On the Riemann curvature on the product of two spheres

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Abstract. Let S^2 be a unit sphere in R^3 and $S^2 \times S^2$ be a product Manifold. We study the Riemann structure on the product manifold $S^2 \times S^2$, and give some result, on the sectional curvature on $S^2 \times S^2$ and we are going to prove a famous problem in Riemannian manifold to prove the existence of Riemannian metric on $S^2 \times S^2$ with positive sectional curvature.

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1 Introduction

We recall some definitions and fundamental results in Riemannian manifolds.

Definition 1.1. A Riemannian metric on a smooth manifold M is a 2-tensor field $g \in \tau^2(M)$ that is symmetric (i.e.,$g(X,Y) = g(Y,X)$) and positive definite (i.e. $g(X,X)0$ if $X \neq 0$). A Riemannian metric thus determines an inner product on each tangent space T_pM, which is typically written $<X,Y> := g(X,Y)$ for all $X,Y \in T_pM$. A manifold together with a given Riemannian metric is called a Riemannian manifold. We often use the word metric to refer to a Riemannian metric when there is no chance of confusion. We denote a Riemannian manifold by (M,g).

Proposition 1.1. *(An atlas for a product manifold). If $\{(U_\alpha, x_\alpha)\}$ and $\{(V_\beta, y_\beta)\}$ are C^∞ atlases for the manifolds M and N of dimensions m and n, respectively, then the collection $\{(U_\alpha \times V_\beta, x_\alpha \times y_\beta)\}$ of charts is a C^∞ atlas on $M \times N$. Therefore,$M \times N$ is a C^∞ manifold of dimension mn.[13]*

Definition 1.2. If (M_1, g_1) and (M_2, g_2) be arbitrary Riemannian manifolds, then we can define a natural Riemannian metric $g = g_1 \oplus g_2$ called the product metric, defined by $g(X_1 + X_2, Y_1 + Y_2) = g_1(X_1, Y_1) + g_2(X_2, Y_2)$, where $X_i, Y_i \in T_{p_i}M_i$ under the natural identification $T_{(p_1,p_2)}M_1 \times M_2 = T_{p_1}M_1 \oplus T_{p_2}M_2$. Local coordinates $(x^1, ..., x^n)$ for M_1 and $(x^{n+1}, ..., x^{n+m})$ for M_2 gives the coordinates $(x^1, ..., x^{n+m})$ for $M_1 \times M_2$. In terms of these coordinates, the product metric has the local expression,

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$g = \sum_{i,j=1}^{n+m} g_{ij} dx^i dx^j$, where (g_{ij}) is the block diagonal matrix

$$(g_{ij}) = \begin{pmatrix} (g_1)_{ij} & 0 \\ 0 & (g_2)_{ij} \end{pmatrix}.$$

It is important in Riemannian manifold to study the relationship between curvature and metric structures. In this section we give a Riemannian metric on S^2 and consider the curvature tensor on S^2 ; in particular we compute sectional curvature on S^2. So we review and discuss some important preliminaries.Let $M = S^2$, (unit sphere) we define a local coordinate (differentiable structure) with spherical coordinate on $M = S^2$.

Let $X_i : U_i \subseteq R^2 \to S^2$ $(i = 1, 2, 3)$ be defined as follows:

$$X_1(\theta, \varphi) = \big(sin(\theta)cos(\varphi), sin(\theta)sin(\varphi), cos(\theta)\big),$$

$$U_1 = \{(\theta, \varphi) \in R^2 | 0 < \theta < \pi, 0 < \varphi < 2\pi\},$$

$$X_2(\theta, \varphi) = \big(sin(\theta)cos(\varphi), sin(\theta)sin(\varphi), cos(\theta)\big),$$

$$U_2 = \{(\theta, \varphi) \in R^2 | 0 < \theta < \pi, -\pi < \varphi < \pi\},$$

$$X_3(\theta, \varphi) = \big(cos(\theta), sin(\theta)cos(\varphi), sin(\theta)sin(\varphi)\big),$$

$$U_3 = \{(\theta, \varphi) \in R^2 \mid \ 0 < \theta < \pi, \ \ 0 < \varphi < 2\pi\}.$$

Hence we have a differentiable structure on S^2. The relevant Riemannian metrics is $ds^2 = d\theta^2 + sin^2(\theta)d\varphi^2$.

