Curvature properties of Siklos metric

Absos Ali Shaikh, Lovejoy Das,
Haradhan Kundu and Dhyanesh Chakraborty

Abstract. Siklos spacetime represents exact gravitational waves propagating on the anti-de-Sitter universe with negative cosmological constant and it is conformally related to pp-wave spacetime. The object of this paper is to investigate the curvature restricted geometric structures admitting by the Siklos spacetime and it is shown that such spacetime is $Ein(2)$, quasi-Einstein and its conformal 2-forms are recurrent. It is also shown that this spacetime satisfies various pseudosymmetric type curvature conditions such as pseudosymmetry, semisymmetry due to conformal curvature tensor and Ricci generalized conformally pseudosymmetry. The curvature properties of Siklos spacetime in vacuum has also been investigated. As special case, we have evaluated the curvature properties of Kaigorodov spacetime and Defrise's spacetime. Finally, we make comparison between the curvature properties of Siklos spacetime and pp-wave spacetime.

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1 Introduction

In 1985, Siklos [61] obtained a class of gravitational waves with a non-zero cosmological constant during the study of exact solutions of Einstein field equations and named these waves as 'Lobatchevski Plane Waves'. Physically these spacetimes represent exact gravitational waves propagating in the anti-de-Sitter universe with negative cosmological constant [38] and are conformally related to pp-waves ([24], [63]). The Weyl tensor of Siklos spacetime has Petrov type N and in vacuum the solutions belong to a special class of non-twisting, non-expanding and shere free solutions of Kundt type [24]. Moreover, it admits a non-covariantly constant null Killing vector field.

The Siklos metric and its properties have been studied by many authors to describe its physical properties. Podolský [38] studied the geodesics for physical interpretation and came out with an interesting result that the direction of propagation of wave surfaces rotates with an angular velocity $\omega = \sqrt{\frac{-\Lambda}{3}}$, where Λ is a negative cosmological

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constant. Blagojecvić and Cvetković [5] considered the extension of vacuum Siklos waves in Poincaré guage theory. Baleanu [3] examined the existence of dual metrics and non-generic supercharges admitted by Siklos metric. The Siklos metric in higher dimension could be found in [61].

There are various forms of Siklos spacetime metric in different coordinate systems. The Siklos metric with respect to Poincaré coordinates $x^\mu = (u, v, x, y)$ is given by

$$ds^2 = \frac{l^2}{x^2}(H(x,y,u)du^2 + 2dudv + dx^2 + dy^2), \tag{1.1}$$

where $l = \sqrt{-3/\Lambda}$ and $H = H(x, y, u)$ is any nowhere vanishing smooth function. From (1.1) it is obvious that the Siklos metric is conformally related to pp-wave metric with scaling factor l^2/x^2. The metric (1.1) reduces to anti-de-Sitter spacetime for $H = 0$. It is easy to check that (1.1) satisfies vacuum Einstein field equations if H obeys the equation

$$H_{xx} - \frac{2}{x}H_x + H_{yy} = 0. \tag{1.2}$$

We call the metric (1.1) with the condition (1.2) as a vacuum Siklos metric. We see that $H = x^3$ is a simplest solution of the equation (1.2) and thus we have a vacuum Siklos type metric as

$$ds^2 = \frac{l^2}{x^2}(x^3du^2 + 2dudv + dx^2 + dy^2) \tag{1.3}$$

which was first independently discovered by Kaigorodov [28] in 1963 in the form

$$ds^2 = (dx^4)^2 + e^{2x^4/l}[2dx^1dx^3 + (dx^2)^2] \pm e^{-x^4/l}(dx^3)^2. \tag{1.4}$$

The transformation between the Kaigorodov and the Siklos coordinates is given by

$$x^1 = lv,\ x^2 = ly,\ x^3 = lu,\ x^4 = -l\ \ln|x|\,. \tag{1.5}$$

It is noteworthy to mention that, in general, the Siklos spacetime is inhomogeneous whereas the Kaigorodov spacetime is actually homogeneous. The metric (1.1) also includes the Defrise's spacetime [11] as its special case for $H = x^{-2}$. It may be noted that Defrise's spacetime is non-vacuum solution.

