Noether invariants, nonholonomic systems and nonlinear constraints

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Abstract. The aim of the paper is to construct Noether invariants for lagrangian non-holonomic dynamics with affine or nonlinear constraints, considered to be adapted to a foliation on the base space. A set of illustrative examples are given, including linear and nonlinear Appell mechanical systems.

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A geometric setting to study nonlinear constraints for nonholonomic spaces is that of a foliation; it is motivated by [3], and used in [14], a form followed effectively in this paper.

We give a way to construct Noether invariants of nonholonomic spaces in the general cases of affine and of nonlinear constraints, using infinitesimal symmetries. The case of non-conservative Lagrangian systems of type (2.1) is given by Theorems 2.2 and 2.5, it covers the case of nonholonomic spaces with affine constraints. The general case of a dynamics given by a generalized nonconservative Lagrangian system of type (3.1) is given by Theorems 3.2 and 3.3, it covers the general case of nonholonomic spaces with nonlinear constraints. In both cases there are given some illustrative examples of infinitesimal symmetries and Noether invariants: linear and nonlinear Appell constraints, as well as the Appell-Hammel dynamic system in an elevator.

1 Symmetries of Lagrangians in a foliate setting

If $\mathcal{F}$ is a foliation on the manifold M, then we can consider a foliation $\mathcal{F}^{\mathbb{R}}$ on $\mathbb{R} \times M$, where the real parameter is added to the transverse part. Specifically, using coordinates on M, $(x^u, x^{\bar{u}})$, coordinates (x^u) are *tangent* coordinates (on the leaves of the foliation), while the coordinates $(x^{\bar{u}})$ are *transverse* coordinates. If the foliation $\mathcal{F}$ is simple, i.e. its leaves are the fibers of a submersion $f : M \longrightarrow \bar{M}$, then $(x^{\bar{u}})$ are coordinated on $\bar{M}$ and (x^u) are coordinates on the leaf $f^{-1}(x^{\bar{u}})$, giving together the coordinates $(x^u, x^{\bar{u}})$ on M. Thus the parameter $t \in \mathbb{R}$ in coordinates $(t, x^u, x^{\bar{u}})$ on $\mathbb{R} \times M$ is transverse, as $(x^{\bar{u}})$.

We follow some ideas used in [15], or, more generally, but giving the same infinitesimal result, as in [6], adapted here in the case of a regular Lagrangians and a

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nonconservative system (see also [10, Sect. 7.1]). We give here a global form for the invariant objects, using foliations.

Let $L : N\mathcal{F}^{\mathbb{R}} \to \mathbb{R}$ be a Lagrangian, where a real parameter can be (non-necessarily) involved. This Lagrangian can be obtained as $L = L' \circ C$, using a Lagrangian $L' : \mathbb{R} \times TM \longrightarrow \mathbb{R}$ and a nonlinear constraint $C : \mathbb{R} \times N\mathcal{F} \to \mathbb{R} \times TM$.

One say that the Lagrangian action of L is *invariant* under a set of an ε–parameter local group of foliate transformations

$$\text{(1.1)} \qquad \begin{cases} \bar{t}(t) = t + \varepsilon\tau(t, x^u(t), x^{\bar{u}}(t), y^{\bar{u}}(t)) + o(\varepsilon^2), \\ \bar{x}^u(t) = x^u(t) + \varepsilon\xi^u(t, x^u(t), x^{\bar{u}}(t), y^{\bar{u}}(t)) + o(\varepsilon^2), \\ \bar{x}^{\bar{u}}(t) = x^{\bar{u}}(t) + \varepsilon\xi^{\bar{u}}(t, x^u(t), x^{\bar{v}}(t), y^{\bar{v}}(t)) + o(\varepsilon^2), \end{cases}$$

if there is an other Lagrangian $\Lambda : N\mathcal{F}^{\mathbb{R}} \to \mathbb{R}$ such that

$$\text{(1.2)} \qquad L\left(\bar{t}, \bar{x}^u(\bar{t}), \bar{x}^{\bar{u}}(\bar{t}), \frac{d\bar{x}^{\bar{u}}}{d\bar{t}}(\bar{t})\right)\frac{d\bar{t}}{dt}$$
$$= L\left(t, x^u(t), x^{\bar{u}}(t), \frac{dx^{\bar{u}}}{dt}(t)\right) + \varepsilon\frac{d}{dt}\Lambda\left(t, x^u(t), x^{\bar{u}}(t), \frac{dx^{\bar{u}}}{dt}(t)\right) + o(\varepsilon^2).$$

We say that the local group is *infinitesimally exact* if the vector field

(1.3)
$$X_0 = \tau(t, x^u, x^{\bar{u}}, y^{\bar{u}})\frac{\partial}{\partial t} + \xi^u(t, x^u, x^{\bar{u}}, y^{\bar{u}})\frac{\partial}{\partial x^u} + \xi^{\bar{u}}(t, x^u, x^{\bar{u}}, y^{\bar{u}})\frac{\partial}{\partial x^{\bar{u}}} = \tau\frac{\partial}{\partial t} + X_0^{(t)},$$

called an *infinitesimal action*, is a global foliated vector field on $\pi^*_{N\mathcal{F}^{\mathbb{R}}}T(M \times \mathbb{R})$.

