Certain types of $(LCS)_n$-manifold and the case of the Riemannian soliton

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Abstract. Although the results discussed in the present study have been studied separately and independently by many geometers, in the present work the results are more generalised in a single stroke for different semisymmetric conditions based on the study of equivalence of geometric structures (initiated by Shaikh and Kundu in 2013) by considering different conditions into various groups or classes in $(LCS)_n$-manifolds. The present paper aims to investigate the nature of Ricci tensor in D-homothetically deformed $(LCS)_n$-manifold under various groups of semisymmetric structures.

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1 Introduction

Herewith and in the sequel, the symbols ∇ and ∇^d stand for the Levi-Civita connection and the D-homothetically deformed connection respectively. Also, the symbols r, Q, S, R, C, E, P, K, $\mathcal{M}$ and $\mathcal{W}_i$ stand for scalar curvature, Ricci operator, Ricci tensor, curvature tensor, conformal curvature tensor[8], concircular curvature tensor[30], projective curvature tensor[30], conharmonic curvature tensor[12], M-projective curvature tensor[18], $\mathcal{W}_i$-curvature tensor([18], [19], [20]) and $\mathcal{W}_i^*$-curvature tensor, $i = 1, 2, ..., 9$ with respect to ∇ respectively, whereas we use the same symbols with superscript d for the same notions with respect to ∇^d.

Definition 1.1. Let T and D be two tensors of type (0, 4). A semi-Riemannian (or Riemannian) manifold is said to be T-semisymmetric type if $D(Y, Z) \cdot T = 0$ for all Y, Z $\in \chi(M)$, the set all vector fields of the manifold M where $D(X, Y)$ acts on T as derivation of tensor algebra. The above condition is often written as $D \cdot T = 0$ .Especially, if we consider $D = T = R$, then the manifold is called semisymmetric [26]. Details about the semisymmetry and other conditions of semisymmetry type are available in : [4], [22], [5], [2], [17], [27] and also references therein.

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In 2013, Kundu and Shaikh ([24], Table 2) investigated the equivalency of the various geometric structures. They have pointed out the following conditions

i) $E \cdot P = 0$, $E \cdot R = 0$, $E \cdot E = 0$, $E \cdot P^* = 0$, $E \cdot \mathcal{M} = 0$, $E \cdot \mathcal{W}_i = 0$ and $E \cdot \mathcal{W}_i^* = 0$ (for all $i = 1, 2, ...9$) are equivalent and named such a class by G_1;

ii) $R \cdot P = 0$, $R \cdot R = 0$, $R \cdot E = 0$, $R \cdot P^* = 0$, $R \cdot \mathcal{M} = 0$, $R \cdot \mathcal{W}_i = 0$ and $R \cdot \mathcal{W}_i^* = 0$ (for all $i = 1, 2, ...9$) are equivalent and named such a class by G_2;

iii) $R \cdot C = 0$ and $R \cdot K = 0$ are equivalent and named such a class by G_3;

iv) $E \cdot C = 0$ and $E \cdot K = 0$ are equivalent and named such a class by G_4;

where

$$C(X,Y) = R(X,Y) - \frac{1}{n-2}\left[\frac{r}{(n-1)}(X \wedge_g Y) + (X \wedge_g QY) + (QX \wedge_g Y)\right], \tag{1.1}$$

$$P(X,Y) = R(X,Y) - \frac{1}{n-1}(X \wedge_S Y), \tag{1.2}$$

$$\mathcal{W}_2(X,Y) = R(X,Y) - \frac{1}{(n-1)}\left[(QX \wedge_g Y) + (X \wedge_g QY) - (X \wedge_S Y)\right], \tag{1.3}$$

$$\mathcal{W}_2^*(X,Y) = R(X,Y) + \frac{1}{(n-1)}\left[(QX \wedge_g Y) + (X \wedge_g QY) - (X \wedge_S Y)\right], \tag{1.4}$$

$$E(X,Y) = R(X,Y) - \frac{r}{n(n-1)}(X \wedge_g Y), \tag{1.5}$$

$$K(X,Y) = R(X,Y) - \frac{1}{n-2}\left[(X \wedge_g QY) + (QX \wedge_g Y)\right], \tag{1.6}$$

$$\mathcal{M}(X,Y) = R(X,Y) - \frac{1}{2(n-1)}\left[(X \wedge_g QY) + (QX \wedge_g Y)\right], \tag{1.7}$$