In the literature of differential geometry there are various geometric structures arose due to the generalization of locally symmetric manifolds ([6], [7]). Some of the important geometric structures are semisymmetric manifolds by Szabó ([65], [66], [67]), pseudosymmetric manifolds by Adamó and Deszcz [1], weakly symmetric manifolds by Tamássy and Binh [69], recurrent manifolds by Ruse ([40], [41], [42] and also [71]), quasi generalized recurrent manifolds by Shaikh and Roy [57], weakly generalized recurrent manifolds by Shaikh and Roy ([43], [58]), hyper generalized recurrent manifolds by Shaikh and Patra ([56], [60]), super generalized recurrent manifolds by Shaikh et. al. [59] etc. We mention that by geometric structures we mean the curvature restricted geometric structures obtained by imposing covariant derivatives of first order or higher orders on several curvature tensors. It is noteworthy to mention that among the above geometric structures the notion of pseudosymmetry is more significant due to its application in relativity and cosmology ([10], [21], [23] and also references therein). Several spacetimes realize various pseudosymmetric type structures (see, [2], [21], [29], [44], [48], [54], [55]). Our objective in this paper is to investigate such kind of geometric structures admitting by Siklos metric (1.1). It is shown that this spacetime is pseudosymmetric and also satisfies the pseudosymmetric type condition $R.C = \frac{1}{3}Q(S, C)$. Moreover it is quasi-Einstein and semisymmetric due to conformal curvature tensor.

The paper is planned as follows. Section 2 deals with the preliminaries of various curvature restricted geometric structures. Section 3 is concerned with the calculation of components of different curvature tensors and investigation of curvature restricted geometric structures admitting by Siklos spacetime. Finally, in Section 4 we make the comparison between the curvature properties of Siklos spacetime and pp-wave spacetime.

2 Preliminaries

In this section we will describe some useful notations and definitions of various curvature restricted geometric structures which are essential tools to investigate the geometric structures of the Siklos spacetime. For this we consider M as an n-dimensional ($n \geq 3$) connected semi-Riemannian smooth manifold equipped with a semi-Riemannian metric g. Let ∇, R, S and κ be respectively the Levi-Civita connection, the Reimann-Christoffel curvature tensor, the Ricci tensor and the scalar curvature of M. Also let $C^\infty(M)$, $\chi(M)$ and $\chi^*(M)$ be respectively the algebra of all smooth functions, the Lie algebra of all smooth vector fields and the Lie algebra of all smooth 1-forms on M. Now we define the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X, Y)$, $\mathcal{C}(X, Y)$, $\mathcal{P}(X, Y)$, $\mathcal{W}(X, Y)$ and $\mathcal{K}(X, Y)$ over $\chi(M)$ on M as follows:

$$
\begin{aligned}
(X \wedge_A Y)Z &= A(Y, Z)X - A(X, Z)Y,\\
\mathcal{R}(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,\\
\mathcal{C}(X, Y)Z &= \mathcal{R}(X, Y)Z - \frac{1}{(n-2)}(X \wedge_g \mathcal{L}Y + \mathcal{L}X \wedge_g Y - \frac{\kappa}{n-1} X \wedge_g Y)Z,\\
\mathcal{P}(X, Y)Z &= \mathcal{R}(X, Y)Z - \frac{1}{(n-1)}(X \wedge_S Y)Z,\\
\mathcal{W}(X, Y)Z &= \mathcal{R}(X, Y)Z - \frac{\kappa}{n(n-1)}(X \wedge_g Y)Z,\\
\mathcal{K}(X, Y)Z &= \mathcal{R}(X, Y)Z - \frac{1}{(n-2)}(X \wedge_g \mathcal{L}Y + \mathcal{L}X \wedge_g Y)Z
\end{aligned}
$$

where A is a symmetric $(0, 2)$-tensor, $\mathcal{L}$ is the Ricci operator defined by $g(X, \mathcal{L}Y) = S(X, Y)$ and X, Y, $Z \in \chi(M)$. From the above endomorphisms we have the following $(0, 4)$- curvature tensors namely Gaussian curvature tensor G, the Riemann-Christoffel curvature tensor R, the Weyl conformal curvature tensor C, the projective curvature tensor P, the concircular curvature tensor W and the conharmonic curvature tensor