The *energy* of L is the Lagrangian $\mathcal{E}(L) = D_v(L) - L$, where D_v is the vertical derivation $D_v(L) = y^{\bar{v}}\dfrac{\partial L}{\partial y^{\bar{v}}}$. Let us consider the differential form $\delta(L) \in \mathcal{X}^*(N\mathcal{F}))$, given by $\delta L = dL + \mathcal{E}(L)d\tau$, called the *Cartan form* of L. Using local coordinates, we have

$$\begin{aligned} \mathcal{E}(L) &= \frac{\partial L}{\partial y^{\bar{v}}}(t, x^u, x^{\bar{u}}, y^{\bar{u}})y^{\bar{v}} - L(t, x^u, x^{\bar{u}}, y^{\bar{u}}), \\ \delta L &= \frac{\partial L}{\partial t}dt + \frac{\partial L}{\partial x^u}dx^u + \frac{\partial L}{\partial x^{\bar{u}}}dx^{\bar{u}} + \frac{\partial L}{\partial y^{\bar{u}}}dy^{\bar{u}} + \left(y^{\bar{v}}\frac{\partial L}{\partial y^{\bar{v}}} - L\right)dt., \end{aligned}$$

An *almost transverse semi-spray* S is a vector field on $\mathbb{R} \times N\mathcal{F}$ that projects on the C–Liouville-type section. Using local coordinates, S has the form

$$\text{(1.4)} \qquad S = \frac{\partial}{\partial t} + C^u\left(x^u, x^{\bar{u}}, y^{\bar{u}}\right)\frac{\partial}{\partial x^u} + y^{\bar{u}}\frac{\partial}{\partial x^{\bar{u}}} + S^{\bar{u}}(t, x^u, x^{\bar{u}}, y^{\bar{u}})\frac{\partial}{\partial y^{\bar{u}}}.$$

The integral integral curves of the vector field S are solutions of the system of differential equations

$$\begin{aligned} \frac{dx^u}{dt} &= C^u\left(x^u, x^{\bar{u}}, y^{\bar{u}}\right), \\ \frac{dx^{\bar{u}}}{dt} &= y^{\bar{u}}, \\ \frac{dx^u}{dt} &= S^{\bar{u}}(t, x^u, x^{\bar{u}}, y^{\bar{u}}). \end{aligned}$$

We denote below by $\frac{d}{dt}$ the action of S on real functions on $T(N\mathcal{F}^{\mathbb{R}})$.

We say that the Lagrangian L is *invariant up to a gauge term*, if for every almost transverse semi-spray S, corresponding to some non-linear constraints, there is an S–related vector field $X \in \mathcal{X}(N\mathcal{F}^{\mathbb{R}})$, called an *infinitesimal symmetry*, and a Lagrangian $\Lambda : N\mathcal{F}^{\mathbb{R}}$, called the *gauge term*, such that

$$\delta L(X) = S(\Lambda). \tag{1.5}$$

Proposition 1.1. *If the Lagrangian action of a regular Lagrangian on $N\mathcal{F}^{\mathbb{R}}$ is invariant under (1.1) and the local group is infinitesimally exact, then L is invariant up to a gauge term Λ, having as a infinitesimal symmetry given by an infinitesimal action.*

We have

$$\begin{aligned} & \tau\frac{\partial L}{\partial t} + \xi^u\frac{\partial L}{\partial x^u} + \xi^{\bar{u}}\frac{\partial L}{\partial x^{\bar{u}}} + \frac{d\xi^{\bar{u}}}{dt}\frac{\partial L}{\partial y^{\bar{u}}} + \frac{d\tau}{dt}\left(L - \frac{dx^{\bar{u}}}{dt}\frac{\partial L}{\partial y^{\bar{u}}}\right) \\ = & \frac{d}{dt}\Lambda\left(t, x^u(t), x^{\bar{u}}(t), \frac{dx^{\bar{u}}}{dt}(t)\right). \end{aligned} \tag{1.6}$$

Notice that the action of the operator $\frac{d}{dt} \in \mathcal{X}\left(\mathbb{R}\times N\mathcal{F}\right)$ has the form