$$\mathcal{W}_0^*(X,Y) = R(X,Y) + \frac{1}{(n-1)}(X \wedge_g QY),, \tag{1.8}$$

$$\mathcal{W}_0(X,Y) = R(X,Y) - \frac{1}{(n-1)}(X \wedge_g QY) \tag{1.9}$$

$$\mathcal{W}_1(X,Y) = R(X,Y) - \frac{1}{(n-1)}(X \wedge_S Y), \tag{1.10}$$

$$\mathcal{W}_1^*(X,Y) = R(X,Y) + \frac{1}{(n-1)}(X \wedge_S Y), \tag{1.11}$$

$$\mathcal{W}_3(X,Y) = R(X,Y) - \frac{1}{(n-1)}(Y \wedge_g QX), \tag{1.12}$$

$$\mathcal{W}_3^*(X,Y) = R(X,Y) + \frac{1}{(n-1)}(Y \wedge_g QX), \tag{1.13}$$

$$\mathcal{W}_5(X,Y) = R(X,Y) - \frac{1}{(n-1)}\left[(X \wedge_g QY) - (X \wedge_S Y)\right], \tag{1.14}$$

$$\mathcal{W}_5^*(X,Y) = R(X,Y) + \frac{1}{(n-1)}\left[(X \wedge_g QY) - (X \wedge_S Y)\right], \tag{1.15}$$

$$\mathcal{W}_7(X,Y) = R(X,Y) + \frac{1}{(n-1)}\left[(QX \wedge_g Y) - (X \wedge_S Y)\right], \tag{1.16}$$

$$\mathcal{W}_7^*(X,Y) = R(X,Y) - \frac{1}{(n-1)}\left[(QX \wedge_g Y) - (X \wedge_S Y)\right], \tag{1.17}$$

$$\mathcal{W}_4(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[g(X,Z)QY - g(X,Y)QZ], \tag{1.18}$$

$$\mathcal{W}_4^*(X,Y)Z = R(X,Y)Z + \frac{1}{(n-1)}[g(X,Z)QY - g(X,Y)QZ], \tag{1.19}$$

$$\mathcal{W}_6(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}[S(Y,Z)X - g(X,Y)QZ]. \tag{1.20}$$

where

$$(X \wedge_B Y)Z = B(Y,Z)X - B(X,Z)Y. \tag{1.21}$$

The present paper is structured as follows: In Section 2, we briefly recall some known results for $(LCS)_n$-manifolds and the properties of D-homothetic deformed $(LCS)_n$-manifold. In Section 3, we study D-homothetic deformed $(LCS)_n$-manifolds belonging to the class G_i $(i = 1,2,3,4)$ and we show that a D-homothetic deformed $(LCS)_n$-manifold belonging to the classes G_1, G_2, G_3 and G_4 are Einstein manifold. In section-4, we find out the conditions for which the Riemann soliton in $(LCS)_n$-manifold belonging to the various classes are sometimes expanding, steady and some other times shrinking.

2 Properties of the $(LCS)_n$-manifold

Let (M^n, g) be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, the characteristic vector field of the manifold. In a $(LCS)_n$-manifold, the following relations hold [[21], [22]]:

$$(\nabla_X \eta)(Y) = \alpha\{g(X,Y) + \eta(X)\eta(Y)\} \qquad (\alpha \neq 0) \tag{2.1}$$

$$\nabla_X \alpha = (X\alpha) = \alpha(X) = \rho\eta(X) \tag{2.2}$$

$$\phi X = \frac{1}{\alpha}\nabla_X \xi \tag{2.3}$$

$$\phi X = X + \eta(X)\xi \tag{2.4}$$

$$\eta(\xi) = -1, \quad \phi \circ \xi = 0, \tag{2.5}$$

$$\eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.6}$$

$$\eta(R(X,Y)Z) = (\alpha^2 - \rho)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.7}$$

$$R(X,Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.8}$$

$$R(\xi,X)Y = (\alpha^2 - \rho)[g(X,Y)\xi - \eta(Y)X], \tag{2.9}$$

$$S(X,\xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{2.10}$$

for any vector fields X, Y, Z.

Definition 2.1. [28] Let M^n be an $(LCS)_n$-manifold with structure (ϕ, ξ, η, g). If the Lorentzian concircular structure (ϕ, ξ, η, g) on M^n is transformed into (ϕ^d, ξ^d, η^d, g^d) such that

$$\phi^d = \phi, \;\; \xi^d = \frac{1}{b}\xi, \;\; \eta^d = b\eta, \;\; g^d = bg - b(b-1)\eta \otimes \eta, \tag{2.11}$$

where b is a positive constant, then the transformation is called a D-homothetic deformation.

The D-homothetic deformation are studied by various authors in [23], [6], [14], [15], [16], [25].

The relation between the ∇ of g and ∇^d of g^d is given by ([1]),

$$\nabla^d_X Y = \nabla_X Y - \frac{(b-1)\alpha}{b} g(\phi X, \phi Y)\xi, \tag{2.12}$$

for any vector fields X, Y on M^n.