Theorem 1.2. *With the C^∞ structure $\{(X_i, U_i)\}_{i=1}^3$ and Riemannian metric $ds^2 = d\theta^2 + sin^2(\theta)d\varphi^2$,on S^2, the sectional curvature of S^2 is positive constant, $k(\sigma) = 1$.*

Proof. Let $\left\{X_\theta = \dfrac{\partial X}{\partial \theta}, X_\varphi = \dfrac{\partial X}{\partial \varphi}\right\}$ be a base of T_pS^2, the tangent space of S^2 at $p \in S^2$. We have $g_{11} = g_{\theta\theta} = 1$, $g_{12} = g_{\theta\varphi} = g_{\varphi\theta} = g_{21} = 0$ and $g_{22} = g_{\varphi\varphi} = sin^2(\theta)$. We compute Christoffel symbols, Riemannian curvature and sectional curvature of S^2. We have:

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & sin^2(\theta) \end{pmatrix} \qquad and \qquad (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{sin^2(\theta)} \end{pmatrix}$$

Now from

$$\gamma^i_{jk} := \frac{1}{2} g^{is} \big(\frac{\partial g_{ks}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{sj}}{\partial x^k}\big)$$

, we obtain the component of γ^i_{jk} as follows: $\gamma^1_{22} = -\dfrac{1}{2} sin(2\theta)$, $\gamma^2_{12} = \gamma^2_{21} = cot(\theta)$ and other terms vanishes. By computing the component terms of the curvature R in (X, U) from the formula [14]:

$$R^s_{ijk} = \frac{\partial \gamma^s_{ik}}{\partial x^j} - \frac{\partial \gamma^s_{ij}}{\partial x^k} + \sum_l \gamma^l_{ik} \gamma^s_{lj} - \sum_l \gamma^l_{ij} \gamma^s_{lk} \quad ; \qquad 1 \le i, j, k, s \le 2,$$

we get $R^1_{221} = -sin^2(\theta), R^1_{212} = sin^2(\theta), R^2_{121} = 1, R^2_{112} = -1$ and other terms, vanishes. Then we have [7],

$$R_{ijkl} = \sum_l R^l_{iks} g_{jl},$$

where,

$$R_{1212} = -sin^2(\theta), R_{1221} = sin^2(\theta), R_{2121} = -sin^2(\theta), R_{2112} = sin^2(\theta)$$

and other terms vanishes. Now, we compute the sectional curvature $k(\sigma)$ of S^2. Let $< X, Y >$ be the 2-dimensional subspace of T_pS^2 then

$$k(\sigma) = \frac{g(R(X,Y)Y, X)}{g(X,X)g(Y,Y) - \big[g(X,Y)\big]^2}.$$

Suppose that $\chi = u^i \dfrac{\partial}{\partial X_i}$, $Y = v^j \dfrac{\partial}{\partial X_j}$, $g_{ij} = g(\dfrac{\partial}{\partial X_i}, \dfrac{\partial}{\partial X_j})$ then we get

$$k(\sigma) = \frac{R_{ijks} u^i u^s v^j v^k}{[u^i u^j g_{ij}][v^i v^j g_{ij}] - [g_{ij} u^i v^j]^2}$$

$$= \frac{[u^1 v^2 - u^2 v^1]^2 sin^2(\theta)}{[(u^1)^2 + (u^2 sin(\theta))^2][(v^1)^2 + (v^2 sin(\theta))^2] - [u^1 v^1 + u^2 v^2 sin^2(\theta)]^2}.$$

Now we simplify the denominator as follows:

$$[(u^1)^2 + (u^2 sin(\theta))^2][(v^1)^2 + (v^2 sin(\theta))^2] - [u^1 v^1 + u^2 v^2 sin^2(\theta)]^2$$

$$= (u^1 v^1)^2 + (u^1 v^2 sin(\theta))^2 + (u^2 v^1 sin(\theta))^2 + (u^2 v^2 sin^2(\theta))^2$$

$$-[(u^1 v^1)^2 + 2u^1 v^1 u^2 v^2 sin^2(\theta) + (u^2 v^2 sin^2(\theta))^2]$$

$$= (u^1 v^2 - u^2 v^1)^2 sin^2(\theta).$$

Hence, $k(\sigma) = 1$. □

2 The sectional curvature on $S^2 \times S^2$

Since $\{(X_i, U_i)\}_{i=1}^3$ is a differential structure on S^2, $\{(X_i \times X_j, U_i \times U_j)\}_{i,j=1}^3$ is a differential structure on $S^2 \times S^2$ [13].