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial}{\partial t} + C^u(x^u, x^{\bar{u}}, \frac{dx^u}{dt})\frac{\partial}{\partial x^u} + \frac{dx^{\bar{u}}}{dt}\frac{\partial}{\partial x^{\bar{u}}} + \frac{d^2x^{\bar{u}}}{dt^2}\frac{\partial}{\partial y^{\bar{u}}} \\ &= \frac{\partial}{\partial t} + C^u\frac{\partial}{\partial x^u} + y^{\bar{u}}\frac{\partial}{\partial x^{\bar{u}}} + \frac{dy^{\bar{u}}}{dt}\frac{\partial}{\partial y^{\bar{u}}}. \end{aligned} \tag{1.7}$$

The regularity conditions on Lagrangians and Lagrangian actions we consider in that follows is to verify the hypothesis of Proposition 1.1 above. Thus an *allowed Lagrangian action* corresponds to a regular Lagrangian on $N\mathcal{F}^{\mathbb{R}}$, the Lagrangian action is invariant under (1.1) and the local group is infinitesimally exact.

The existence of Λ in formula (1.5) rise the problem if it depends or not on S. If Λ comes from a local group action, as in hypothesis of Proposition 1.1, then Λ does not depend on S. In this case, taking into account the formula (1.4), the equality (1.6) implies

$$\begin{aligned} & \tau\frac{\partial L}{\partial t} + \xi^u\frac{\partial L}{\partial x^u} + \xi^{\bar{u}}\frac{\partial L}{\partial x^{\bar{u}}} + \left(\frac{\partial\xi^{\bar{u}}}{\partial t} + C^u\frac{\partial\xi^{\bar{u}}}{\partial x^u} + y^{\bar{v}}\frac{\partial\xi^{\bar{u}}}{\partial x^{\bar{v}}}\right)\frac{\partial L}{\partial y^{\bar{u}}} + \\ & \left(\frac{\partial\tau}{\partial t} + C^u\frac{\partial\tau}{\partial x^u} + y^{\bar{v}}\frac{\partial\tau}{\partial x^{\bar{v}}}\right)\left(L - \frac{dx^{\bar{u}}}{dt}\frac{\partial L}{\partial y^{\bar{u}}}\right) \\ = & \frac{\partial\Lambda}{\partial t} + C^u\frac{\partial\Lambda}{\partial x^u} + y^{\bar{v}}\frac{\partial\Lambda}{\partial x^{\bar{v}}}. \end{aligned} \tag{1.8}$$

and

$$\frac{\partial\xi^{\bar{v}}}{\partial y^{\bar{u}}}\frac{\partial L}{\partial y^{\bar{v}}} + \frac{\partial\tau}{\partial y^{\bar{u}}}\left(L - \frac{dx^{\bar{u}}}{dt}\frac{\partial L}{\partial y^{\bar{u}}}\right) = \frac{\partial\Lambda}{\partial y^{\bar{u}}}. \tag{1.9}$$

The two relations (1.8) and (1.9) are called Killing equations in the classical case [15], or [11] in the nonholonomic case.

2 The case of nonconservative Lagrangian systems

A *nonconservative Lagrangian system* has the form

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial y^{\bar{u}}} &= \frac{\partial L}{\partial x^{\bar{u}}} + Q_{\bar{u}}(t,T,x^u,x^{\bar{u}},y^{\bar{u}}), \\ \frac{dx^u}{dt} &= C^u(t,T,x^u,x^{\bar{u}},y^{\bar{u}}), \\ \frac{dT}{dt} &= \tau. \end{aligned} \tag{2.1}$$

This is the case of nonholonomic spaces with affine constraints, as studied for example in [2, 14].

But we can extend the definition above to the case when

$$Q_{\bar{u}} = Q_{\bar{u}}(t,T,x^u,x^{\bar{u}},y^{\bar{u}},Y^{\bar{u}} = \frac{dy^{\bar{u}}}{dt}).$$

The dependence can be considered by higher order derivatives, but we need here only a particular case in the next section.

Two global forms associated with a Lagrangian L are $d_vL \in \Gamma(\pi^*_{N\mathcal{F}^{\mathbb{R}}}(N^*\mathcal{F}))$ (the vertical differential) and $H_vL \in \Gamma(\pi^*_{N\mathcal{F}^{\mathbb{R}}}(N^*\mathcal{F}\otimes N^*\mathcal{F}))$ (the vertical Hessian), given by $d_vL(Z) = Z(L)$ and $H_vL(Z_1, Z_2)$, for vertical lifts Z, Z_1 and Z_2. In local coordinates,

$$d_vL = \frac{\partial L}{\partial y^{\bar{u}}}dx^{\bar{u}}, H_vL = \frac{\partial^2 L}{\partial y^{\bar{u}}\partial y^{\bar{v}}}dx^{\bar{u}} \otimes dx^{\bar{v}}.$$

If the vertical Hessian of L is non-degenerated, then L is *regular* and the nonconservative system (2.1) is said to be also *regular*.

