & S^*\mathbb{R}^n \\ & & \downarrow \hat{p} & & \downarrow p & & \\ & & \overline{Gr_1(\Lambda)} & \xrightarrow{i} & \overline{Gr_1(\mathbb{R}^n)} & & \end{array}$$

obtained by combining diagram (2.1) and (4.4). By the definitions of the maps in (4.7), we know the diagram is commutative. By Theorem 2.3, we have  $p^*\omega_0 = \varphi_F^*\omega$  and  $\hat{p}^*\hat{\omega}_0 = \varphi_{F^*}\hat{\omega}$ . Combining with (4.6) and the commutativity of the diagram (4.7), we obtain the desired claim  $i^*\omega_0 = \hat{\omega}_0$ .  $\square$

Suppose  $N$  is a hypersurface in  $(\mathbb{R}^n, F)$ , then we have the following

**Proposition 4.2.**  $vol_{n-1}(N) = \frac{1}{2\epsilon_{n-1}} \int_{l \in \overline{Gr_1(\mathbb{R}^n)}} \#(N \cap l) |\omega_0^{n-1}|$ , where  $\omega_0$  is the symplectic form on  $\overline{Gr_1(\mathbb{R}^n)}$ .

This idea of intrinsic proof is given by Prof. J. H. G. Fu.

*Proof.* It suffices to prove the claim in the case when  $N$  is affine. Without loss of generality, assume  $N \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n$  is compact and convex with smooth boundary.

Consider the following diagram

$$(4.8) \quad S^*N \xrightarrow{\hat{i}} S^*\mathbb{R}^{n-1} \xrightarrow{\varphi_{F^*}} S\mathbb{R}^{n-1} \xrightarrow{i} S\mathbb{R}^n \xrightarrow{\varphi_F} S^*\mathbb{R}^n \xrightarrow{\pi} \overline{Gr_1(\mathbb{R}^n)},$$

where  $i$  and  $k$  are embeddings, and  $\pi := p \circ \varphi_F^{-1} = p \circ \varphi_{F^*}$  is a projection from diagram (2.1).

As  $N$  is a  $(n-1)$ -dimensional manifold, the canonical contact form  $\hat{\alpha}$  on  $S^*N$  is defined as  $\hat{\alpha}_\xi(X) := \xi(\hat{\pi}_0^*X)$  for  $X \in T_\xi S^*N$ , where  $\hat{\pi}_0 : S^*N \rightarrow N$  is the projection.

Let  $j = \varphi_F \circ i \circ \varphi_{F^*}$ , then

$$(4.9) \quad (j^*\alpha)_{\hat{i}(\xi)}(\hat{i}_*X) = ((\varphi_F \circ i \circ \varphi_{F^*})^*\alpha)_{\hat{i}(\xi)}(\hat{i}_*X) = \xi(\pi_0^*X) = \hat{\alpha}_\xi(X).$$

for any  $X \in T_\xi S^*N$ , which implies  $(\hat{i} \circ j)^*\alpha = \hat{i}^*j^*\alpha = \hat{\alpha}$ , and then  $(i \circ j)^*\omega = \hat{\omega}$  where  $\hat{\omega} := d\hat{\alpha}$  and  $\omega$  is introduced in Section 1.2.

Applying Stokes' theorem, we have

$$(4.10) \quad \begin{aligned} \int_{D^*N} \hat{\omega}^{n-1} &= \int_{\partial(D^*N)} \hat{\alpha} \wedge \hat{\omega}^{n-2} = \int_{S^*N} \hat{\alpha} \wedge \hat{\omega}^{n-2} + \int_{\hat{\pi}_0^{-1}(\partial N)} \hat{\alpha} \wedge \hat{\omega}^{n-2} \\ &= \int_{S^*N} \hat{\alpha} \wedge \hat{\omega}^{n-2} \end{aligned}$$

since the degree of  $\hat{\alpha} \wedge \hat{\omega}^{n-2}$  on the component measuring perturbations of base points is larger than the dimension of the base manifold, and

$$(4.11) \quad \begin{aligned} \int_{S_+^* \mathbb{R}^n \cap \pi_0^{-1}(N)} \omega^{n-1} &= \int_{\partial(S_+^* \mathbb{R}^n \cap \pi_0^{-1}(N))} \alpha \wedge \omega^{n-2} \\ &= \int_{S^* N} \hat{i}^* j^* \alpha \wedge \hat{i}^* j^* \omega^{n-2} + \int_{\hat{\pi}_0^{-1}(\partial N)} \hat{i}^* j^* \alpha \wedge \hat{i}^* j^* \omega^{n-2} \\ &= \int_{S^* N} \hat{\alpha} \wedge \hat{\omega}^{n-2}, \end{aligned}$$

where

$$(4.12) \quad S_+^* \mathbb{R}^n = \{ \xi \in S^* \mathbb{R}^n : \xi(v_0) \geq 0, v_0 \text{ satisfies } dF(v_0)(v) = 0 \text{ for all } v \in S \mathbb{R}^{n-1} \}.$$