K as follows:

$$\begin{aligned}G(X_1,X_2,X_3,X_4) &= g((X_1\wedge_g X_2)X_3,X_4),\\ R(X_1,X_2,X_3,X_4) &= g(\mathcal{R}(X_1,X_2)X_3,X_4),\\ C(X_1,X_2,X_3,X_4) &= g(\mathcal{C}(X_1,X_2)X_3,X_4),\\ P(X_1,X_2,X_3,X_4) &= g(\mathcal{P}(X_1,X_2)X_3,X_4),\\ W(X_1,X_2,X_3,X_4) &= g(\mathcal{W}(X_1,X_2)X_3,X_4),\\ K(X_1,X_2,X_3,X_4) &= g(\mathcal{K}(X_1,X_2)X_3,X_4)\end{aligned}$$

where $X_1,\ X_2,\ X_3,\ X_4 \in \chi(M)$. Again the Ricci tensor of level k is defined by $S^k(X,Y) = S(X,\mathcal{L}^{k-1}Y)$.
We can easily operate the endomorphism $\mathcal{B}(X,Y)$ on a $(0,k)$-tensor H, $k \geq 1$, and obtain the tensor $B\cdot H$ given by

$$\begin{aligned}(B\cdot H)(X_1,X_2,\cdots,X_k;X,Y) &= (\mathcal{B}(X,Y)\cdot H)(X_1,X_2,\cdots,X_k)\\ &= -H(\mathcal{B}(X,Y)X_1,X_2,\cdots,X_k)-\cdots-H(X_1,X_2,\cdots,\mathcal{B}(X,Y)X_k),\end{aligned}$$

where B is the associated $(0,4)$-tensor corresponding to the endomorphism $\mathcal{B}(X,Y)$. Again if A is a symmetric $(0,2)$ tensor then for $\mathcal{B}(X,Y) = X\wedge_A Y$ we define a $(0,k+2)$-tensor $Q(A,H)$, called *Tachibana tensor* [68], by

$$\begin{aligned}Q(A,H)(X_1,X_2,\cdots,X_k;X,Y) &= ((X\wedge_A Y).H)(X_1,X_2,\cdots,X_k)\\ &= -H((X\wedge_A Y)X_1,X_2,\cdots,X_k)-\cdots-H(X_1,X_2,\cdots,(X\wedge_A Y)X_k)\\ &= A(X,X_1)H(Y,X_2,\cdots,X_k)+\cdots+A(X,X_k)H(X_1,X_2,\cdots,Y)\\ &\quad -A(Y,X_1)H(X,X_2,\cdots,X_k)-\cdots-A(Y,X_k)H(X_1,X_2,\cdots,X).\end{aligned}$$

Definition 2.1. ([7], [12], [16], [18], [48], [49], [50], [54], [55], [66]) A semi-Riemannian manifold M is said to be H- semisymmetric due to B if $B\cdot H = 0$ and it is said to be H-pseudosymmetric type due to B if $B\cdot H = L_H Q(g,H)$, where L_H is some scalar function on the set $\{x\in M : Q(g,H)_x \neq 0\}$.

In particular, a H-semisymmetric manifold due to B is said to be semisymmetric (resp., Ricci, conformally, concircularly and conharmonically semisymmetric) if $B = R$ and $H = R$ (resp., S, C, W and K). Again a H-pseudosymmetric manifold due to B is said to be Dezcz pseudosymmetric (resp., Ricci, conformally, concircularly and conharmonically pseudosymmetric) if $B = R$, $H = R$ (resp., S, C, W and K).

Definition 2.2. A semi-Riemannian manifold is said to be a k-quasi-Einstein manifold if rank $(S-\alpha g) = k$, $0 \leq k \leq (n-1)$, for a scalar α. The manifold is called Einstein(resp., quasi-Einstein) if $k = 0$ (resp., $k = 1$). If $\alpha = 0$, then a quasi-Einstein manifold is called Ricci simple.

For two symmetric $(0,2)$ tensors E and A, we define their Kulkarni-Nomizu product $E\wedge A$ by ([15], [25])

$$\begin{aligned}(E\wedge A)(X_1,X_2,X_3,X_4) &= E(X_1,X_4)A(X_2,X_3)+E(X_2,X_3)A(X_1,X_4)\\ &- E(X_1,X_3)A(X_2,X_4)-E(X_2,X_4)A(X_1,X_3).\end{aligned}$$

Definition 2.3. A semi-Reimannian manifold is said to be a generalized Roter type manifold ([17], [18], [20], [47], [48], [50], [51], [53]) if

$$R = a_1 g \wedge g + a_2 g \wedge S + a_3 S \wedge S + a_4 g \wedge S^2 + a_5 S \wedge \ S^2 + a_6 S^2 \wedge S^2$$

holds for some $a_i \in C^\infty(M)$, $1 \leq i \leq 6$. It reduces to Roter type manifold for $a_4 = a_5 = a_6 = 0$ ([13], [14], [15], [19] and [26]).