Proposition 2.1. *If the Lagrangian L is regular, then curves on M that are solutions of a nonconservative Lagrangian system (2.1) are exactly the integral curves of an almost transverse semi-spray L (called* the canonical semispray *of the nonconservative Lagrangian system).*

In the autonomous case (i.e. $\frac{\partial L}{\partial t} = 0$), the term $\frac{\partial}{\partial t}$ does not appear in the formula (1.4) of an almost transverse semi-spray.

If S is an almost transverse semi-spray, one say that an $h \in \mathcal{F}(N\mathcal{F}^{\mathbb{R}}))$ is S–invariant if $S(h) = 0$.

Theorem 2.2. *Consider an allowed Lagrangian action that corresponds to a regular Lagrangian L on $N\mathcal{F}^{\mathbb{R}}$, let S be the canonical semi-spray of a nonconservative Lagrangian system (2.1) and, supplementary, let us suppose that there is an $f \in \mathcal{F}(N\mathcal{F}^{\mathbb{R}})$ such that*

$$\frac{df}{dt} = Q_{\bar{u}}\left(\xi^{\bar{u}} - \tau y^{\bar{u}}\right) - \left(\xi^u - \tau C^u\right)\frac{\partial L}{\partial x^u}, \tag{2.2}$$

where Γ is the Liouville vector field. Then the function $h = \tau\mathcal{E}(L) - d_vL(\xi) + \Lambda + f$ is an S–invariant.

In the case $Q = 0$, one recover the case of a Lagrangian system.

Corollary 2.3. *Consider an allowed Lagrangian action that corresponds to a regular Lagrangian on $N\mathcal{F}^{\mathbb{R}}$, let S be the canonical semi-spray of the Lagrangian system and let us suppose that there is an $f \in \mathcal{F}(N\mathcal{F}^{\mathbb{R}})$ such that*

$$\frac{df}{dt} = (\tau C^u - \xi^u) \frac{\partial L}{\partial x^u}. \tag{2.3}$$

Then the function $h = \tau\mathcal{E}(L) - d_v L(\xi) + \Lambda + f$ is an S–invariant.

Corollary 2.4. *Consider an allowed Lagrangian action that corresponds to a regular Lagrangian on $N\mathcal{F}^{\mathbb{R}}$, let S be the canonical semi-spray of the Lagrangian system, let us suppose that the infinitesimal symmetry is compatible with constraints and also $\tau = 1$. Then the function $h = \mathcal{E}(L) - d_v L(\xi) + \Lambda$ is an S–invariant.*

The case when $L : N\mathcal{F} \to \mathbb{R}$, or $L : \widetilde{N\mathcal{F}} \to \mathbb{R}$, is the case when the Lagrangian function does not depend on the parameter t. We look now to some special situations below.

But the existence of a function f is not always possible, even locally (see below, in an example at the very end of this section). Thus we state the following variant of Theorem 2.2, that is true in this case.

Theorem 2.5. *Consider an allowed Lagrangian action that corresponds to a regular Lagrangian L on $N\mathcal{F}^{\mathbb{R}}$, let S be the canonical semi-spray of a nonconservative Lagrangian system (2.1). Then*

$$\frac{d}{dt}\left(\tau\mathcal{E}(L) - d_v L(\xi) + \Lambda\right) + Q_{\bar{u}}\left(\xi^{\bar{u}} - \tau y^{\bar{u}}\right) - \left(\xi^u - \tau C^u\right)\frac{\partial L}{\partial x^u} = 0. \tag{2.4}$$

Corollary 2.3 have the following general form:

Corollary 2.6. *Consider an allowed Lagrangian action that corresponds to a regular Lagrangian L on $N\mathcal{F}^{\mathbb{R}}$, let S be the canonical semi-spray of the Lagrangian system given by L. Then*

$$\frac{d}{dt}\left(\tau\mathcal{E}(L) - d_v L(\xi) + \Lambda\right) + \left(\tau C^u - \xi^u\right)\frac{\partial L}{\partial x^u} = 0. \tag{2.5}$$

As an example, we consider the *linear Appell constraints* as in [14, Example 5.1] (see also, for example, [13]). The manifold is $M = \mathbb{R}^3 \times T^2$ and the foliation is the simple foliation defined by the fibers of the canonical projection $\mathbb{R}^3 \times T^2 \to T^2$. Consider the coordinates (x^1, x^2, x^3) on $\mathbb{R}^3$ and $(x^{\bar{1}}, x^{\bar{2}})$ on T^2. The linear Appell constraints are given by the formulas

$$C^1 = Ry^{\bar{1}} \cos x^{\bar{2}}, C^2 = Ry^{\bar{1}} \sin x^{\bar{2}}, C^3 = ry^{\bar{1}}. \tag{2.6}$$