Definition 2.1. Let (M_1, g_1), (M_2, g_2) be arbitrary Riemannian manifolds and $M = M_1 \times M_2$. We define a natural Riemannian metric as a product metrics by direct sum of two metrics $g = g_1 \oplus g_2$ as follows:
$g_{(p_1,p_2)}(X_1+X_2, Y_1+Y_2) = (g_1)_{p_1}(X_1, Y_1)+(g_2)_{p_2}(X_2, Y_2) \quad \forall X_i, Y_i \in T_{p_i}M_i; i = 1, 2.$
also we get the tangent space on $M = M_1 \times M_2$ as a direct sum of $T_{p_1}M_1$, and $T_{p_2}M_2$. Let $T_{(p_1,p_2)}M = T_{p_1}M_1 \oplus T_{p_2}M_2$. If $(x^1, \ldots, x^n)$, $(x^{n+1}, \ldots, x^{n+m})$ is the local coordinate on M_1, M_2, respectively. Then the standard local coordinate on

product manifold $M = M_1 \times M_2$ can be show by $(x^1, \dots, x^{n+m})$. In this system of local coordinates, the product metric is defined by:

$$g = \sum_{i,j=1}^{n+m} g_{ij} dx_i \otimes dx_j.$$

The matrix of components of the above product metric is given by:

$$(g_{ij}) = \begin{pmatrix} (g_1)_{ij} & 0 \\ 0 & (g_1)_{ij} \end{pmatrix}.$$

Corollary 2.1. *If $M_1 = M_2 = S^2$ and $g_1 = g_2 = ds^2 = d\theta^2 + sin^2(\theta) d\varphi^2$ then the product manifold $S^2 \times S^2$ with the product metric $g = g_1 \oplus g_2 = ds^2 \oplus ds^2$ is a Riemannian manifold [9].*

Theorem 2.2. *Product Riemannian manifold $(S^2 \times S^2, g); g = g_1 \oplus g_2 = ds^2 \oplus ds^2 = d\theta_1^2 + sin^2(\theta_1) d\varphi_1^2 + d\theta_2^2 + sin^2(\theta_2) d\varphi_2^2$ has positive sectional curvature.*

Proof. It is easy to show that the matrix of the product metric

$$g = \sum_{i,j=1}^{n+m} g_{ij} dx^i \otimes dx^j.$$

is:

$$(g_{ij}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & sin^2(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & sin^2(\theta_2) \end{bmatrix}.$$

Obviously, $det(g_{ij}) = sin^2(\theta_1) sin^2(\theta_2)$. Of course for any charts $\{(X_i \times X_j, U_i \times U_j)\}_{i,j=1}^3$ on $S^2 \times S^2$, we have: $\forall \theta; sin(\theta) \neq 0, therefordet(g_{ij}) > 0$. It follows that g_{ij} is inverseable. Hence (g^{ij}) is equal to

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{sin^2(\theta_1)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{sin^2(\theta_2)} \end{bmatrix}$$

we take $x^1 = \theta_1$, $x^2 = \varphi_1$, $x^3 = \theta_2$, $x^4 = \varphi_2$, we get the sectional curvature on $S^2 \times S^2$ in 4-steps.In the first step compute the Christoffel symbols γ^i_{jk}.

$$\gamma^i_{jk} = \frac{1}{2} g^{is} \left(\frac{\partial g_{sj}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^s} + \frac{\partial g_{ks}}{\partial x^j} \right)$$

$$\gamma^1_{22} = -\frac{sin(2\theta_1)}{2},$$
$$\gamma^2_{12} = \gamma^2_{21} = cot(\theta_1),$$
$$\gamma^3_{44} = -\frac{sin(2\theta_2)}{2}$$
$$\gamma^4_{34} = \gamma^4_{43} = cot(\theta_2)$$

and the other terms vanishes.
Second step : computing the components of the curvature R

$$R^s_{ijk}=\frac{\partial\gamma^s_{ik}}{\partial x^j}-\frac{\partial\gamma^s_{jk}}{\partial x^i}+\sum_l\gamma^l_{ik}\gamma^s_{lj}-\sum_l\gamma^l_{jk}\gamma^s_{ik};\qquad 1\le i,j,k,s\le 4,$$

$$\begin{array}{lll} R^1_{122}=-sin^2(\theta_1) & , & R^1_{212}=sin^2(\theta_1),\\ R^2_{121}=1 & , & R^2_{211}=-1,\\ R^3_{344}=-sin^2(\theta_2) & , & R^3_{434}=sin^2(\theta_2),\\ R^4_{343}=1 & , & R^4_{433}=-1,\end{array}$$

and the other terms vanishes.
Third step: computing

$$R_{ijkl}=\sum_s R^s_{ijk}g_{ls}$$

.

$$\begin{array}{lll} R_{1221}=-sin^2(\theta_1) & , & R_{2121}=sin^2(\theta_1),\\ R_{1212}=sin^2(\theta_1) & , & R_{2112}=-sin^2(\theta_1),\\ R_{3443}=-sin^2(\theta_2) & , & R_{4343}=sin^2(\theta_2),\\ R_{3434}=sin^2(\theta_2) & , & R_{4334}=-sin^2(\theta_2),\end{array}$$

and other terms vanishes.

Fourth step:Let
$u=u^1\oplus u^2\in T_(p_1,p_2)(S^2\times S^2)=T_{\big((\theta_1,\varphi_1),(\theta_2,\varphi_2)\big)}(S^2\times S^2)$,
$v=v^1\oplus v^2\in T_(p_1,p_2)(S^2\times S^2)=T_{\big((\theta_1,\varphi_1),(\theta_2,\varphi_2)\big)}(S^2\times S^2)$.