Therefore,

$$(4.13) \quad \int_{D^* N} \hat{\omega}^{n-1} = \int_{S_+^* \mathbb{R}^n \cap \pi_2^{-1}(N)} \omega^{n-1}.$$

Now let us consider the ‘‘upper’’ half space of geodesics in  $(\mathbb{R}^n, F)$ ,

$$(4.14) \quad \overline{Gr_1^+(\mathbb{R}^n)} := \{ l(x, \eta) : dF(\eta)(\eta_0) \geq 0, \eta_0 \text{ satisfies } dF(\eta_0)(v) = 0 \text{ for all } v \in S \mathbb{R}^{n-1} \}.$$

Since  $\pi^* \omega_0 = \omega$ , then we get

$$(4.15) \quad \begin{aligned} \int_{S_+^* \mathbb{R}^n \cap \pi_0^{-1}(N)} \omega^{n-1} &= \int_{S_+^* \mathbb{R}^n \cap \pi_0^{-1}(N)} \pi^* \omega_0^{n-1} \\ &= \int_{\pi^{-1}(l) \in S_+^* \mathbb{R}^n \cap \pi_0^{-1}(N)} \#(N \cap l) \omega_0^{n-1} \\ &= \int_{l \in \overline{Gr_1^+(\mathbb{R}^n)}} \#(N \cap l) \omega_0^{n-1}. \end{aligned}$$

Combining with (4.13), we obtain

$$(4.16) \quad \begin{aligned} vol_{n-1}(N) &= \frac{1}{\epsilon_{n-1}} \int_{D^* N} \hat{\omega}^{n-1} \\ &= \frac{1}{\epsilon_{n-1}} \int_{l \in \overline{Gr_1^+(\mathbb{R}^n)}} \#(N \cap l) \omega_0^{n-1} \\ &= \frac{1}{2\epsilon_{n-1}} \int_{l \in \overline{Gr_1(\mathbb{R}^n)}} \#(N \cap l) |\omega_0^{n-1}|, \end{aligned}$$

that finishes the proof.  $\square$

## 5 $k$ -th Holmes-Thompson volume and Crofton measures

Let us introduce a general fact first. Busemann constructed all projective metrics  $F$  for projective Finsler space  $(\mathbb{R}^n, F)$ , and it was also proved in [16] by Schneider using spherical harmonics.

**Theorem 5.1.** (Busemann) *Suppose  $F$  is a projective metric on  $\mathbb{R}^n$ , then  $F(x, v) = \int_{\xi \in S^{n-1}} |\langle \xi, v \rangle| f(\xi, \langle \xi, x \rangle) \Omega_0$  for any  $(x, v) \in T \mathbb{R}^n$ , where  $\Omega_0$  is the Euclidean volume form on  $S^{n-1}$  and  $f$  is some continuous function on  $S^{n-1} \times \mathbb{R}$ .*

In fact, for the case that  $(\mathbb{R}^n, F)$  is Minkowski, we can use a theorem on surjectivity of cosine transform,

$$(5.1) \quad \mathcal{C}(f)(\cdot) = \int_{\xi \in S^{n-1}} |\langle \xi, \cdot \rangle| f(\xi) \Omega_0,$$

of even functions from Chapter 3 of [5],

**Theorem 5.2.** *For any even  $C^{2[(n+3)/2]}$  function  $g$  on  $S^{n-1}$ ,  $n \geq 2$ , where  $[\cdot]$  is the greatest integer function, there is an even function  $f$  on  $S^{n-1}$  such that  $\mathcal{C}(f) = g$ .*

From it we directly obtain that there exists an even function  $f$  on  $S^{n-1}$ , such that

$$(5.2) \quad L(\overline{xy}) = \frac{1}{4} \int_{\xi \in S^{n-1}} |\langle \xi, \overline{xy} \rangle| f(\xi) \Omega_0.$$

On the other hand, for any  $v = \overline{xy}$ ,  $x, y \in (\mathbb{R}^2, F)$ , by Proposition 3.1 we know

$$(5.3) \quad F(x, v) = \frac{1}{\omega_{n-1}} \int_{l \in \overline{Gr_1(\mathbb{R}^2)}} \#(\overline{xy} \cap l) |\omega_0|.$$

In fact, there is a relation between  $\Omega_0$  and  $\omega_0$ . Considering the following double fibration

$$(5.4) \quad \overline{Gr_1(\mathbb{R}^n)} \xleftarrow{\pi_1} \mathcal{I} \xrightarrow{\pi_2} \overline{Gr_{n-1}(\mathbb{R}^n)},$$

where  $\mathcal{I} = \left\{ (l, H) \in \overline{Gr_1(\mathbb{R}^n)} \times \overline{Gr_{n-1}(\mathbb{R}^n)} : l \subset H \right\}$ , we have