The Lagrangian is

$$L = \frac{1}{2}\alpha\left(\left(y^1\right)^2 + \left(y^2\right)^2\right) + \frac{1}{2}\beta\left(y^3\right)^2 + \frac{1}{2}I_1\left(y^{\bar{1}}\right)^2 + \frac{1}{2}I_2\left(y^{\bar{2}}\right)^2 + \gamma x^3.$$

and the constraints are given by (2.6). The induced Lagrangian has the form

$$\begin{aligned} L_c(x^1, x^2, x^3, x^{\bar{1}}, x^{\bar{2}}, y^{\bar{1}}, y^{\bar{2}}) &= \frac{1}{2}\left(I_1 + \alpha R^2 + \beta r^2\right)\left(y^{\bar{1}}\right)^2 + \frac{1}{2}I_2\left(y^{\bar{2}}\right)^2 + \gamma x^3 \\ &= \frac{1}{2}\alpha''\left(y^{\bar{1}}\right)^2 + \frac{1}{2}I_2\left(y^{\bar{2}}\right)^2 + \gamma x^3, \end{aligned}$$

An infinitesimal symmetry is given by $\xi^{\bar{u}} = y^{\bar{u}}$, $\xi^3 = 0$, $\tau = \tau_0 = const.$ and $\Lambda = \frac{1}{2}\alpha'' \left(y^{\bar{1}}\right)^2 + \frac{1}{2}I_2 \left(y^{\bar{2}}\right)^2$, since it verifies Killing relations (1.8) and (1.9), thus also (1.6). We have $Q_{\bar{u}} = \gamma r \delta_{\bar{u}\bar{1}}$, thus the equation (2.2) has the form

$$\frac{df}{dt} = \gamma r \delta_{\bar{u}\bar{1}} y^{\bar{u}}(1-\tau) + \gamma r \tau y^{\bar{1}} = r\gamma y^{\bar{1}}.$$

But $y^{\bar{1}} = \frac{dx^{\bar{1}}}{dt}$, thus one can take $f = r\gamma x^{\bar{1}} + c$ and we can use Theorem 2.2. It follows that $h = -(1+\tau_0)\Lambda + \tau_0 x^3 + r\gamma x^{\bar{1}} + c$ is an invariant along the integral curves of the linear Appell constraints system, where c is a real constant.

Another infinitesimal symmetry is given by $\xi^{\bar{u}} = \frac{1}{2}x^{\bar{u}}$, $\xi^3 = -x^3$, $\tau = t$ and $\Lambda = 0$, since relation (1.6) holds; relations (1.8) and (1.9) also hold, thus this infinitesimal symmetry is a Killing one.

The equation (2.2) has the form

$$\frac{df}{dt} = \frac{r\gamma}{2}x^{\bar{1}} + \gamma x^3.$$

Since $x^{\bar{1}}$ and x^3 can not be in the form $\frac{dg}{dt}$, in this case we can not find a global function f to satisfy the above relation, as in the case of the previous symmetry. In this case, we can not use Theorem 2.2, but Theorem 2.5. Since $\tau\mathcal{E}(L) - d_v L(\xi) + \Delta = (t-1)\,\Delta - t\gamma x^3$,where $\Delta = \frac{1}{2}\alpha'' \left(y^{\bar{1}}\right)^2 + \frac{1}{2}I_2 \left(y^{\bar{2}}\right)^2$, it follows that along the integral curves of the linear Appell constraints system, we have

$$\frac{d}{dt}\left((t-1)\,\Lambda - t\gamma x^3\right) + \frac{r\gamma}{2}x^{\bar{1}} + \gamma x^3 = 0.$$

3 The case of generalized nonconservative Lagrangian systems

A *generalized nonconservative Lagrangian system* is a dynamic system of the form

$$(3.1)\qquad \begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial y^{\bar{u}}} &= \frac{\partial L}{\partial x^{\bar{u}}} + b_{\bar{u}\bar{v}}(t,T,x^u,x^{\bar{u}},y^{\bar{u}})\frac{dy^{\bar{v}}}{dt} + Q_{\bar{u}}(t,T,x^u,x^{\bar{u}},y^{\bar{u}}),\\ \frac{dx^u}{dt} &= C^u(t,T,x^u,x^{\bar{u}},y^{\bar{u}}),\\ \frac{dT}{dt} &= \tau. \end{aligned}$$

We say that the Lagrangian L is *quasi-regular* if the matrix $\left(\frac{\partial^2 L}{\partial y^{\bar{u}}\partial y^{\bar{v}}} - b_{\bar{u}\bar{v}}\right)$ has an inverse in every point.