In view of (2.11), (2.12) and definition of Riemannian curvature tensor, Ricci tensor, scalar curvature, one can easily bring out the followings:

Proposition 2.1. *If a Lorentzian concircular structure (ϕ, ξ, η, g) on M^n is transformed into (ϕ^d, ξ^d, η^d, g^d) under a D-homothetic deformation, then R, R^d, S, S^d, r and r^d are related by*

$$\begin{aligned} & R^d(X,Y)Z \\ = \; & R(X,Y)Z - \frac{(b-1)\alpha^2}{b}[g(\phi Y, \phi Z)X - g(\phi X, \phi Z)Y], \end{aligned} \tag{2.13}$$

$$S^d(X,Y) = S(X,Y) - \frac{(n-1)(b-1)\alpha^2}{b} g(\phi Y, \phi Z), \tag{2.14}$$

$$r^d = r - \frac{(b-1)\alpha^2}{b}(n-1)^2, \tag{2.15}$$

for any vector fields X, Y, Z on M^n.

Now we shall find out some properties of a D-homothetically deformed structure (ϕ^d, ξ^d, η^d, g^d) of a Lorentzian concircular structure manifold M^n as follows:

Proposition 2.2. *Under a D-homothetic deformation of a Lorentzian concircular structure (ϕ, ξ, η, g) on M^n is transformed into (ϕ^d, ξ^d, η^d, g^d), then for any vector fields X, Y, Z on M^n, we have*

$$\phi^d X = X + \eta^d(X)\xi^d, \tag{2.16}$$

$$\eta^d(\xi^d) = -1, \tag{2.17}$$

$$\phi^d \xi^d = 0, \;\; \eta^d \circ \phi^d = 0, \tag{2.18}$$

$$g^d(\phi^d X, \phi^d Y) = g^d(X,Y) + \eta^d(X)\eta^d(Y), \tag{2.19}$$

$$g^d(X, \xi^d) = \eta^d(X), \tag{2.20}$$

$$\nabla^d_X \xi^d = \frac{\alpha}{b}[X + \eta^d(X)\xi^d], \tag{2.21}$$

$$(\nabla^d_X \eta^d)(Y) = \frac{\alpha}{b}[g^d(X,Y) + \eta^d(X)\eta^d(Y)], \tag{2.22}$$

$$S^d(X, \xi^d) = \frac{(\alpha^2 - b\rho)}{b^4}[b(b+n-1)-1]\eta^d(X), \tag{2.23}$$

$$\eta^d(R^d(X,Y)Z) = \frac{(\alpha^2 - b\rho)}{b^4}[g^d(Y,Z)\eta^d(X) - g^d(X,Z)\eta^d(Y)], \tag{2.24}$$

$$R^d(\xi^d, X)Y = \frac{(\alpha^2 - b\rho)}{b^4}[g^d(X,Y)\xi^d - \eta^d(Y)X], \tag{2.25}$$

$$R^d(X,Y)\xi^d = \frac{(\alpha^2 - b\rho)}{b^4}[\eta^d(Y)X - \eta^d(X)Y]. \tag{2.26}$$

for any vector fields X and Y on M^n.

In view of (2.24) from (1.1), (1.5), (1.2), (1.6) and (1.21) one can easily bring out the followings:

$$\begin{aligned} & g^d(C^d(X,Y)Z, \xi^d) \\ = \; & \eta^d(C^d(X,Y)Z)) \\ = \; & (\frac{(\alpha^2 - b\rho)}{b^4} - \frac{(\alpha^2 - b\rho)[b(b+n-1)-1]}{b^4(n-2)} \\ & - \frac{r^d}{(n-1)(n-2)})(g^d(Y,Z)\eta^d(X) - g^d(X,Z)\eta^d(Y)) \\ & - \frac{1}{n-2}(S^d(Y,Z)\eta^d(X) - S^d(X,Z)\eta^d(Y)), \end{aligned} \tag{2.27}$$

$$\begin{aligned} & g^d(K^d(X,Y)Z, \xi^d) \\ = \; & \eta^d(K^d(X,Y)Z)) \\ = \; & (\frac{(\alpha^2 - b\rho)}{b^4} - \frac{(\alpha^2 - b\rho)[b(b+n-1)-1]}{b^4(n-2)}) \\ & (g^d(Y,Z)\eta^d(X) - g^d(X,Z)\eta^d(Y)) \\ & - \frac{1}{n-2}(S^d(Y,Z)\eta^d(X) - S^d(X,Z)\eta^d(Y)), \end{aligned} \tag{2.28}$$

$$\begin{aligned} & g^d(E^d(X,Y)Z,\xi^d) \\ = \ & \eta^d(E^d(X,Y)Z)) \\ = \ & (\frac{(\alpha^2-b\rho)}{b^4}-\frac{r^d}{n(n-1)})[g^d(Y,Z)\eta^d(X)-g^d(X,Z)\eta^d(Y)], \end{aligned} \tag{2.29}$$

$$\begin{aligned} & g^d(P^d(X,Y)Z,\xi^d) \\ = \ & \eta^d(P^d(X,Y)Z)) \\ = \ & \frac{(\alpha^2-b\rho)}{b^4}[g^d(Y,Z)\eta^d(X)-g^d(X,Z)\eta^d(Y)] \\ - \ & \frac{1}{n-2}[S^d(Y,Z)\eta^d(X)-S^d(X,Z)\eta^d(Y)]. \end{aligned} \tag{2.30}$$

