Finsler Spaces and their applications to Field Theory

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Abstract. In this paper, we have shown how a gauge transformation takes a Lorentz space to a space which is Finslerian. Note that the inverse transformation takes a metric from a Finsler metric back to the Lorentz metric. This demonstrates a generalized equivalence whereby a transformation exists which produces a local inertial frame along the world line of a particle. This means that the motion of a particle along a curved path might not only be due to a gravitational field derived from a metric, but might also be due to other metric produced fields.

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1 Introduction

The main benefit is a physical understanding of how a Finsler space might describe a space which contains a non-gravitational field. That is, it has been shown how a gauge transformation takes a Lorentz space to a space which is Finslerian. Note that the inverse transformation takes a metric from a Finsler metric back to the Lorentz metric. This demonstrates a generalized equivalence whereby a transformation exists which produces a local inertial frame along the world line of a particle. This means that the motion of a particle along a curved path might not only be due to a gravitational field derived from a metric, but might also be due to other metric produced fields. It will be shown shortly exactly how this occurs. First, though, some standard Finsler results are presented. A significant point is that these results are developed in terms of a coordinate transformation of the base space M. This contrasts with the gauge transformation just depicted which is a vertical diffeomorphism, a transformation in the fiber space. The gauge transformation is used to get to the Finsler space. The connections given so far describe the transition to that space. The coordinate transformation deals with the properties of the resulting Finsler space. It describes the translation (sometimes called the transplantation) as one moves from one point to another in the space.

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2 Finsler Gauge Transformations

If a particle in a space-time moves along a curved, non-geodesic path, then it is said that the particle is under the influence of some external force. In such a case, an external force term is added to the equation of motions to explain the path of motion. Alternative point of view is that motion can be explained by a new metric, which would result from a gauge transformation. In this way, physical force fields can be geometrized, and general relativistic idea of spacetime curvature determining the path of the particle will also include fields other than gravitation. For this purpose a class of gauge transformations which act on tangent space is considered [2].

Under these kind of transformations, the tangent vector y^μ transforms as

$$\bar{y}^i = \widetilde{Y}^i_j y^j \tag{2.1}$$

where $i, j, ... = 0, 1, 2, 3$ are indices corresponding the space components, and

$$\widetilde{Y}^i_j = \frac{\partial \bar{y}^i}{\partial y^j}, \tag{2.2}$$

$$\widetilde{Y}^i_k Y^k_j = \delta^i_j \tag{2.3}$$

where,$Y^k_j = \frac{\partial y^k}{\partial \bar{y}^j}$ is the inverse transformation, and these transformations (Y^k_j) are called Y transformations.

Even though the transformation does not act on the base space coordinates, it will seen to produce changes in the base space. Thus, these transformations also depend on the base coordinates, such as

$$\widetilde{Y}^i_j = \widetilde{Y}^i_j(x, y) \tag{2.4}$$

The Y transformation of the metric tensor is given as

$$\bar{g}_{ij}(x, y) = Y^\alpha_i(x, y) Y^\beta_j(x, y) g_{\alpha\beta}(x, y) \tag{2.5}$$

Under this transformation, Finsler metric function is invariant, such as

$$\begin{aligned} \bar{F}^2(x, \bar{y}) &= \bar{g}_{ij}\bar{y}^i\bar{y}^j \\ &= g_{\alpha\beta}(x, y) Y^\alpha_i Y^\beta_j \widetilde{Y}^i_k \widetilde{Y}^j_l y^k y^l \\ &= F^2(x, y). \end{aligned} \tag{2.6}$$

Here y^j is the contravariant vector and the covariant vector associated with it is y_i. where $y_i = g_{ij}y^j$. Covariant vector y_i transforms as

$$\bar{y}_i = Y^\alpha_i y_i. \tag{2.7}$$

Since

$$\begin{aligned} \frac{\partial \bar{y}_i}{\partial \bar{y}^j} &= \bar{g}_{ij} \\ &= Y^\alpha_i Y^\beta_j g_{\alpha\beta} + Y^\beta_j \frac{\partial Y^\alpha_i}{\partial y^\beta} y_\alpha \end{aligned} \tag{2.8}$$

The Y transformation of the Finslerian metric tensor does not yield a tensor unless

$$\frac{\partial Y_i^\alpha}{\partial y^\beta} y_\alpha = 0 \tag{2.9}$$

The condition (2.9) is called as the metric condition [8].