Notice that the definition of a generalized nonconservative Lagrangian system can be extended by replacing, in the first relation (3.1), $b_{\bar{u}\bar{v}}(t,T,x^u,x^{\bar{u}},y^{\bar{u}})\frac{dy^{\bar{v}}}{dt} + Q_{\bar{u}}(t,T,x^u,x^{\bar{u}},y^{\bar{u}})$ by $Q_{\bar{u}}(t,T,x^u,x^{\bar{u}},y^{\bar{u}},\frac{dy^{\bar{u}}}{dt})$ and then, the non-singularity of the matrix $\left(\dfrac{\partial^2 L}{\partial y^{\bar{u}} \partial y^{\bar{v}}} - b_{\bar{u}\bar{v}}\right)$ be replaced by the non-singularity of the matrix $\left(\dfrac{\partial^2 L}{\partial y^{\bar{u}} \partial y^{\bar{v}}} - \dfrac{\partial Q_{\bar{u}}}{\partial y^{\bar{v}}}\right)$. But we do not need this general case in this paper, since the case of nonlinear constraints for nonholonomic fit in the case we study here.

An analogous of Proposition 3.1 is the following result.

Proposition 3.1. *If the Lagrangian L is quasi-regular, then the curves on M that are solutions of a generalized nonconservative Lagrangian system (3.1) are exactly the integral curves of an almost transverse semisprayy/ L (called* the canonical semispray *of the generalized nonconservative Lagrangian system).*

Theorem 2.2 extends in the following way, having a similar conclusion.

Theorem 3.2. *Consider an allowed Lagrangian action that corresponds to a regular Lagrangian L on $N\mathcal{F}^{\mathbb{R}}$, let S be the canonical semi-spray of a generalized nonconservative Lagrangian system (3.1) and, supplementary, let us suppose that there is an $f \in \mathcal{F}(N\mathcal{F}^{\mathbb{R}})$ such that*

$$\frac{df}{dt} = \left(Q_{\bar{u}} + b_{\bar{u}\bar{v}} S^{\bar{v}}\right)\left(\xi^{\bar{u}} - \tau y^{\bar{u}}\right) - \frac{\partial L}{\partial x^u}\left(\xi^u - \tau C^u\right). \tag{3.2}$$

Then the function $h = \tau \mathcal{E}(L) - d_v L(\xi) + \Lambda + f$ is an S–invariant.

Similarly, Theorem 2.5 extends in the same way, as follows.

Theorem 3.3. *Consider an allowed Lagrangian action that corresponds to a regular Lagrangian L on $N\mathcal{F}^{\mathbb{R}}$, let S be the canonical semi-spray of a generalized nonconservative Lagrangian system (3.1). Then along its integral curves, we have:*

$$\frac{d}{dt}\left(\tau \mathcal{E}(L) - d_v L(\xi) + \Lambda\right) + \left(Q_{\bar{u}} + b_{\bar{u}\bar{v}} S^{\bar{v}}\right)\left(\xi^{\bar{u}} - \tau y^{\bar{u}}\right) - \frac{\partial L}{\partial x^u}\left(\xi^u - \tau C^u\right) = 0. \tag{3.3}$$

As an example, we consider the *nonlinear Appell constraints* as in [14, Example 5.2] (see also, for example, [13]). The manifold is $M = \mathbb{R}^3$ and the foliation is the simple foliation defined by the fibers of the canonical projection on the first two coordinates, $\mathbb{R}^3 \to \mathbb{R}^2$, $\left(x^{\bar{1}}, x^{\bar{2}}, x^1\right) \to \left(x^{\bar{1}}, x^{\bar{2}}\right)$. The nonlinear Appell constraint is given by the formula

$$C^1 = \alpha \sqrt{\left(y^{\bar{1}}\right)^2 + \left(y^{\bar{2}}\right)^2}. \tag{3.4}$$

The Lagrangian is

$$L = \frac{\beta}{2}\left(\left(y^{\bar{1}}\right)^2 + \left(y^{\bar{2}}\right)^2\right) + \frac{\gamma}{2}\left(y^1\right)^2 + \delta x^1 \tag{3.5}$$

and the induced Lagrangian has the form

$$L_c(x^1, x^2, x^3, x^{\bar{1}}, x^{\bar{2}}, y^{\bar{1}}, y^{\bar{2}}) = \frac{\alpha'}{2}\left(\left(y^{\bar{1}}\right)^2 + \left(y^{\bar{2}}\right)^2\right) + \delta x^1,$$

where $\alpha' = \dfrac{\beta + \alpha^2 \gamma}{2}$.

