On Finsler spaces with rational spray coefficients

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Abstract. In the present paper, we consider a class of Finsler metrics F for which geodesic spray coefficients are rational function in y. It is proved that if a Finsler metric of this class is Einstein (weak-Einstein) then it is Ricci-flat. It is also shown that if such class of Finsler metric is of scalar flag curvature with isotropic S-curvature then it must be Ricci-flat. Further it is proved that such Finsler metrics with isotropic Berwald curvature (isotropic mean Berwald curvature) reduces to a Berwald metric (weakly Berwald metric).

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1 Introduction

Recently, Y. Yu and Y. You [21] have studied the m-th root Finsler metric and shown that fundamental metric tensor g_{ij} of m-th root Finsler metrics are not necessarily rational functions in y while the spray coefficients are rational functions in y. Using the rational spray coefficients they have proved that if m-th root Finsler metric is an Einstein metric or weak-Einstein, then it reduces to Ricci-flat with scalar flag curvature $K = 0$. A. Tayebi et. al. [15, 16, 17] have used the rational spray coefficients of m-th root Finsler metric to characterize different curvature properties of m-th root Finsler metric. They have proved that every m-th root Finsler metric with isotropic Berwald curvature (isotropic mean Berwald curvature, isotropic Landsberg curvature, isotropic mean Landsberg curvature) reduces to Berwald metric (weakly Berwald metric, Landsberg metric, weakly Landsberg metric respectively). Further they have proved that every m-th root metric with almost vanishing H-curvature has vanishing H-curvature. Also B. Tiwari and G. Prajapati [18] have studied the Kropina change of m-th root Finsler metric and shown that the spray coefficients of such metric are rational functions in y. Kropina metrics, m-Kropina metrics, generalized Kropina metrics, Kropina change of m-th root metrics, m-th root metrics etc., are examples of Finsler metrics whose fundamental tensor g_{ij} are not necessarily rational functions in y while its geodesic spray coefficients are rational functions in y.

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Motivating from above study, we consider the Finsler metric in the following form

$$F = \sqrt{\overline{g}_{ij} y^i y^j} \tag{1.1}$$

where

$$\overline{g}_{ij} = \eta(x,y) a_{ij}(x,y), \tag{1.2}$$

$\eta(x,y)$ is an irrational function in y and $a_{ij}(x,y)$ are rational functions in y. m-th root metrics and Kropina change of m-th root Finsler metrics are examples of Finsler metrics of the form defined by the equation (1.1). In the present paper we see that under certain condition on η, the spray coefficients of such metrics are also rational function in y.

The Einstein metrics are solutions to Einstein field equation in General Relativity, which closely connect Riemannian geometry with gravitation in General Relativity. In many occasions, S.S. Chern had asked the following question

"Does every smooth manifold admit an Einstein Finsler metric ?"

This problem is extremely involved and remains open. One effective approach to the above problem is to consider some special Finsler metrics. Motivated by Chern's open question, Finsler Einstein metrics have been recently received more attentions, see [9, 11, 12, 22, 19] etc. Up to now, most known Einstein Finsler metrics are either of Randers type or Ricci flat. C. Robles studied a special class of Einstein Finsler metrics, that is, Einstein Randers metrics, and proved that for a Randers metric on a 3-dimensional manifold, it is Einstein if and only if it has constant flag curvature. E. Guo, X. Mo and X. Zhang give an explicit construction of a Einstein Finsler metrics of non-constant flag curvature in terms of navigation representation [10]. Z. Shen and C. Yu, using certain transformation, have constructed a large class of Einstein metrics [14]. We consider the Finsler metric F defined by the equation (1.1) and proved that if Finsler metric F is Einstein then it also reduces to Ricci flat. More precisely, we have established the following results

Theorem 1.1. *Let $\overline{F}$ be a non-Riemannian Finsler metric on a manifold of dimension $n \geq 2$ given by equation (1.1) such that components of grad $(\log \eta)$ are rational function in y. If $\overline{F}$ is Einstein metric, then it is Ricci-flat.*

Theorem 1.2. *Let $\overline{F}$ be a non-Riemannian Finsler metric on a manifold of dimension $n \geq 2$ given by equation (1.1) such that components of grad $(\log \eta)$ are rational function in y. If $\overline{F}$ is a weak Einstein metric, then it is Ricci-flat.*