**Proposition 5.3.**  *$GT(f\Omega_0 \wedge dr) = \omega_0$ , where  $GT$  is the Gelfand transform for the double fibration (5.4) and  $r$  is the Euclidean distance of a hyperplane  $H$  to the origin.*

*Proof.* For any  $x, y$  in any 2-plane  $\Pi \subset \mathbb{R}^n$ , we know that the length of  $\overline{xy}$ ,

$$(5.5) \quad L(\overline{xy}) = \frac{1}{4} \int_{l \in \overline{Gr_1(\Pi)}} \#(\overline{xy} \cap l) |\omega_0|_{\overline{Gr_1(\Pi)}}.$$

where  $\overline{Gr_1(\Pi)} := \left\{ l \in \overline{Gr_1(\mathbb{R}^n)} : l \subset \Pi \right\}$ .

Let  $I = \left\{ l \in \overline{Gr_1(\Pi)} \subset \overline{Gr_1(\mathbb{R}^n)} : \overline{xy} \cap l \neq \emptyset \right\}$  and  $G_H := \pi_1(\pi_2^{-1}(H))$  for  $H \in \overline{Gr_{n-1}(\mathbb{R}^n)}$ . By the fundamental theorem of Gelfand transform, Theorem 1.2,

$$(5.6) \quad \int_{H \in \overline{Gr_{n-1}(\mathbb{R}^n)}} \#(I \cap G_H) |f\Omega_0 \wedge dr| = \int_I |GT(f\Omega_0 \wedge dr)|.$$

Therefore,

$$(5.7) \quad \begin{aligned} \int_{l \in \overline{Gr_1(\Pi)}} \#(\overline{xy} \cap l) |GT(f\Omega_0 \wedge dr)| &= \int_I |GT(f\Omega_0 \wedge dr)| \\ &= \int_{H \in \overline{Gr_{n-1}(\mathbb{R}^n)}} \#(I \cap G_H) |f\Omega_0 \wedge dr| \\ &= \int_{\xi \in S^{n-1}} |\langle \xi, \overline{xy} \rangle| f(\xi) \Omega_0 \end{aligned}$$

since  $\overline{Gr_{n-1}(\mathbb{R}^n)} \cong S^{n-1} \times \mathbb{R}$ . By (5.2) and (5.5) we thus obtain

$$(5.8) \quad \int_{l \in \overline{Gr_1(\Pi)}} \#(\overline{xy} \cap l) |GT(f\Omega_0 \wedge dr)| = \int_{l \in \overline{Gr_1(\Pi)}} \#(\overline{xy} \cap l) |\omega_0|,$$

which implies  $GT(f\Omega_0 \wedge dr)|_{\overline{Gr_1(\Pi)}} = \omega_0|_{\overline{Gr_1(\Pi)}}$  for any plane  $\Pi \subset \mathbb{R}^n$  by the injectivity of cosine transform (5.1). (In Chapter 3 of [5] Groemer shows by using condensed harmonic expansion and Parseval's equation, that  $\mathcal{C}(f_1) = \mathcal{C}(f_2)$  iff  $f_1^+ = f_2^+$ , where  $f_1^+(v) = \frac{f_1(v) + f_1(-v)}{2}$  and similarly for  $f_2^+$ , for any bounded integrable functions  $f_1$  and  $f_2$  on  $S^{n-1}$ .)

Now define a basis for  $T_l \overline{Gr_1(\mathbb{R}^n)}$ , the tangent space of  $\overline{Gr_1(\mathbb{R}^n)}$  at  $l \in \overline{Gr_1(\mathbb{R}^n)}$ .

Note that  $\overline{Gr_1(\mathbb{R}^n)} \stackrel{\psi}{\simeq} TS_F^{n-1}$  from Section 2. Let  $\{e_i : i = 1, \dots, n\}$  be the basis for  $\mathbb{R}^n$ , and curve  $\overline{\gamma}_i$  with  $\overline{\gamma}_i(t) = l + te_i$  for  $i = 1, \dots, n-1$ , where  $l \in \overline{Gr_1(\mathbb{R}^n)}$ , and then define  $\overline{e}_i := \overline{\gamma}'_i(0)$  for  $i = 1, \dots, n-1$ . Let  $l(x, \xi)$  be a line in  $\overline{Gr_1(\mathbb{R}^n)}$  passing through  $x$  with direction  $\xi$  and  $r_i(t)(\xi)$  for be the rotation about origin with the direction from  $e_n$  towards  $e_i$  for time  $t$ , then let  $v_i(t)$  be the parallel transport from  $\psi(l(x, \xi))$  along on  $r_i(t)(\xi)$  on  $S_F^{n-1}$ , and then define curves  $\overline{\gamma}_i(t) = \psi^{-1}(v_i(t))$  for  $i = 1, \dots, n-1$ , thus we can define  $\overline{e}_i := \overline{\gamma}'_i(0)$ . Then  $\{\overline{e}_i, \overline{e}_j : i, j = 1, \dots, n-1\}$  is a basis for  $T_l \overline{Gr_1(\mathbb{R}^n)}$ .