Definition 2.4. ([4], [48], [51]) A semi-Reimannian manifold M is said to be Ein(2), Ein(3) and Ein(4) if

$$\begin{aligned} S^2 + \mu_1 S + \mu_2 g &= 0, \\ S^3 + \mu_3 S^2 + \mu_4 S + \mu_5 g &= 0 \; and \\ S^4 + \mu_6 S^3 + \mu_7 S^2 + \mu_8 S + \mu_9 g &= 0 \end{aligned}$$

holds respectively for some $\mu_i \in C^\infty(M)$, $1 \leq i \leq 9$.

Definition 2.5. ([22], [27]) A semi-Riemannian manifold M is said to be of Codazzi type (resp., cyclic parallel) Ricci tensor if

$$\nabla_{X_1} S(X_2, X_3) = \nabla_{X_2} S(X_1, X_3)$$

$$(resp., \nabla_{X_1} S(X_2, X_3) + \nabla_{X_2} S(X_3, X_1) + \nabla_{X_3} S(X_1, X_2) = 0)$$

holds on M.

Definition 2.6. A semi-Riemannian manifold M is said to be weakly symmetric [69] if

$$\begin{aligned} \nabla_X R(X_1, X_2, X_3, X_4) &= \Pi(X) R(X_1, X_2, X_3, X_4) + \Phi(X_1) R(X, X_2, X_3, X_4) \\ &+ \overline{\Phi}(X_2) R(X_1, X, X_3, X_4) + \Psi(X_3) R(X_1, X_2, X, X_4) + \overline{\Psi}(X_4) R(X_1, X_2, X_3, X) \end{aligned}$$

holds $\forall\ X, X_i \in \chi(M)$ $(i = 1, 2, 3, 4)$ and some 1-forms $\Pi, \Phi, \overline{\Phi}, \Psi$ and $\overline{\Psi}$ on $\{x \in M : R_x \neq 0\}$. In particular, if $\frac{1}{2}\Pi = \Phi = \overline{\Phi} = \Psi = \overline{\Psi}$ (resp., $\Phi = \overline{\Phi} = \Psi = \overline{\Psi} = 0$), then the manifold is called Chaki pseudosymmetric manifold [8] (resp., recurrent manifold [40]).

It is also noted that the notion of Chaki pseudosymmetry is different from Deszcz pseudosymmetry. For details about the weak symmetry and its interrelation with Deszcz pseudosymmetry, we refer the reader to see [45] and also references therein.

Definition 2.7. ([32], [33]) A symmetric $(0, 2)$ tensor A on a semi-Riemannian manifold is said to be Riemann compatible if

$$R(\mathcal{A}X_1, X, X_2, X_3) + R(\mathcal{A}X_2, X, X_3, X_1) + R(\mathcal{A}X_3, X, X_1, X_2) = 0$$

holds, where $\mathcal{A}$ is the endomorphism corresponding to A defined as $g(\mathcal{A}X_1, X_2) = A(X_1, X_2)$. Again an 1-form Φ is said to be Riemann compatible if $\Phi \otimes \Phi$ is Riemann compatible.

In the similar manner, we can define conformal compatibility, concircular compatibility and conharmonic compatibility.

Definition 2.8. Let B be a $(0,4)$ tensor and A be a $(0,2)$ tensor on M. Then the corresponding curvature 2-forms $\Omega^m_{(B)l}$ ([4], [30]) are recurrent if ([34], [35], [36])

$$\begin{aligned}&(\nabla_{X_1}B)(X_2,X_3,X_4,X)+(\nabla_{X_2}B)(X_3,X_1,X_4,X)+(\nabla_{X_3}B)(X_1,X_2,X_4,X)=\\ &\quad \Pi(X_1)B(X_2,X_3,X_4,X)+\Pi(X_2)B(X_3,X_1,X_4,X)+\Pi(X_3)B(X_1,X_2,X_4,X)\end{aligned}$$

and the 1-forms $\Lambda_{(A)l}$ ([64]) are recurrent if

$$(\nabla_{X_1}A)(X_2,X)-(\nabla_{X_2}A)(X_1,X)=\Pi(X_1)A(X_2,X)-\Pi(X_2)A(X_1,X)$$

for some 1-form Π.