Theorem 1.3. *Let $\overline{F}$ be a non-Riemannian Finsler metric on a manifold of dimension $n \geq 2$ given by equation (1.1) such that components of grad $(\log \eta)$ are rational function in y. If $\overline{F}$ is of scalar flag curvature $K(x,y)$ and isotropic S-curvature, then $K = 0$.*

In Finsler geometry, Berwald curvature is an important non-Riemannian quantity. A Finsler metric is called Berwald metric if Berwald curvature vanishes. Berwald metrics are just a bit more general than Riemannian and locally Minkowskian metrics.

It is proved that on a Berwald space, the parallel translation along any geodesic preserves the Minkowski functionals. Thus the most easily described characteristic of a Berwald space is that all tangent spaces are linearly isometric to a common Minkowski space. In [8], Cheng and Shen introduce a new class of non-Riemannian Finsler metrics called the isotropic Berwald metrics which contains the class of Berwald metrics. Recently, S. Bacso and B. Szilagyi [4] gave an example for the weakly Berwald Finsler space, and a sufficient condition for the existence of a weakly Berwald Finsler space of Kropina type was also determined. R. Yoshikawa and K. Okubo [20] obtained the conditions for generalized Kropina spaces and Matsumoto spaces to be weakly-Berwald spaces and Berwald spaces. In this paper we have proved the following result

Theorem 1.4. *Let $\overline{F}$ be a non-Riemannian Finsler metric on the manifold of dimension $n \geq 2$ given by equation (1.1) such that components of grad $(\log \eta)$ are rational function in y. Suppose that $\overline{F}$ has isotropic Berwald curvature (isotropic mean Berwald curvature). Then $\overline{F}$ reduces to a Berwald metric (weakly Berwald metric).*

In [1], Akbar-Zadeh introduces the non-Riemannian quantity H which is obtained from the mean Berwald curvature by the covariant horizontal differentiation along geodesics. It is proved that for a Finsler manifold of scalar flag curvature K with dimension $n \geq 3, K$ is constant if and only if H vanishes.

Theorem 1.5. *Let $\overline{F}$ be a non-Riemannian Finsler metric on the manifold of dimension $n \geq 2$ given by equation (1.1) such that components of grad $(\log \eta)$ are rational function in y. Suppose that $\overline{F}$ has almost vanishing H-curvature. Then H vanishes identically.*

2 Preliminaries

Let M be an n-dimensional C^{∞}-manifold. Denote by T_xM the tangent space at $x \in M$ and by $TM := \bigcup_{x \in M} T_xM$ the tangent bundle of M. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$.
Definition: A Finsler metric on M is a function $F : TM \to [0, \infty)$ with the following properties:
(i) F is C^{∞} on TM_0,
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM and
(iii) the Hessian of $\frac{F^2}{2}$ with element $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0.

The pair $F^n = (M, F)$ is called a Finsler space of dimension n. F is called fundamental function and g_{ij} is called the fundamental tensor of the Finsler space F^n. By the homogeneity of F, we have $F(x, y) = \sqrt{g_{ij}(x, y)y^i y^j}$. An important class of Finsler metrics are *Riemann metrics*, which are in the form of $F(x, y) = \sqrt{g_{ij}(x)y^i y^j}$. Another important class of Finsler metrics are *Minkowski metrics*, which are in the form of $F(x, y) = \sqrt{g_{ij}(y)y^i y^j}$.
In local coordinates, the geodesics of a Finsler metric $F = F(x, y)$ are characterized by