Here we have four cases to discuss.

First of all, one can obtain the fact

$$(5.9) \quad GT(f\Omega_0 \wedge dr)(\overline{e}_i, \overline{e}_i) = \omega_0(\overline{e}_i, \overline{e}_i)$$

by choosing a plane  $\Pi_i$  with the tangent space of  $\overline{Gr_1(\Pi_i)}$  spanned by  $\overline{e}_i$  and  $\overline{e}_i$  for  $i = 1, \dots, n-1$ .

On the other hand, in the double fibration (5.4),  $\pi_2|_{\pi_1^{-1}(L_{ij})}$ , in which  $L_{ij}$  is be the lines in  $\overline{Gr_1(\mathbb{R}^n)}$  obtained by translation along  $\overline{e}_i$  or  $\overline{e}_j$  for  $i, j = 1, \dots, n-1$ , is not a submersion from  $\pi_1^{-1}(L_{ij})$  to  $\overline{Gr_{n-1}(\mathbb{R}^n)}$ . Precisely, choose  $\tilde{e}_i$  and  $\tilde{e}_j$  in  $T_{(l, H)}\mathcal{I}$ ,  $l \subset H$  such that  $d\pi_1(\tilde{e}_i) = \overline{e}_i$  and  $d\pi_1(\tilde{e}_j) = \overline{e}_j$ , moreover,  $d\pi_2(\tilde{e}_i)$  and  $d\pi_2(\tilde{e}_j)$  are linearly dependent in  $T_H \overline{Gr_{n-1}(\mathbb{R}^n)}$ . Therefore  $\pi_{1*}\pi_2^*(f\Omega_0 \wedge dr)_l(\tilde{e}_i, \tilde{e}_j) = \int_{\pi_1^{-1}(l)} \pi_2^*(f\Omega_0 \wedge dr)(\tilde{e}_i, \tilde{e}_j) = 0$  for  $i, j = 1, \dots, n-1$ , and obviously  $\omega_0(\overline{e}_i, \overline{e}_j) = 0$ , thus

$$(5.10) \quad GT(f\Omega_0 \wedge dr)(\overline{e}_i, \overline{e}_j) = \omega_0(\overline{e}_i, \overline{e}_j) = 0$$

for  $i, j = 1, \dots, n-1$ .

For the case of  $\overline{e}_i$  and  $\overline{e}_j$ ,  $i \neq j$ ,  $i = 1, \dots, n-1$ . Let  $\overline{L}_{ij}$  be the lines in  $\overline{Gr_1(\mathbb{R}^n)}$  obtained by translation along  $\overline{e}_i$  or rotation along  $\overline{e}_j$ . Again,  $\pi_2|_{\overline{L}_{ij}}$  in (5.4) is not a submersion from  $\pi_1^{-1}(\overline{L}_{ij})$  to  $\overline{Gr_{n-1}(\mathbb{R}^n)}$  either, and it also can be explained precisely as the above case, therefore  $\pi_{1*}\pi_2^*(f\Omega_0 \wedge dr)(\overline{e}_i, \overline{e}_j) = 0$  for  $i, j = 1, \dots, n-1$ , and obviously  $\omega_0(\overline{e}_i, \overline{e}_j) = 0$ , thus

$$(5.11) \quad GT(f\Omega_0 \wedge dr)(\overline{e}_i, \overline{e}_j) = \omega_0(\overline{e}_i, \overline{e}_j) = 0$$

for  $i \neq j$ ,  $i, j = 1, \dots, n-1$ .



Similarly for the last case of  $\bar{e}_i$  and  $\bar{e}_j$ ,  $i, j = 1, \dots, n-1$ ,

$$(5.12) \quad GT(f\Omega_0 \wedge dr)(\bar{e}_i, \bar{e}_j) = \omega_0(\bar{e}_i, \bar{e}_j) = 0.$$

So we have  $GT(f\Omega_0 \wedge dr) = \omega_0$  on  $\overline{Gr_1(\mathbb{R}^n)}$ .  $\square$

One can use the diagonal intersection map and Gelfand transform (see, for instance, [15] and [3]) to construct Crofton measure for the  $k$ -th Holmes-Thompson volume.