An infinitesimal symmetry is given by $\xi^{\bar{u}} = y^{\bar{u}}$, $\xi^1 = 0$, $\tau = \tau_0 (= const.)$ and $\Lambda = \dfrac{\alpha'}{2}\left(\left(y^{\bar{1}}\right)^2 + \left(y^{\bar{2}}\right)^2\right)$. We have $Q_{\bar{u}} = \frac{\partial C^1}{\partial y^{\bar{u}}}\frac{\partial L}{\partial x^1} = \frac{\delta \alpha y^{\bar{u}}}{\sqrt{\Delta}}$.

The equation (3.2) has the form

$$\frac{df}{dt} = \delta C^1,$$

thus we can take $f = \delta x^1 + c$, where c is a real constant, since $\frac{dx^1}{dt} = C^1$. Thus $h = \tau_0(\Lambda - \delta x^1) - 2\Lambda + \Lambda + \delta x^1 + c = (1-\tau_0)(\delta x^1 - \Lambda) + c$, or

$$h = (1-\tau_0)(\delta x^1 - \Lambda) + c. \tag{3.6}$$

We obtain a non-trivial invariant for $\tau_0 \neq 1$.

A classical example of time dependent nonlinear constraint is the *Appell-Hammel dynamic system in an elevator* considered in [9], having the time dependent constraints

$$\alpha^2\left(\left(y^{\bar{1}}\right)^2 + \left(y^{\bar{2}}\right)^2\right) - \left(y^1 - v^0(t)\right)^2 = 0. \tag{3.7}$$

It is easy to see that the above Appell example corresponds to the particular case when $v^0(t) = 0$.

We take $C^1\left(t, y^{\bar{1}}, y^{\bar{2}}\right) = v^0(t) + \alpha\sqrt{\left(y^{\bar{1}}\right)^2 + \left(y^{\bar{2}}\right)^2}$ and the Lagrangian (3.5), as in the case of a nonlinear Appell system, a particular case of this example, when $v^0(t) = 0$. The induced Lagrangian in this case is

$$\begin{aligned} & L_c\left(x^1, x^{\bar{1}}, x^{\bar{2}}, y^{\bar{1}}, y^{\bar{2}}\right) \\ = \; & \frac{\beta + \alpha^2\gamma}{2}\left(\left(y^{\bar{1}}\right)^2 + \left(y^{\bar{2}}\right)^2\right) + \gamma v^0\sqrt{\left(y^{\bar{1}}\right)^2 + \left(y^{\bar{2}}\right)^2} + \delta x^1 + \frac{1}{2}\left(v^0\right)^2, \end{aligned}$$

and the pseudo-curvature is $R_V = \dfrac{\partial L}{\partial y^u}\left[C_V, \left[C_V, \dfrac{\partial}{\partial y^{\bar{u}}}\right]\right]^u = 0$.

Let us denote $\Delta = \left(y^{\bar{1}}\right)^2 + \left(y^{\bar{2}}\right)^2$ and $\alpha'' = \beta + \alpha^2\gamma$, thus the induced Lagrangian has the form

$$L_c = \frac{\alpha''}{2}\Delta + \gamma v^0\sqrt{\Delta} + \delta x^1 + \frac{1}{2}\left(v^0\right)^2.$$

An infinitesimal symmetry is given by $\xi^{\bar{u}} = \tau_0 y^{\bar{u}}$, $\xi^1 = \left(u - \dfrac{\tau_0}{\delta}\right)v^0\dot{v}^0$, $\tau = \tau_0 =$ *const.*, $u =$ *const.* and $\Lambda = \tau_0\left(\dfrac{\alpha''\Delta}{2} + \gamma v^0\sqrt{\Delta}\right) + u\frac{\left(h^0\right)^2}{2}$.

We have $Q_{\bar{u}} = \dfrac{\partial C^1}{\partial y^{\bar{u}}}\dfrac{\partial L_c}{\partial x^1} + \dfrac{\partial^2 C^u}{\partial t \partial y^{\bar{u}}}\dfrac{\partial L}{\partial y^u} = \dfrac{\alpha\delta y^{\bar{u}}}{\sqrt{\Delta}}$, $S^{\bar{u}} = \dfrac{\left(a\delta + \gamma\dot{v}^0\right)y^{\bar{u}}}{\sqrt{\Delta}}$,$b_{\bar{u}\bar{v}} = \dfrac{\partial L}{\partial y^1}\dfrac{\partial^2 C^1}{\partial y^{\bar{u}}\partial y^{\bar{v}}}$, $b_{\bar{u}\bar{v}}S^{\bar{u}} = 0$, and the equation (3.2) has the form

$$\begin{aligned} \frac{df}{dt} &= -\delta\left(\left(u - \frac{\tau_0}{\delta}\right)v^0\dot{v}^0 - \tau_0 C^1\right) + \left(\tau_0^2 - \tau_0\right)\left(\gamma\dot{v}^0\sqrt{\Delta} + v^0\dot{v}^0\right) \\ &= v^0\dot{v}^0\left(\tau_0^2 - \delta u\right) + \tau_0\delta C^1 \end{aligned}$$