Definition 2.9. ([39], [50], [54], [70]) Let $L(M)$ be the vector space formed by all 1-forms θ on M satisfying

$$\theta(X_1)B(X_2,X_3,X_4,X_5)+\theta(X_2)B(X_3,X_1,X_4,X_5)+\theta(X_3)B(X_1,X_2,X_4,X_5)=0$$

where B is a $(0,4)$ tensor. Then M is said to be a B-space by Venzi if $dimL(M)\geq 1$.

3 Curvature restricted geometric structures

In terms of Poincaré coordinates (u,v,x,y) the non-zero components of the metric tensor of the Siklos metric (1.1) are given by

$$g_{11}=\frac{l^2}{x^2}H,\quad g_{12}=g_{21}=g_{33}=g_{44}=\frac{l^2}{x^2}. \tag{3.1}$$

Then by a straightforward calculation we get the non-zero components (upto symmetry) of its Riemann curvature tensor R, Ricci tensor S and scalar curvature κ as given by

$$\left\{\begin{array}{c} -R_{1212}=R_{1323}=R_{1424}=R_{3434}=-\frac{l^2}{x^2},\\ R_{1313}=-\frac{l^2}{2x^4}(x^2H_{xx}-xH_x+2H),\\ R_{1314}=-\frac{l^2}{2x^2}H_{xy},\ R_{1414}=-\frac{l^2}{2x^4}(x^2H_{yy}-xH_x+2H);\\ S_{11}=\frac{1}{2x^2}(x^2(H_{xx}+H_{yy})-2xH_x+6H),\ S_{12}=S_{33}=S_{44}=\frac{3}{x^2};\\ \text{and } \kappa=\frac{12}{l^2}. \end{array}\right. \tag{3.2}$$

Again the non-zero components (upto symmetry) of ∇R and ∇S are given by

$$\left\{\begin{array}{c} R_{1213,1}=\frac{l^2(xH_{xx}-H_x)}{2x^4},\ R_{1214,1}=-R_{1334,1}=\frac{l^2H_{xy}}{2x^3},\\ R_{1313,1}=\frac{l^2(H_{xu}-xH_{xxu})}{2x^3},\ R_{1313,3}=\frac{l^2(H_x-xH_{xx}+xH_{xxx})}{2x^4},\\ R_{1313,4}=-\frac{l^2(H_{xy}+xH_{xxy})}{2x^3},\ R_{1314,1}=-\frac{l^2H_{xyu}}{2x^2},\\ R_{1314,3}=-\frac{l^2(2H_{xy}+xH_{xxy})}{2x^3},\ R_{1414,1}=\frac{l^2(H_{xu}-xH_{yyu})}{2x^3},\\ R_{1314,4}=-\frac{l^2(H_{yy}+xH_{xyy}-H_{xx})}{2x^3},\ R_{1414,3}=\frac{l^2(H_x-x(2H_{yy}+xH_{xyy}-H_{xx}))}{2x^4},\\ R_{1414,4}=-\frac{l^2(xH_{yyy}-3H_{xy})}{2x^3},\ R_{1434,1}=\frac{l^2(H_x-xH_{yy})}{2x^4}; \end{array}\right. \tag{3.3}$$

$$\left\{\begin{array}{c} S_{11,1}=\frac{1}{2}\left(H_{xxu}-\frac{2}{x}H_{xu}+H_{yyu}\right),\\ S_{11,3}=\frac{1}{2}\left(H_{xyy}+H_{xxx}+\frac{2}{x}H_{yy}-\frac{2}{x^2}H_x\right),\\ S_{11,4}=\frac{1}{2}\left(H_{yyy}-\frac{2}{x}H_{xy}+H_{xxy}\right),\ S_{13,1}=\frac{1}{2x^2}\left(H_{xx}+xH_{yy}-2H_x\right). \end{array}\right. \tag{3.4}$$

Now using (3.1) and (3.2) we can calculate the components of S^2, $g \wedge g, g \wedge S, S \wedge S, g \wedge S^2, S \wedge\ S^2$ and $S^2 \wedge S^2$. From above we can state the following:

Proposition 3.1. *The Siklos metric* (1.1) *is neither Einstein nor* GRT_4 *but (i) quasi-Einstein (Rank* $(S - \frac{3}{l^2}g) = 1$*) and (ii)* $Ein(2)$ $(S^2 = \frac{6}{l^2}S)$*.*

Now using (3.2) we can calculate the components of $R \cdot R$, $Q(g, R)$ and $Q(S, R)$ and hence we can state the following:

Proposition 3.2. *The Siklos metric* (1.1) *is neither weakly symmetric nor Ricci generalized pseudosymmetric but a pseudosymmetric manifold of constant type* $(R \cdot R = \frac{1}{l^2}Q(g, R))$*.*