Let  $\Omega_{n-1} := f\Omega_0 \wedge dr$  and define a map

$$(5.13) \quad \begin{aligned} \pi : \overline{Gr_{n-1}(\mathbb{R}^n)} \setminus \Delta_k &\rightarrow \overline{Gr_{n-k}(\mathbb{R}^n)} \\ \pi((H_1, \dots, H_k)) &= H_1 \cap \dots \cap H_k, \end{aligned}$$

where  $\Delta_k = \{(H_1, \dots, H_k) : \dim(H_1 \cap \dots \cap H_k) > n-k\}$  and then let  $\Omega_{n-k} := \pi_* \Omega_{n-1}^k$ .

Now consider the following double fibration,

$$(5.14) \quad \overline{Gr_1(\mathbb{R}^n)} \xleftarrow{\pi_{1,k}} \mathcal{I}_k \xrightarrow{\pi_{2,k}} \overline{Gr_{n-k}(\mathbb{R}^n)},$$

where  $\mathcal{I}_k = \{(l, S) \in \overline{Gr_1(\mathbb{R}^n)} \times \overline{Gr_{n-k}(\mathbb{R}^n)} : l \subset S\}$ . Then we have the following proposition about the Gelfand transform on (5.14)

**Proposition 5.4.**  $GT(\Omega_{n-k}) = \omega_0^k$  for  $1 \leq k \leq n-1$ .

*Proof.* Let

$$(5.15) \quad \mathcal{H} := \{(l, (H_1, H_2, \dots, H_k)) \in \overline{Gr_1(\mathbb{R}^n)} \times \overline{Gr_{n-1}(\mathbb{R}^n)}^k : l \subset H_1 \cap \dots \cap H_k\}$$

and consider the following diagram

$$(5.16) \quad \begin{array}{ccccc} \overline{Gr_1(\mathbb{R}^n)} & \xleftarrow{\pi_{1,k}} & \mathcal{I}_k & \xrightarrow{\pi_{2,k}} & \overline{Gr_{n-k}(\mathbb{R}^n)} \\ & \tilde{\pi}_1 \swarrow & \uparrow \tilde{\pi} & \uparrow \pi & \\ & & \mathcal{H} & \xrightarrow{\tilde{\pi}_2} & \overline{Gr_{n-1}(\mathbb{R}^n)}^k, \end{array}$$

in which  $\tilde{\pi} : \mathcal{H} \rightarrow \mathcal{I}_k$  is defined by  $\tilde{\pi}((l, (H_1, H_2, \dots, H_k))) = (l, H_1 \cap H_2 \cap \dots \cap H_k)$ .

Note that

$$(5.17) \quad \pi_{1*} \pi_{2*}^* \Omega_{n-1} = \omega_0,$$

by Proposition 5.3.

For the lower part of the diagram (5.16),

$$(5.18) \quad \overline{Gr_1(\mathbb{R}^n)} \xleftarrow{\tilde{\pi}_1} \mathcal{H} \xrightarrow{\tilde{\pi}_2} \overline{Gr_1(\mathbb{R}^n)}^k,$$

By manipulating the map  $\tilde{\pi}_2 = \underbrace{\pi_2 \times \dots \times \pi_2}_k$ , the product of  $k$  copies of the map  $\pi_2$ ,

applying Fubini theorem for (5.17) and using the fact that  $\tilde{\pi}_1 \times \tilde{\pi}_2 : \mathcal{H} \rightarrow \overline{Gr_1(\mathbb{R}^n)} \times \overline{Gr_1(\mathbb{R}^n)}^k$  is an immersion, one can infer  $\tilde{\pi}_{1*} \tilde{\pi}_2^* \Omega_{n-1}^k = \omega_0^k$ .

Thus, by the commutativity of the diagram (5.16) we obtain  $\pi_{1,k*} \pi_{2,k}^* \Omega_{n-k} = \omega_0^k$ .

$\square$

In order to study the  $k$ -th Holmes-Thompson volume, one can restrict on some  $k + 1$ -dimensional flat subspace. So fix  $S \in \overline{Gr_{k+1}(\mathbb{R}^n)}$  and then define a map by intersection

$$(5.19) \quad \begin{aligned} \pi_S : \overline{Gr_{n-k}(\mathbb{R}^n)} \setminus \Delta(S) &\rightarrow \overline{Gr_1(S)} \\ \pi_S(H^{n-k}) &= H^{n-k} \cap S \end{aligned}$$

for  $H^{n-k} \in \overline{Gr_{n-k}(\mathbb{R}^n)} \setminus \Delta(S)$ , where

$$(5.20) \quad \Delta(S) := \left\{ H^{n-k} \in \overline{Gr_{n-k}(\mathbb{R}^n)} : \dim(H^{n-k} \cap S) > 0 \right\}.$$

Then we have the following proposition

**Proposition 5.5.**  $(\pi_S)_* \Omega_{n-k} = \omega_0^k|_{\overline{Gr_1(S)}}$ , for  $1 \leq k \leq n - 1$ .

*Proof.* From Proposition 5.4, we know that  $\pi_{1,k*} \pi_{2,k}^* \Omega_{n-k} = \omega_0^k$  for the double fibration  $\overline{Gr_{n-k}(\mathbb{R}^n)} \xleftarrow{\pi_{2,k}} \mathcal{L}_k \xrightarrow{\pi_{1,k}} \overline{Gr_1(\mathbb{R}^n)}$ .