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx^i}{dt}) = 0, \tag{2.1}$$

where

$$G^i = \frac{1}{4} g^{il} \left[2(g_{jl})_{x^k} y^j y^k - (g_{ij})_{x^l} y^i y^j \right] \tag{2.2}$$

are called spray coefficients. For a tangent vector $y \in T_xM$, in local coordinate system, define $B_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ and $E_y : T_xM \otimes T_xM \to \mathbb{R}$ by $B_y(u, v, w) = B^i_{jkl}(x, y) u^j v^k w^l \frac{\partial}{\partial x^i}$ and $E_y(u, v) = E_{jk}(x, y) u^j v^k$ respectively, where $B^i_{jkl} = [G^i]_{y^j y^k y^l}$, $E_{jk} = \frac{1}{2} B^m_{mjk} = \frac{1}{2} \frac{\partial^2}{\partial y^j \partial y^k} [\frac{\partial G^m}{\partial y^m}]$, $u = u^i \frac{\partial}{\partial x^i}$, $v = v^i \frac{\partial}{\partial x^i}$ and $w = w^i \frac{\partial}{\partial x^i}$. Then $B = \{B_y | y \in TM_0\}$ and $E = \{E_y | y \in TM_0\}$ are called Berwald curvature and mean Berwald curvature respectively. A Finsler metric is called Berwald metric and weakly Berwald metric if $B = 0$ and $E = 0$ respectively. In view of equations (1.2) and (2.2), the spray coefficients

$$\overline{G}^i = \frac{1}{4} \left[\frac{\eta_{x^l}}{\eta} y^i y^l + 2a^{il} (a_{jl})_{x^k} y^j y^k - a^{il} (a_{ij})_{x^l} y^i y^j \right] \tag{2.3}$$

Proposition 2.1. *Let $\overline{F}$ be a non-Riemannian Finsler metric on a manifold of dimension $n \geq 2$ given by equation (1.1). Then the spray coefficients $\overline{G}^i$ of $\overline{F}^n$ are rational functions in y if and only if components of grad $(\log \eta)$ are rational functions in y.*

3 Einstein metrics

For a Finsler metric F, the Riemann curvature $R_y = R^i_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i_k = 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial x^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}. \tag{3.1}$$

The metric F is said to be of scalar curvature of there is a scalar function $K = K(x, y)$ such that

$$R^i_k = K(x, y) F^2 \left\{ \delta^i_k - \frac{F_k y^i}{F} \right\} \tag{3.2}$$

The Ricci curvature is trace of the Riemann curvature, $Ric = R^k_k$. In view of the definition of Riemann curvature, Ricci curvature and Proposition 2.1, we have following lemma,

Lemma 3.1. *Let $\overline{F}$ be a non-Riemannian Finsler metric on a manifolds of dimension $n \geq 2$ given by equation (1.1) such that components of grad $(\log \eta)$ are rational function in y. Then R^i_k and $Ric = R^k_k$ are rational functions in y.*

A Finsler metric $F = F(x, y)$ on an n-dimensional manifold M is called an Einstein metric if there is a scalar function $\lambda = \lambda(x)$ on M such that $Ric = (n-1)\lambda F^2$. F is said to be Ricci constant (resp. flat) if $\lambda =$ constant (resp. zero).

By definition, every 2-dimensional Riemann metric is an Einstein metric, but generally not of Ricci constant. In dimension $n \geq 3$, the second Schur Lemma ensures that every Riemannian Einstein metric must be Ricci constant. In particular, in dimension $n = 3$, a Riemann metric is Einstein if and only if it is of constant sectional

curvature.
Proof of Theorem 1.1 By Lemma 3.1, Ric is a rational function in y. Suppose $\overline{F}$ is an Einstein metric, that is, $Ric = (n-1)\lambda\overline{F}^2$ and $\overline{F}^2$ is not a rational function because of η. Therefore $\lambda = 0$. □

In Finsler geometry, the flag curvature is an analogue of the sectional curvature from Riemannian geometry. A natural problem is to study and characterize Finsler metrics of constant flag curvature. There are only three local Riemannian metrics of constant sectional curvature, up to a scaling. However there are lots of non-Riemannian Finsler metrics of constant flag curvature. For example, the Funk metric is positively complete and non-reversible with $K = \frac{1}{4}$ and the Hilbert-Klein metric is complete and reversible with $K = -1$ [13]. Clearly, if a Finsler metric is of constant flag curvature, then it is an Einstein metric. Thus we obtain

Corollary 3.2. *Let $\overline{F}$ be a non-Riemannian Finsler metric on a manifold of dimension $n \geq 2$. If $\overline{F}$ is of constant flag curvature K, then $K = 0$.*