It is of interest to ask how many of the known Finsler metrics can be obtained by this sort of gauge transformation. At this point one can only list those for which a specific Y matrix is known: Randers, Kropina, Beil, Weyl, and metrics where Y gives a conformal transformation. Obviously, nonlinear metrics are not included. What does this gauge transformation mean physically? It can be interpreted as what happens when a nongravitational field is turned on in a region of space. For example, the field could be electromagnetic. A metric has also been given for the electroweak field $SU(2) \times U(1)$[7]. The gauge transformation could also be interpreted as a distortion or deformation of the original Lorentz space. In other words, the gauge field twists or distorts the space. The relative effect is, by the way, a torsion rather than a curvature. Although, remarkably, the final outcome is a curved space. The torsion interpretation has been advocated by Holland[9], who relates the transformation to nonholonomic frames. The nonholonomic frame viewpoint is explained in a very useful new paper by Bucataru[8].

It is obvious that Y transformations, when Y^i_j is a function of x only, that is

$$Y^i_j = Y^i_j(x). \tag{2.10}$$

satisfy the metric condition. These type of transformations are called K-group or linear transformations [8].

Y transformations can be interpreted as the transformations from an original space where there exists no external field, to a space that also contains external fields which are turned on by some physical potentials contained in Y^i_j [6].

It is now time to get to some specific physics using the above developments. There are several gauge transformations which might give useful results. R.G.Beil[4] had studied Y transformations for general Finsler (α, β)-metrics.

$$Y^i_j = \sqrt{a}\delta^i_j - \frac{1}{B^2}(\sqrt{a} - \sqrt{a + bB^2})B^i B_j. \tag{2.11}$$

where B_j is a vector which can be associated to a physical potential, and $B^2 = g_{ij}B^iB^j$. Here b is a constant depending on the physical space that will be geometrized. The inverse transformation is given by the inverse of the matrix (2.11), such as

$$\widetilde{Y}^j_i = \frac{1}{\sqrt{a}}\delta^j_i - \frac{1}{B^2}\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{a + bB^2}}\right)B^j B_i. \tag{2.12}$$

Remark: If we take $a(x, y) = 1$ and $b(x, y) = k$, the nonholonomic Finsler frame (2.11) is the frame used by R.G.Beil in [4], formula (5.1).
We further use to give some specific examples.

Example:1 The Y transformations for Finsler space with Matsumoto metric, are given by

$$
\begin{aligned}
Y^i_j &= \sqrt{\frac{\alpha^2(\alpha-2\beta)}{(\alpha-\beta)^3}}\delta^i_j - \frac{\alpha^2}{B^2\alpha^2-\beta^2} \\
&\cdot \left\{\sqrt{\frac{\alpha^2(\alpha-2\beta)}{(\alpha-\beta)^3}} \pm \sqrt{\frac{-\alpha^2(-\alpha\beta^2+2\alpha\beta^2+2\beta^3+B^2\alpha^3-4B^2\alpha^2\beta)}{(\alpha-\beta)^4\beta}}\right\} \\
&\left(B^i - \frac{\beta y^i}{\alpha^2}\right)\left(B_j - \frac{\beta y_j}{\alpha^2}\right). \qquad (2.13)
\end{aligned}
$$

$$
\widetilde{Y}^j_i = \delta^j_i - \frac{1}{C^2}\left(1 \pm \sqrt{\frac{\beta(\alpha-\beta)^3C^2}{\alpha^4}}\right)B^jB_i; \qquad (2.14)
$$

where

$$
C^2 = \frac{\alpha^2(\alpha-2\beta)B^2}{(\alpha-\beta)^3} - \frac{(\alpha-4\beta)(B^2\alpha^2-\beta^2)^2}{\beta(\alpha-\beta)^4}.
$$

Example:2 The Y transformations for Finsler space with Infinite series of (α,β) metric, are given by

$$
\begin{aligned}
Y^i_j &= \sqrt{\frac{2(c_1\beta^2+\alpha^2)}{\beta^2}}\delta^i_j \\
&- \frac{1}{16}\left[\left\{\frac{\beta^6\left(\sqrt{\frac{2(c_1\beta^2+\alpha^2)}{\beta^2}}+\sqrt{\frac{2(\beta^4c_1-\alpha^2\beta^2+2\alpha^4B^2)}{\beta^4}}\right)}{\alpha^2(\alpha^2B^2-\beta^2)}\right\}\right. \qquad (2.15) \\
&\left.\left(\frac{4y^i}{\beta^2}-\frac{4\alpha^2B^i}{\beta^3}\right)\left(\frac{4y_j}{\beta^3}-\frac{4\alpha^2B_j}{\beta^3}\right)\right]
\end{aligned}
$$

$$
\widetilde{Y}^j_i = \delta^j_i - \frac{1}{C^2}\left(1 \pm \sqrt{1-\frac{C^2\beta^4}{\alpha^4-c_1^2\beta^4}}\right)B^iB_j \qquad (2.16)
$$

where

$$
C^2 = \frac{2(c_1\beta^2+\alpha^2)}{\beta^2} + \frac{4(\alpha^2B^2-\beta^2)^2}{\beta^4}.
$$