Therefore, one can obtain by the definition of the intersection map (5.19)

$$(5.21) \quad (\pi_S)_* \Omega_{n-k} = \pi_{1,k*} \pi_{2,k}^* \Omega_{n-k}|_{\overline{Gr_1(S)}} = \omega_0^k|_{\overline{Gr_1(S)}}.$$

□

Finally, one can obtain the following theorem about Holmes-Thompson volumes.

**Theorem 5.6.** (Alvarez) Suppose  $N$  is a  $k$ -dimensional submanifold in  $(\mathbb{R}^n, F)$ . Then  $\text{vol}_k(N) = \frac{1}{2\epsilon_k} \int_{P \in \overline{Gr_{n-k}(\mathbb{R}^n)}} \#(N \cap P)|\Omega_{n-k}|$  for  $1 \leq k \leq n - 1$ .

*Proof.* By Proposition 4.2, the claim is true for hypersurface case.

It is sufficient to show the claim for the case when  $N \subset S$  for some  $S \in \overline{Gr_{k+1}(\mathbb{R}^n)}$ . We obtain by Proposition 4.2 and Proposition 5.5,

$$(5.22) \quad \begin{aligned} \text{vol}_k(N) &= \frac{1}{2\epsilon_k} \int_{l \in \overline{Gr_1(S)}} \#(N \cap l)|\omega_0^k| \\ &= \frac{1}{2\epsilon_k} \int_{l \in \overline{Gr_1(S)}} \#(N \cap l)|(\pi_S)_* \Omega_{n-k}| \\ &= \frac{1}{2\epsilon_k} \int_{P \in \overline{Gr_{n-k}(\mathbb{R}^n)}} \#(N \cap P)|\Omega_{n-k}|. \end{aligned}$$

as desired. □

## 6 Length and related issues

The classic Crofton formula is

$$(6.1) \quad \text{Length}(\gamma) = \frac{1}{4} \int_0^\infty \int_0^{2\pi} \#(\gamma \cap l(r, \theta)) d\theta dr$$

for any rectifiable curve in Euclidean plane, where  $\theta$  is the angle of the normal of the oriented line  $l$  to the  $x$ -axis and  $r$  is its distance to the origin. Let us denote the affine 1-Grassmannians (lines) in  $\mathbb{R}^2$  by  $\overline{Gr_1(\mathbb{R}^2)}$ .

As for Minkowski plane, it is a normed two dimensional space with a norm  $F(\cdot) = \|\cdot\|$ , in which the unit disk is convex and  $F$  has some smoothness.

Two of the key tools used to obtain the Crofton formula for Minkowski plane are the cosine transform and Gelfand transform. Let us explain them one by one first and see their connection next. A fact from spherical harmonics about cosine transform is there is some even function on  $S^1$  such that

$$(6.2) \quad F(\cdot) = \frac{1}{4} \int_{S^1} |\langle \xi, \cdot \rangle| g(\xi) d\xi,$$

if  $F$  is an even  $C^4$  function on  $S^1$ . A good reference for this is [5]. As for Gelfand transform, it is the transform of differential forms and densities on double fibrations, for instance,  $\mathbb{R}^2 \xleftarrow{\pi_1} \mathcal{I} \xrightarrow{\pi_2} \overline{Gr_1(\mathbb{R}^2)}$ , where  $\mathcal{I} := \{(x, l) \in \mathbb{R}^2 \times \overline{Gr_1(\mathbb{R}^2)} : x \in l\}$  is the incidence relations and  $\pi_1$  and  $\pi_2$  are projections. A formula one can take as an example of the fundamental theorem of Gelfand transform is the following

$$(6.3) \quad \int_{\gamma} \pi_{1*} \pi_2^* |\Omega| = \int_{l \in \overline{Gr_1(\mathbb{R}^2)}} \#(\gamma \cap l) |\Omega|,$$

where  $\Omega := g(\theta) d\theta \wedge dr$ . But we give a direct proof here.