$= \frac{d}{dt}\left(\frac{\left(v^0\right)^2}{2}\left(\tau_0^2 - \delta u\right) + \tau_0 \delta x^1\right)$, thus we can take

$$f = \frac{\left(v^0\right)^2}{2}\left(\tau_0^2 - \delta u\right) + \tau_0 \delta x^1 + c,$$

where c is a real constant. We have $\mathcal{E}(L) = y^{\bar{u}}\frac{\partial L}{\partial y^{\bar{u}}} - L = y^{\bar{u}}\left(\alpha'' y^{\bar{u}} + \gamma v^0 \frac{y^{\bar{u}}}{\sqrt{\Delta}}\right) - \frac{\alpha''\Delta}{2} - \gamma v^0\sqrt{\Delta} - \delta x^1 - \frac{\left(v^0\right)^2}{2} = \frac{\alpha''\Delta}{2} - \delta x^1 - \frac{\left(v^0\right)^2}{2}$

Thus $h = \frac{\alpha''\Delta}{2} - \delta x^1 - \frac{\left(v^0\right)^2}{2} - y^{\bar{u}}\left(\alpha'' y^{\bar{u}} + \gamma v^0 \frac{y^{\bar{u}}}{\sqrt{\Delta}}\right) + \frac{\alpha''\Delta}{2} + \gamma v^0\sqrt{\Delta} + \frac{\left(v^0\right)^2}{2}\left(\tau_0^2 - \delta u\right) + \tau_0\delta x^1 + c = \frac{\left(v^0\right)^2}{2}\left(\tau_0^2 - 1 - \delta u\right) + \left(\tau_0 - 1\right)\delta x^1 + c.$

We consider now the case when $v^0(t) = v_0^0 t + c_0$, where v_0^0 and c_0 are real constants. An infinitesimal symmetry is given by $\xi^{\bar{u}} = y^{\bar{u}}$, $\xi^1 = \frac{\gamma}{\delta} v_0^0 \sqrt{\Delta}$, $\tau = 0$ and $\Lambda = \frac{\alpha''\Delta}{2} + \gamma v^0\sqrt{\Delta}$, where $\Delta = \left(y^{\bar{1}}\right)^2 + \left(y^{\bar{2}}\right)^2$. We have $Q_{\bar{u}} = \frac{\alpha\delta y^{\bar{u}}}{\sqrt{\Delta}}$ and the equation (3.2) has the form

$$\begin{aligned}
\frac{df}{dt} &= Q_{\bar{u}}\xi^{\bar{u}} - \xi^u \frac{\partial L_c}{\partial x^u} = \frac{\alpha\delta y^{\bar{u}}}{\sqrt{\Delta}} y^{\bar{u}} - \delta\frac{\gamma}{\delta} v_0^0\sqrt{\Delta} = \left(\alpha\delta - \gamma v_0^0\right)\sqrt{\Delta} \\
&= \frac{\alpha\delta - \gamma v_0^0}{\alpha}\left(v^0 + \alpha\sqrt{\Delta}\right) - \frac{\left(\alpha\delta - \gamma\right)}{\alpha}\left(v_0^0 t + c_0\right) \\
&= \frac{d}{dt}\left(\frac{\alpha\delta - \gamma v_0^0}{\alpha}\left(x^1 - v_0^0\frac{t^2}{2} - c_0 t\right) + c\right)
\end{aligned}$$

thus we can take $f = \delta'\left(x^1 - v_0^0\frac{t^2}{2} - c_0 t\right) + c_1$, where c is a real constant and $\delta' = \frac{\delta\alpha - \gamma v_0^0}{\alpha}$. Thus

$h = -\alpha''\sqrt{\Delta} - \gamma v^0\sqrt{\Delta} + \frac{\alpha''\Delta}{2} + \gamma v^0\sqrt{\Delta} + \delta'\left(x^1 - v_0^0\frac{t^2}{2} - c_0 t\right) + c_1 = -\frac{\alpha''\Delta}{2} + \delta' x^1 - \delta'\left(v_0^0\frac{t^2}{2} + c_0 t\right) + c_1.$

For $v_0^0 = c_0 = 0$, we obtain a non-trivial invariant, similar to the invariant (3.6) of a nonlinear Appell system, obtained previously.

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