3 Charged Classical Particle in Finsler Space-time

In this section, an original metric tensor is used to produce the Finsler metric function by a specific Y transformation. The original metric is assumed to be Minkowskian for simplicity. In this case gravitational field effects are neglected, but even in the presence of electromagnetic fields alone, the physical space-time can be described as curved Finsler space-time. The results calculated are same as usual classical electrodynamics which is based on the flat Minkowski space-time, with an additional electromagnetic field [4, 2].

3.1 Geodesic Equation

The original metric is chosen as ordinary Minkowskian metric η_{ij} in the form

$$\eta_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.1}$$

After applying Y transformation (2.11) to this metric, the resulting metric will be

$$\bar{g}_{ij} = \eta_{ij} + kB_iB_j. \tag{3.2}$$

In this case, vector B_i is related to electromagnetic vector potential A_i. The contravariant form of the metric tensor (3.1) can be written as

$$\bar{g}^{ij} = \eta^{ij} - k(1 + kB^2)B^iB^j, \tag{3.3}$$

where $B^2 = \eta_{ij}B^iB^j$, so that

$$\bar{g}^{ik}\bar{g}_{kj} = \delta^i_j. \tag{3.4}$$

If we calculate the geodesic equation resulting from the new metric (3.1), we get

$$\frac{dy^i}{d\tau} + kB_my^m(\frac{\partial B_i}{\partial x^\alpha} - \frac{\partial B_\alpha}{\partial x^i})y^\alpha = 0, \tag{3.5}$$

where $y^\alpha = \frac{dx^\alpha}{d\tau}$, and τ is the proper time.

Since we deal with the geometrization of electrodynamics, with conditions

$$B_iy^i = \frac{e}{mck} \tag{3.6}$$

and

$$B_i = A_i, \tag{3.7}$$

where e is the charge of the electron, m is the mass of the electron and c is the velocity of light and k is a constant and will be determined by the field equations.
The geodesic equation (3.4) will take the form

$$\frac{dy^i}{d\tau} + \frac{e}{mc}F_{ij}y^j = 0, \tag{3.8}$$

where

$$F_{ij} = (\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}) \tag{3.9}$$

is the electromagnetic field tensor.

The geodesic equation (3.7) is identical with the Lorentz equation in Minkowskian space-time, with corresponding velocity y^i.

An important point is that the laws of physics must be invariant under arbitrary gauge transformations. If we consider an electromagnetic gauge transformation

$$\bar{A}_i = A_i + \frac{\partial \Lambda(x)}{\partial x^i} \tag{3.10}$$

where $\Lambda(x)$ is any arbitrary function, the metric tensor (3.1) is invariant and the geodesic equation (3.4) remains unchanged.

3.2 Field Equations

By introducing the condition (3.5), the velocity dependent metric (3.1) reduces to a Riemannian metric. So field equations are calculated by Riemann geometry. The Ricci tensor for the metric (3.1) is calculated as

$$\begin{aligned}
R_{mn} = &- \frac{1}{4}k^2\bar{g}^{\alpha n}\bar{g}^{li}F_{pl}F_{\alpha i}B_mB_n - \frac{1}{2}k\bar{g}^{il}F_{nl}F_{mi} \\
&- \frac{1}{2}k^2(1+kB^2)^{-1}\eta^{il}F_{\alpha i}B^\alpha(B_mB_{l,n} + B_nB_{l,m}) \\
&- \frac{1}{2}k^2(1+kB^2)^{-1}\bar{g}^{il}B_{i,l}B^\alpha(B_mF_{n\alpha} + B_nF_{m\alpha}) \\
&+ \frac{1}{2}k\bar{g}^{il}(F_{ml,i}B_n + F_{nl,i}B_m) \\
&- \frac{1}{2}k(1+kB^2)^{-1}\eta^{il}B_{n,l}B_{m,i} \\
&- k[\frac{1}{2}(1+kB^2)^{-1}\eta^{il} - k(1+kB^2)^{-2}B^iB^l]B_{l,n}B_{i,m} \\
&+ \frac{1}{2}(1+kB^2)^{-1}\bar{g}^{il}B_{i,l}(B_{l,m} + B_{m,l}) \\
&+ \frac{1}{2}k(1+kB^2)^{-1}B^\alpha(F_{\alpha n,m} + F_{\alpha m,n}),
\end{aligned} \tag{3.11}$$