Example 3.1. Let

$$\overline{F} = \frac{\left[\sum_{i=1}^{n}(y^i)^4\right]^{\frac{k+1}{4}}}{(y^j)^k}, \text{ for any fixed j, } 1 \leq j \leq n.$$

By direct computation, we get $\overline{G}^i = 0$ and $R^i_k = 0$. Thus the flag curvature of $\overline{F}$ is zero.
Example 3.2. Let

$$\overline{F} = \frac{\left[\sum_{i=1}^{n}(x^i)^2(y^i)^4\right]^{\frac{k+1}{4}}}{(x^j y^j)^k}, \text{ for any fixed j, } 1 \leq j \leq n.$$

By direct computation, we get $\overline{G}^i = \frac{(y^i)^2}{4kx^i}$ and $R^i_k = 0$. Thus the flag curvature of $\overline{F}$ is zero. It is known that every Berwald metric with $K = 0$ is locally Minkowskian. So $\overline{F}$ is locally Minkowskian.

4 Weak Einstein metrics

A weakly Einstein metric is the generalization of the Einstein metric. A Finsler metric F is called a weakly Einstein metric if its Ricci curvature Ric is in the form $Ric = (n-1)(\frac{3\theta}{F} + \lambda)F^2$, where θ is a 1-form and $\lambda = \lambda(x)$ is a scalar function. In general, a weak Einstein metric is not necessarily an Einstein metric and vice versa.
Proof of Theorem 1.2 Suppose $\overline{F}$ is a weak Einstein metric, then

$$Ric = (n-1)(3\theta\overline{F} + \lambda\overline{F}^2),$$

where θ is an 1-form and $\lambda = \lambda(x)$ is a scalar function. By Lemma 3.1 Ric is rational function in y. Therefore we have
If $\lambda \neq 0$, we get

$$\overline{F} = \frac{-3(n-1)\theta \pm \sqrt{9(n-1)^2\theta^2 + 4(n-1)\lambda Ric}}{2(n-1)\lambda}.$$

On the other hand, $\overline{F} = \sqrt{\eta a_{ij}y^iy^j}$, so we get

$$\sqrt{\eta a_{ij}y^iy^j} = \frac{-3(n-1)\theta \pm \sqrt{9(n-1)^2\theta^2 + 4(n-1)\lambda Ric}}{2(n-1)\lambda}.$$

Here the left hand side is purely irrational. Then the right hand side will be irrational if and only if $\theta = 0$. Thus we have that $\overline{F}$ is an Einstein metric. Using Theorem 1.1, we obtain that Ric identically vanishes. □

5 Scalar flag curvature

For a tangent plane $P = span(y, u)$, and for y and u are linearly independent vectors of tangent space T_xM of M at point $x \in M$, the flag curvature $K = K(P, u)$ depends on plane P as well as vector $u \in P$.

(a) A Finsler metric F is of scalar flag curvature if for any non-zero vector $y \in T_xM$, $K = K(x, y)$ is independent of P containing $y \in T_xM$.

(b) F is called of almost isotropic flag curvature if

$$K = \frac{3c_{x^m}y^m}{F} + \lambda, \tag{5.1}$$

where $c = c(x)$ and $\lambda = \lambda(x)$ are some scalar functions on M.

(c) F is of weakly isotropic flag curvature if

$$K = \frac{3\theta}{F} + \lambda, \tag{5.2}$$

where θ is a 1-form and $\lambda = \lambda(x)$ is a scalar function.
Clearly, if a Finsler metric is of weakly isotropic flag curvature, then it is a weak Einstein metric.

Lemma 5.1. *Let $\overline{F}$ be a non-Riemannian Finsler metric on a manifold of dimension $n \geq 2$. If $\overline{F}$ is of almost isotropic flag curvature K, then $K = 0$.*

The S-curvature $S = S(x, y)$ in Finsler geometry was introduced by Shen [13] as a non-Riemannian quantity, defined as

$$S(x, y) = \frac{d}{dt}[\tau(\sigma(t), \dot{\sigma}(t))]_{|t=0} \tag{5.3}$$

where $\tau = \tau(x, y)$ is a scalar function on $T_xM\backslash\{0\}$, called distortion of F and $\sigma = \sigma(t)$ is the geodesic with $\sigma(0) = x$ and $\dot{\sigma}(0) = y$.
A Finsler metric F is called of isotropic S-curvature if

$$S = (n + 1)cF, \tag{5.4}$$

for some scalar function $c = c(x)$, on M.