Since the Siklos metric is pseudosymmetric, we can state the following results:

Corollary 3.3. *The Siklos metric* (1.1) *satisfies (i)* $R \cdot S = \frac{1}{l^2}Q(g, S)$*, (ii)* $R \cdot C = \frac{1}{l^2}Q(g, C)$*, (iii)* $R \cdot P = \frac{1}{l^2}Q(g, P)$*, (iv)* $R \cdot W = \frac{1}{l^2}Q(g, W)$*, (v)* $R \cdot K = \frac{1}{l^2}Q(g, K)$*, (vi)* $W \cdot R = 0$*, (vii)* $W \cdot S = 0$*, (viii)* $W \cdot C = 0$*, (ix)* $W \cdot W = 0$ *and (x)* $W \cdot K = 0$*.*

Again since the Siklos metric is not Ricci generalized pseudosymmetric, then we get the following:

Corollary 3.4. *The Siklos metric* (1.1) *is neither (i) R-space by Venzi nor (ii)* $K_n(2)$*.*

Since the Siklos metric is not weakly symmetric, then we get the following:

Corollary 3.5. *The Siklos metric* (1.1) *is neither (i) recurrent nor (ii) pseudosymmetric in the sense of Chaki.*

Again using components of R, S and κ we get the non-zero components (upto symmetry) of conformal curvature tensor C as follows:

$$C_{1313} = -C_{1414} = -\frac{l^2}{4x^2}(H_{xx} - H_{yy}),\ C_{1314} = -\frac{l^2}{2x^2}H_{xy}. \tag{3.5}$$

Then by a straightforward calculation using (3.3)-(3.5) we can find out the components of ∇C, $R \cdot C$, $C \cdot R$, $Q(g, C)$ and $Q(S, C)$, from which we get the following:

Proposition 3.6. *The Siklos metric* (1.1) *is not conformally recurrent but it is (i) a C-space by Venzi, (ii)* $CK_n(2)$*-space with 1-fom of recurrency*

$$\Pi = \left\{1, 0, \frac{1}{x} - \frac{\alpha_4 - 2\alpha_2\alpha_3}{4\alpha_1^2 + \alpha_2^2}, \frac{\alpha_2\alpha_3 - 2\alpha_1\alpha_4}{4\alpha_1^2 + \alpha_2^2}\right\},$$

where $\alpha_1 = H_{xy}$, $\alpha_2 = H_{yy} - H_{xx}$, $\alpha_3 = H_{xxy} + H_{yyy}$, $\alpha_4 = H_{xxx} + H_{xyy}$,
(iii)Ricci generalized conformally pseudosymmetric, (iv) semisymmetric due to conformal curvature tensor and satisfies the pseudosymmetric type condition $R \cdot R - Q(S, R) = -\frac{2}{l^2}Q(g, C)$*.*

Since the Siklos metric satisfies $C \cdot R = 0$ and $R \cdot C = \frac{1}{3}Q(S, C)$, we get the following:

Corollary 3.7. *The Siklos metric* (1.1) *satisfies (i)* $C \cdot S = 0$, *(ii)* $C \cdot C = 0$, *(iii)* $C \cdot P = 0$, *(iv)* $C \cdot W = 0$, *(v)* $C \cdot K = 0$, *(vi)* $K \cdot R = -\frac{2}{l^2}Q(g,R)$, *(vii)* $K \cdot S = -\frac{2}{l^2}Q(g,S)$, *(viii)* $K \cdot C = -\frac{2}{l^2}Q(g,C)$, *(ix)* $K \cdot P = -\frac{2}{l^2}Q(g,P)$, *(x)* $K \cdot W = -\frac{2}{l^2}Q(g,W)$, *(xi)* $K \cdot K = -\frac{2}{l^2}Q(g,K)$ *and (xii)* $P \cdot C = 0$.