where $_{,i}$ denotes $\frac{\partial}{\partial x^i}$.
The curvature scalar, $R = \bar{g}^{mn}R_{mn}$ is found to be

$$\begin{aligned}
R = &- \frac{1}{4}k^2B^2(1+kB^2)^{-1}\bar{g}^{\alpha l}\bar{g}^{pi}F_{pl}F_{\alpha i} \\
&- \frac{1}{2}k\bar{g}^{mn}\bar{g}^{il}F_{nl}F_{mi} \\
&+ 2k(1+kB^2)^{-1}\bar{g}^{il}B^mF_{ml,i} \\
&- k(1+kB^2)^{-1}\bar{g}^{il}\bar{g}^{mn}(B_{n,l}B_{m,i} - B_{i,l}B_{m,n}) \\
&- \frac{1}{2}k^2(1+kB^2)^{-2}\eta^{il}B^mB^nF_{mi}F_{nl}.
\end{aligned} \tag{3.12}$$

If the two highest order terms of these equations are considered, then Einstein tensor can be written as

$$\begin{aligned}
G_{mn} &= R_{mn} - \frac{1}{2}\bar{g}_{mn}R \\
&= \frac{1}{8}k\bar{g}^{mn}\bar{g}^{pi}F_{\alpha i}F_{lp} \\
&- \frac{1}{2}k^2\bar{g}^{\alpha l}\bar{g}^{pi}F_{pl}F_{\alpha i}B_mB_n \\
&- \frac{1}{2}k\bar{g}^{il}F_{nl}F_{mi} \\
&- \frac{1}{8}kB^{-2}\bar{g}^{\alpha l}\bar{g}^{pi}F_{\alpha i}F_{lp}B_mB_n \\
&- \frac{1}{2}kB^{-2}\eta^{il}F_{\alpha i}B^{\alpha}(B_mB_{l,n} + B_nB_{l,m}) \\
&+ \frac{1}{2}k\bar{g}^{il}(F_{ml,i}B_n + F_{nl,i}B_m) \\
&- kB^{-2}\bar{g}^{il}B^{\alpha}F_{\alpha n,i}B_mB_n \\
&+ kB^{-2}\bar{g}^{il}\bar{g}^{\alpha p}(B_{\alpha,l}B_{p,i} - B_{i,l}B_{p,\alpha})B_mB_n \\
&+ \frac{1}{4}kB^{-4}\eta^{il}B^{\alpha}B^{p}F_{\alpha i}F_{pl}B_mB_n
\end{aligned} \tag{3.13}$$

Again by taking the highest order terms and by same simplifications, equation (3.12) reduces to

$$\begin{aligned}
G_{mn} &= \frac{1}{2}k^2\bar{g}^{\kappa l}\bar{g}^{rs}F_{rl}F_{s\kappa}B_iB_j \\
&+ \frac{1}{2}k(g^{\kappa l}F_{i\kappa}F_{lj} + \frac{1}{4}g_{ij}g^{\kappa l}g^{rs}F_{rl}F_{s\kappa})
\end{aligned} \tag{3.14}$$

It is accepted that the field equations of a particle under the influence of an electromagnetic field will be

$$G_{mn} = 8\pi\kappa c^{-4}(\rho_0 v_m v_n + T_{mn}), \tag{3.15}$$

where κ is the gravitational constant, ρ_0 is the proper matter density and T_{mn} is the electromagnetic energy tensor. From classical Riemannian geometry, the electromagnetic energy tensor is

$$\bar{T}_{mn} = \frac{1}{4\pi}(g^{\alpha l}F_{ml}F_{n\alpha} - \frac{1}{4}\bar{g}_{mn}\bar{g}^{i\alpha}\bar{g}^{j\beta}F_{ij}F_{\alpha\beta}). \tag{3.16}$$

If we compare the electromagnetic energy tensor (3.15) with Einstein tensor (3.12) calculated from metric (3.1), a value for the constant k can be determined as

$$k = \frac{4\kappa}{c^{-4}}. \tag{3.17}$$

By this relation, electromagnetic energy tensor, has appeared as part of Einstein tensor. And also the matter density has appeared as part of curvature. Since everything is expressed in terms of curvature tensor, electromagnetic field is completely geometrized. An important consequence of comparison of equations (3.12) and (3.15) is that the particle mass can be derived from electromagnetic field [3,11].

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