*Proof.* First, consider the case of  $\Omega = d\theta \wedge dr$ . For any  $v \in T_x \gamma$ , since there is some  $v' \in T_{x'} \mathcal{I}$ , such that  $\pi_{1*}(v') = v$ , then

$$(6.4) \quad \begin{aligned} (\pi_{1*} \pi_2^* |\Omega|)_x(v) &= \left( \int_{\pi_1^{-1}(x)} \pi_2^* |\Omega| \right)_x(v) \\ &= \int_{x' \in \pi_1^{-1}(x)} (\pi_2^* |\Omega|)_{x'}(v') \\ &= \int_{S^1} (\pi_2^* |d\theta \wedge dr|)(v') \\ &= \int_{S^1} |dr(\pi_{2*}(v'))| d\theta \\ &= \int_{S^1} |\langle v, \theta \rangle| d\theta \\ &= 4|v|. \end{aligned}$$

So  $\int_{\gamma} \pi_{1*} \pi_2^* |\Omega| = 4 \text{Length}(\gamma) = \int_{l \in \overline{Gr_1(\mathbb{R}^2)}} \#(\gamma \cap l) |\Omega|$  by the classic Crofton formula.

When  $\Omega = f(\theta) d\theta \wedge dr$ , we just need to replace  $d\theta$  by  $g(\theta) d\theta$  in the equalities in the first case.  $\square$

Moreover, from the above proof and (6.2), for any curve  $\gamma(t) : [a, b] \rightarrow \mathbb{R}^2$  differentiable almost everywhere in the Minkowski space,

$$(6.5) \quad \int_{\gamma} \pi_{1*} \pi_2^* |\Omega| = \int_a^b (\pi_{1*} \pi_2^* |\Omega|)(\gamma'(t)) dt = \int_a^b 4F(\gamma'(t)) dt = 4 \text{Length}(\gamma),$$

so then by (6.3) we know

$$(6.6) \quad \text{Length}(\gamma) = \frac{1}{4} \int_{l \in \overline{Gr_1(\mathbb{R}^2)}} \#(\gamma \cap l) |g(\theta) d\theta \wedge dr|$$

for Minkowski plane.

The Holmes-Thompson Area  $HT^2(U)$  of a measurable set  $U$  in a Minkowski plane is defined as  $HT^2(U) := \frac{1}{\pi} \int_{D^*U} |\omega_0|^2$ , where  $\omega_0$  is the natural symplectic form on the cotangent bundle of  $\mathbb{R}^2$  and  $D^*U := \{(x, \xi) \in T^*\mathbb{R}^2 : F^*(\xi) \leq 1\}$ . To study it from the perspective of integral geometry, we need to introduce a symplectic form  $\omega$  on the space of affine lines  $\overline{Gr_1(\mathbb{R}^2)}$ , for which one can see [13].

## 7 Holmes-Thompson area and related issues

Now let's see the Crofton formula for Minkowski plane, which is  $Length(\gamma) = \frac{1}{4} \int_{\overline{Gr_1(\mathbb{R}^2)}} \#(\gamma \cap l) |\omega|$ . To prove this, it is sufficient to show that it holds for any straight line segment

$$(7.1) \quad L : [0, \|p_2 - p_1\|] \rightarrow \mathbb{R}^2, L(t) = p_1 + \frac{p_2 - p_1}{\|p_2 - p_1\|} t,$$

starting at  $p_1$  and ending at  $p_2$  in  $\mathbb{R}^2$ . First, using the diffeomorphism between the circle bundle and co-circle bundle, which is

$$(7.2) \quad \begin{aligned} \varphi_F : S\mathbb{R}^2 &\rightarrow S^*\mathbb{R}^2 \\ \varphi_F(x, \xi) &= (x, dF_\xi), \end{aligned}$$

we can obtain a fact that

$$(7.3) \quad \begin{aligned} \int_{L \times \left\{ \frac{p_2 - p_1}{\|p_2 - p_1\|} \right\}} \varphi_F^* \alpha_0 &= \int_{\varphi_F(L \times \left\{ \frac{p_2 - p_1}{\|p_2 - p_1\|} \right\})} \alpha_0 \\ &= \int_0^{\|p_2 - p_1\|} \alpha_0 dF_{\frac{p_2 - p_1}{\|p_2 - p_1\|}} \left( \left( \frac{p_2 - p_1}{\|p_2 - p_1\|}, 0 \right) \right) dt \\ &= \int_0^{\|p_2 - p_1\|} dF_{\frac{p_2 - p_1}{\|p_2 - p_1\|}} \left( \frac{p_2 - p_1}{\|p_2 - p_1\|} \right) dt, \end{aligned}$$

where  $\alpha_0$  is the tautological one-form, precisely  $\alpha_0 \xi(X) := \xi(\pi_{0*} X)$  for any  $X \in T_\xi T^*\mathbb{R}^2$ , and  $d\alpha_0 = \omega_0$ . Applying the basic equality that  $dF_\xi(\xi) = 1$ , which is derived from the positive homogeneity of  $F$ , for all  $\xi \in S\mathbb{R}^2$ , the above quantity becomes  $\int_0^{\|p_2 - p_1\|} 1 dt$ , which equals to  $\|p_2 - p_1\|$ .