Again since the Siklos metric is not conformally recurrent, we get the following:

Corollary 3.8. *The Siklos metric* (1.1) *is not (i) projectively recurrent, (ii) concircularly recurrent, (iii) conharmonically recurrent.*

Now in view of (3.2), we can evaluate the non-zero components (upto symmetry) of projective curvature tensor P, the concircular curvature tennsor W and the conharmonic curvature tensor K of (1.1) as follows:

$$P_{1211} = \frac{l^2}{6x^4}(x(H_{xx} + H_{yy}) - 2H_x),\ P_{1313} = -\frac{l^2}{6x^2}(x(2H_{xx} - H_{yy}) - H_x),$$

$$P_{1314} = P_{1413} = -P_{1341} = -P_{1431} = -\frac{l^2}{2x^2}H_{xy},\ P_{1331} = \frac{l^2}{2x^2}(xH_{xx} - H_x),$$

$$P_{1414} = \frac{l^2}{6x^2}(x(H_{xx} - 2H_{yy}) + H_x),\ P_{1441} = \frac{l^2}{2x^3}(xH_{yy} - H_x);$$

$$W_{1313} = \frac{l^2}{2x^3}(H_x - xH_{xx}),\quad W_{1314} = \frac{l^2}{2x^2}H_{xy},\quad W_{1414} = \frac{l^2}{2x^3}(H_x - xH_{yy});$$

$$-K_{1212} = K_{1323} = K_{1424} = K_{3434} = \frac{2l^2}{x^4},\quad K_{1313} = \frac{l^2}{4x^4}(8H - x^2(H_{xx} - H_{yy})),$$

$$K_{1314} = \frac{l^2}{2x^2}H_{xy},\quad K_{1414} = \frac{l^2}{4x^4}(8H + x^2(H_{xx} - H_{yy})).$$

Then we can easily calculate the components of ∇P, $K \cdot R$, $K \cdot C$, $K \cdot W$, $K \cdot K$, $Q(g,W)$, $Q(S,W)$, $Q(g,K)$ and $Q(S,K)$, which yields the following:

Proposition 3.9. *The Siklos metric* (1.1) *is (i) P-space by venzi as well as W-space by Venzi with null 1-form* $\{1,0,0,0\}$, *(ii)* $K \cdot C = -\frac{2}{3}Q(S,C)$, *(iii)* $K \cdot W = -\frac{2}{3}Q(S,W)$ *and (iv)* $R \cdot W = \frac{1}{3}Q(S,W)$.

From the above propositions, we get the following theorem on curvature restricted geometric structures admitted by the Siklos metric (1.1).

Theorem 3.10. *The Siklos metric* (1.1) *possesses the following curvature restricted geometric structures:*
(i) $R \cdot R = \frac{1}{l^2}Q(g,R)$ *and hence it is Ricci pseudosymmetric, conformally pseudosymmetric, projectively pseudosymmetric, concircularly pseudosymmetric and conharmonically pseudosymmetric of constant type. Also it satisfies* $W \cdot R = 0$ *and hence* $W \cdot C = W \cdot P = W \cdot S = W \cdot W = W \cdot K = 0$ *(i.e., it satisfies semisymmetric type conditions due to concircular curvature tensor),*
(ii) $C \cdot R = 0$ *and hence* $C \cdot S = C \cdot P = C \cdot C = C \cdot W = C \cdot K = 0$ *(i.e., it satisfies semisymmetric type conditions due to Weyl conformal curvature tensor). Hence it satisfies* $K \cdot R = -\frac{2}{l^2}Q(g,R)$ *and all conharmonically pseudosymmetric type conditions of constant type,*

(iii) $P \cdot C = P \cdot W = 0$ *and hence* $R \cdot C = \frac{1}{3}Q(S, C)$ *and* $K \cdot C = -\frac{2}{3}Q(S, C)$*,*
(iv) C-space, P-space, W-space by Venzi for (1,0,0,0),
(v) If $4\alpha_1^2 + \alpha_2^2 \neq 0$*, then the conformal 2-forms* $\Omega^m_{(C)l}$ *are recurrent with 1-form of recurrency*

$$\Pi = \left\{1, 0, \frac{1}{x} - \frac{\alpha_4 - 2\alpha_2\alpha_3}{4\alpha_1^2 + \alpha_2^2}, \frac{\alpha_2\alpha_3 - 2\alpha_1\alpha_4}{4\alpha_1^2 + \alpha_2^2}\right\}$$

where $\alpha_1 = H_{xy}$, $\alpha_2 = H_{yy} - H_{xx}$, $\alpha_3 = H_{xxy} + H_{yyy}$, $\alpha_4 = H_{xxx} + H_{xyy}$,
(vi) The concircular 2-forms $\Omega^m_{(W)l}$ *are recurrent with 1-form of recurrency* $\{1, 0, 0, 0\}$,
(vii) quasi-Einstein since Rank $(S - \alpha g) = 1$*, for*

$$\alpha = \frac{3}{l^2}, \ \beta = 1, \ \text{and } \eta = \left\{\sqrt{\frac{x(H_{xx} + H_{yy}) - 2H_x}{2x}}, 0, 0, 0\right\}$$

with $\|\eta\| = 0$*,*
(viii) Ein(2) space such that $S^2 = \frac{6}{l^2}S$*.*

Since the Defrise's spacetime metric, vacuum Siklos spacetime metric and Kaigorodov spacetime metric are special cases of Siklos metric (1.1), hence we can state the following:

Corollary 3.11. *The Defrise's spacetime metric i.e., the metric* (1.1) *with* $H = x^{-2}$ *fulfills the following curvature restricted geometric structures:*
(1) it satisfies all the curvature properties of Theorem (3.10) *from (i) to (viii) with different associated 1-forms,*
(2) Ricci tensor is of cyclic parallel,
(3) Ricci tensor is Riemann compatible, Weyl compatible and projective compatible.

Corollary 3.12. *The vacuum Siklos spacetime metric (i.e., the metric* (1.1) *with the condition* (1.2)*) fulfills the following curvature restricted geometric structures:*
(i) Einstein space, i.e., $S = \frac{3}{l^2}g$ *and hence C=P=W,*
(ii) $R \cdot R = \frac{1}{l^2}Q(g, R)$ *and hence it is conformally pseudosymmetric and conharmonically pseudosymmetric,*
(iii) $C \cdot R = 0$ *and hence* $C \cdot C = C \cdot K = 0$ *(i.e., semisymmetric due to Weyl conformal curvature tensor),*
(iv) C-space by Venzi for (1,0,0,0),
(v) the conformal 2-forms $\Omega^m_{(C)l}$ *are recurrent with 1-form of recurrency* $\{1, 0, 0, 0\}$*,*
(vi) $divR = 0$*,* $divC = 0$ *and* $divK = 0$*.*

Corollary 3.13. *The Kaigorodov spacetime metric* (1.3) *satisfies the following curvature restricted geometric structures:*
(1) the curvature properties of Corollary (3.12) *from (i) to (vi),*
(2) Ricci tensor is Riemann compatible and conformal compatible.

4 *pp*-wave metric and Siklos metric

A pp-wave metric is a Lorentzian manifold with a parallel light-like vector field and satisfies a certain curvature condition. In terms of Brinkmann coordinates the metric of the pp-wave is given by

$$ds^2 = H(u,x,y)du^2 + 2dudv + dx^2 + dy^2 \tag{4.1}$$

where $H = H(u,x,y)$ is any nowhere vanishing smooth function. For detailed study about the pp-wave metric we lead the readers to see [24], [31], [37], [62] and the references therein. Recently Shaikh et al. [52] studied the curvature properties of the pp-wave metric and also studied the sufficient condition for which a generalized pp-wave metric turns into pp-wave metric. We note that both the Siklos metric and the pp-wave metric represent exact gravitational waves physically and are related to each other conformally. So, in this section we are interested to draw a useful picture of comparisons between these metrics with respect to their curvature restricted geometric structures.

A. Similarities:
(i) both the metrics are quasi-Einstein,
(ii) both the metrics are semisymmetric due to conformal curvature tensor,
(iii) their conformal 2-forms are recurrent,
(iv) both the metrics are $C-$space by Venzi.

B. Dissimilarities:
(i) the Siklos metric is pseudosymmetric but the pp-wave metric is semisymmetric,
(ii) both the metrics are not roter type but the Siklos metric is $Ein(2)$ whereas pp-wave metric is $Ein(3)$ with vanishing scalar curvature,
(iii) the Ricci tensor of the Siklos metric is neither Reimann compatible nor Weyl compatible but for the pp-wave metric it is both Riemann compatible as well as Weyl compatible,
(iv) Also the Ricci tensor of the pp-wave metric is of recurrent type whereas the Ricci tensor of the Siklos metric does not possess such structure.

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Authors' addresses:

Absos Ali Shaikh, Haradhan Kundu and Dhyanesh Chakraborty
Department of Mathematics, University of Burdwan,
Golapbag, Burdwan 713104, West Bengal, India.
Email: aask2003@yahoo.co.in , aashaikh@mathburuniv.ac.in
kundu.haradhan@gmail.com
dhyanesh2011@gmail.com

Lovejoy Das
Department of Mathematics, Kent State University,
Tuscarawas, 330 University Dr NE,
New Philadelphia, Ohio-44663, USA.
Email: ldas@kent.edu