Theorem 5.2. *[7] Let (M, F) be an n-dimensional Finsler manifold of scalar flag curvature $K(x, y)$. Suppose that the S-curvature is isotropic, $S = (n+1)c(x)F$. Then there is a scalar function $\lambda(x)$ on M such that $K = \frac{3c_{x^m}y^m}{F} + \lambda$. In particular, $c(x) = c$ is a constant if and only if $K = K(x)$ is a scalar function on M.*

In dimension $n \geq 3$, a Finsler metric F is of isotropic flag curvature if and only if F is of constant flag curvature by Schur's Lemma. In general, a Finsler metric of weakly isotropic flag curvature and that of isotropic flag curvature are not equivalent. **Proof of Theorem 1.3** Lemma 5.1 and Theorem 5.2, completes the Theorem 1.3. □

6 Berwald curvature

A Finsler metric F is said to be isotropic Berwald metric if its Berwald curvature is in the following form

$$B^i_{jkl} = c(F_{y^j y^k}\delta^i_l + F_{y^k y^l}\delta^i_j + F_{y^l y^j}\delta^i_k + F_{y^j y^k y^l}y^i), \tag{6.1}$$

and isotropic mean Berwald metric if its mean curvature is in the form $E_{ij} = \frac{n+1}{2}cF_{ij}$, where $c = c(x)$ is a scalar function on M.

Proof of theorem 1.4 Let $\overline{F}$ be a non-Rimannian Finsler metric. Suppose that $\overline{F}$ has isotropic Berwald curvature given by equation (6.1). By Proposition 2.1, the left hand side of equation (6.1) is a rational function in y, while the right hand side is an irrational function, implies that $c = 0$ and hence $\overline{F}$ reduces to Berwald metric.

Moreover, if $\overline{F}$ be of isotropic mean Berwald curvature, that is,

$$\overline{E}_{ij} = \frac{n+1}{2}c\overline{F}_{ij}, \tag{6.2}$$

where $c = c(x)$ is a scalar function on M .
we have

$$\overline{E}_{ij} = \tfrac{(n+1)(k+1)c}{2}\overline{F}$$

$$\times\left[\frac{1}{mA}A_{ij} + \frac{(k+1-m)}{m^2A^2}A_iA_j - \frac{k}{mA\beta}(A_ib_j + A_jb_i) + \frac{k}{\beta^2}b_ib_j\right]. \tag{6.3}$$

Left hand side is rational function in y while right hand side irrational due to presence of $\overline{F}$. Thus equation (6.3) implies either $c = 0$ or

$$\left[\frac{1}{mA}A_{ij} + \frac{(k+1-m)}{m^2A^2}A_iA_j - \frac{k}{mA\beta}(A_ib_j + A_jb_i) + \frac{k}{\beta^2}b_ib_j\right] = 0. \tag{6.4}$$

If equation (6.4) holds, then $\overline{h}_{ij} = 0$ which is not possible. Hence $c = 0$ and $\overline{E}_{ij} = 0$. This completes the proof of Theorem 1.4. □

7 H-curvature

The non-Riemannian quantity $H = H_{ij} dx^i \otimes dx^j$ is defined by

$$H_{ij} := E_{ij|s} y^s \tag{7.1}$$

A Finsler metric is called of almost vanishing H-curvature if $H_{ij} = \frac{n+1}{2}\theta F_{y^i y^j}$, for some 1-form θ on M.

Proof of theorem 1.5 let $\overline{F}$ be of almost vanishing $\overline{H}$-curvature, that is,

$$\overline{H}_{ij} = \frac{n+1}{2}\theta \overline{F}_{y^i y^j}, \tag{7.2}$$

for some 1-form θ on M.

In view of equations (7.1) and (7.2), we have left hand side is rational function in y while right hand side irrational due to presence of $\overline{F}$. Thus equation (7.2) implies $\theta = 0$ and hence $\overline{H}_{ij} = 0$. This completes the proof of Theorem (1.5). □

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