Let  $R := \{ \xi_x \in S^*\mathbb{R}^2 : x \in \overline{p_1 p_2} \}$  and  $T = \left\{ l \in \overline{Gr_1(\mathbb{R}^2)} : l \cap \overline{p_1 p_2} \neq \emptyset \right\}$ , and  $p'$  is the projection (composition) from  $S^*\mathbb{R}^2$  to  $\overline{Gr_1(\mathbb{R}^2)}$ .

Apply the above fact and  $p'^*\omega = \omega_0$ ,

$$(7.4) \quad \begin{aligned} \int_T |\omega| &= \int_{p'(R)} |\omega| = \int_R |p'^*\omega| = \int_R |\omega_0| \\ &= \left| \int_{R^+} \omega_0 \right| + \left| \int_{R^-} \omega_0 \right| \\ &= \left| \int_{\partial R^+} \alpha_0 \right| + \left| \int_{\partial R^-} \alpha_0 \right| \\ &= 4\|p_2 - p_1\|. \end{aligned}$$

Thus we have shown the Crofton formula for Minkowski plane.

Furthermore, combining with (6.6), we have

$$(7.5) \quad \frac{1}{4} \int_{l \in \overline{Gr_1(\mathbb{R}^2)}} \#(\gamma \cap l) |\Omega| = \frac{1}{4} \int_{\overline{Gr_1(\mathbb{R}^2)}} \#(\gamma \cap l) |\omega|,$$

where  $\Omega = g(\theta)d\theta \wedge dr$ . Then, by the injectivity of cosine transform in [5],  $|\Omega| = |\omega|$ .

To obtain the Holmes-Thompson area, one can define a map

$$(7.6) \quad \begin{aligned} \pi : \overline{Gr_1(\mathbb{R}^2)} \times \overline{Gr_1(\mathbb{R}^2)} \setminus \tilde{\Delta} &\rightarrow \mathbb{R}^2 \\ \pi(l, l') &= l \cap l', \end{aligned}$$

where  $\tilde{\Delta} := \{(l, l') : l \parallel l' \text{ or } l = l'\}$ , by the construction of taking intersections in [15]. The following theorem is obtained.

**Theorem 7.1.** For any bounded measurable subset  $U$ ,

$$(7.7) \quad HT^2(U) = \frac{1}{2\pi} \int_{x \in \mathbb{R}^2} \chi(x \cap U) |\pi_* \Omega^2|$$

in which  $HT^2(U)$  denotes the Holmes-Thompson area of  $U$ .

*Proof.* On one hand,

$$(7.8) \quad \frac{1}{\pi} \int_{D^*U} \omega_0^2 = \frac{1}{\pi} \int_{\partial D^*U} \omega_0^2 = \frac{1}{\pi} \int_{S^*U} \alpha_0 \wedge \omega_0.$$

On the other hand, let  $\mathcal{T}_U := \left\{ ((l, l') \in \overline{Gr_1(\mathbb{R}^2)} \times \overline{Gr_1(\mathbb{R}^2)} : l \cap l' \in U) \right\}$ ,

$$(7.9) \quad \frac{1}{\pi} \int_{x \in \mathbb{R}^2} \chi(x \cap U) \pi_* \Omega^2 = \frac{1}{\pi} \int_U \pi_* \omega^2 = \frac{1}{\pi} \int_{\mathcal{T}_U} \omega^2.$$

Let  $\mathbb{T}^*U := \{(\xi_x, \xi'_x) : \xi_x, \xi'_x \in S_x^*U\}$ , then

$$(7.10) \quad (p' \times p')^{-1}(\mathcal{T}_U) = \mathbb{T}^*U \setminus \{(\xi_x, \xi_x) : \xi_x \in S_x^*U\}.$$

Therefore

$$(7.11) \quad \begin{aligned} \frac{1}{\pi} \int_{\mathcal{T}_U} \omega^2 &= \frac{1}{\pi} \int_{\mathbb{T}^*U \setminus \{(\xi_x, \xi_x) : \xi_x \in S_x^*U\}} p'^* \omega^2 \\ &= \frac{1}{\pi} \int_{\mathbb{T}^*U \setminus \{(\xi_x, \xi_x) : \xi_x \in S_x^*U\}} \omega_0^2 \\ &= \frac{2}{\pi} \int_{\{(\xi_x, \xi_x) : \xi_x \in S_x^*U\}} \alpha_0 \wedge \omega_0 \\ &= \frac{2}{\pi} \int_{S^*U} \alpha_0 \wedge \omega_0. \end{aligned}$$

Thus, the claim follows from (7.8),(7.9) and (7.11).  $\square$

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