Where $u^1=(u^1_1,u^1_2),u^2=(u^2_1,u^2_2),v^1=(v^1_1,v^1_2),v^2=(v^2_1,v^2_2)$ and $\sigma=<u,v>$ be the 2- dimensional subspace of $T_(p_1,p_2)(S^2\times S^2)$ is generated by u,v. Then

$$k(\sigma)=\frac{g(R(u,v)v,u)}{g(u,u)g(v,v)-\big[g(u,v)\big]^2}=\frac{v_j(u_iR_{ijkl}u_l)v_k}{g(u,u)g(v,v)-\big[g(u,v)\big]^2}.$$

where $u_1=u^1_1,u_2=u^1_2,u_3=u^2_1,u_4=u^2_2$ and $v_1=v^1_1,v_2=v^1_2,v_3=v^2_1,v_4=v^2_2$.

Let $u=u^1_1\frac{\partial}{\partial\theta_1}+u^1_2\frac{\partial}{\partial\varphi_1}+u^2_1\frac{\partial}{\partial\theta_2}+u^2_2\frac{\partial}{\partial\varphi_2}$ and $v=v^1_1\frac{\partial}{\partial\theta_1}+v^1_2\frac{\partial}{\partial\varphi_1}+v^2_1\frac{\partial}{\partial\theta_2}+v^2_2\frac{\partial}{\partial\varphi_2}$. then we have,

$$g(u,u)=g_1(u^1,u^1)+g_2(u^2,u^2)=\big[(u_1)^2+(u_2sin(\theta_1))^2\big]+\big[(u_3)^2+(u_4sin(\theta_2))^2\big],$$

$$g(v,v)=g_1(v^1,v^1)+g_2(v^2,v^2)=\big[(v_1)^2+(v_2sin(\theta_1))^2\big]+\big[(v_3)^2+(v_4sin(\theta_2))^2\big],$$

$$g(u,v)=g_1(u^1,v^1)+g_2(u^2,v^2)=\big[(u_1v_1)+u_2v_2sin^2(\theta_1)\big]+\big[(u_3v_3)+u_4v_4sin^2(\theta_2)\big].$$

Then the denominator

$$
\begin{aligned}
B &= g(u,u)g(v,v) - \big[g(u,v)\big]^2 \\
&= \Big[\big(u_1^2 + (u_2 sin(\theta_1))^2\big) + \big(u_3^2 + (u_4 sin(\theta_2))^2\big)\Big] \\
&\times \Big[\big(v_1^2 + (v_2 sin(\theta_1))^2\big) + \big(v_3^2 + (v_4 sin(\theta_2))^2\big)\Big] \\
&- \Big[\big(u_1v_1 + u_2v_2 sin^2(\theta_1)\big) + \big(u_3v_3 + u_4v_4 sin^2(\theta_2)\big)\Big]^2 \\
&= (u_1v_3 - u_3v_1)^2 + (u_1v_2 - u_2v_1)^2 sin^2(\theta_1) + (u_1v_4 - u_4v_1)^2 sin^2\theta_2 \\
&+ (u_2v_3 - u_3v_2)^2 sin^2(\theta_1) + (u_3v_4 - u_4v_3)^2 sin^2(\theta_2) \\
&+ (u_2v_4 - u_4v_2)^2 sin^2(\theta_1) sin^2(\theta_2).
\end{aligned}
$$

Obviously B is always positive. The numerator of the fraction is

$$
\begin{aligned}
A &= v_j(u_i R_{ijkl} u_l)v_k \\
&= R_{1221}u_1u_1v_2v_2 + R_{2121}u_2u_1v_1v_2 + R_{1212}u_1u_2v_2v_1 + R_{2112}u_2u_2v_1v_1 \\
&+ R_{3443}u_3u_3v_4v_4 + R_{4343}u_4u_3v_3v_4 + R_{3434}u_3u_4v_4v_3 + R_{4334}u_4u_4v_3v_3 \\
&= [u_1u_2v_2v_1 - u_1u_1v_2v_2 - u_2u_2v_1v_1 + u_2u_1v_1v_2]\, sin^2(\theta_1) \\
&+ [u_3u_4v_4v_3 - u_3u_3v_4v_4 - u_4u_4v_3v_3 + u_4u_3v_3v_4]\, sin^2(\theta_2) \\
&= [u_1v_2 - u_2v_1]^2\, sin^2(\theta_1) + [u_3v_4 - u_4v_3]^2\, sin^2(\theta_2).
\end{aligned}
$$

Then $k(\sigma) = \dfrac{A}{B} > 0$ and this complete the prof. □

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