3 Semi-symmetric structures on D-homothetically deformed $(LCS)_n$-manifold

In this section we consider different types of semi-symmetric structures on D-homothetically deformed $(LCS)_n$-manifolds, namely, D-homothetically deformed $(LCS)_n$-manifolds belonging to the class G_i $(i=1,2,3,4)$.

3.1 D-homothetically deformed $(LCS)_n$-manifolds belonging to the class G_1

Here, we take D-homothetically deformed $(LCS)_n$-manifolds admitting the condition

$$(E^d(X,Y)\cdot R^d)(Z,U)V=0,$$

which implies

$$\begin{aligned} & g^d(E^d(\xi^d,Y)R^d(Z,U)V,\xi^d)=g^d(R^d(E^d(\xi^d,Y)Z,U)V,\xi^d) \\ & +g^d(R^d(Z,E^d(\xi^d,Y)U)V,\xi^d)+g^d(R^d(Z,U)E^d(\xi^d,Y)V,\xi^d). \end{aligned} \tag{3.1}$$

Setting $Y=Z=e_i$ in (3.1) and taking the summation over i, $1\le i\le n$, we get

$$\begin{aligned} & \sum_{i=1}^n g^d(E^d(\xi^d,e_i)R^d(e_i,U)V,\xi^d) \\ = \ & \sum_{i=1}^n g^d(R^d(E^d(\xi^d,e_i)e_i,U)V,\xi^d)+\sum_{i=1}^n g^d(R^d(e_i,E^d(\xi^d,e_i)U)V,\xi^d) \\ & +\sum_{i=1}^n g^d(R^d(e_i,U)E^d(\xi^d,e_i)V,\xi^d). \end{aligned} \tag{3.2}$$

Using (2.16)- (2.26) and (2.29), we obtain

$$\begin{aligned} & \sum_{i=1}^n g^d(E^d(\xi^d,e_i)R^d(e_i,U)V,\xi^d)=[\frac{r^d}{n(n-1)}-\frac{(\alpha^2-b\rho)}{b^4}] \\ & [S^d(U,V)-\frac{(\alpha^2-b\rho)}{b^4}\{g^d(U,V)+\eta^d(U)\eta^d(V)\}], \end{aligned} \tag{3.3}$$

$$\sum_{i=1}^{n} g^d(R^d(E^d(\xi^d, e_i)e_i, U)V, \xi^d) = \frac{(\alpha^2 - b\rho)}{b^4}[\frac{r^d}{n(n-1)} - \frac{(\alpha^2 - b\rho)}{b^4}]$$
$$[b(b+n-1)-1][g^d(U,V) + \eta^d(U)\eta^d(V)], \tag{3.4}$$

$$\sum_{i=1}^{n} g^d(R^d(e_i, E^d(\xi^d, e_i)U)V, \xi^d)$$
$$= -\frac{(\alpha^2 - b\rho)}{b^4}[\frac{r^d}{n(n-1)} - \frac{(\alpha^2 - b\rho)}{b^4}][g^d(U,V) + \eta^d(U)\eta^d(V)], \tag{3.5}$$

$$\sum_{i=1}^{n} g^d(R^d(e_i, U)E^d(\xi^d, e_i)V, \xi^d)$$
$$= -\frac{(\alpha^2 - b\rho)}{b^4}[\frac{r^d}{n(n-1)} - \frac{(\alpha^2 - b\rho)}{b^4}][b(b+n-1)-1]\eta^d(U)\eta^d(V). \tag{3.6}$$

In consequence of (3.3), (3.4), (3.5) and (3.6), the equation (3.2) yields

$$S^d(U,V) = \frac{(\alpha^2 - b\rho)}{b^4}[b(b+n-1)-1]g^d(U,V) \tag{3.7}$$

Contracting over U and V in (3.7) we obtain

$$r^d = \frac{(\alpha^2 - b\rho)}{b^4}\{b(b+n-1)-1\}b(b+n-1). \tag{3.8}$$

Thus we can state the following theorem:

Theorem 3.1. *A D-homothetically deformed (LCS)$_n$-manifold belonging to the class G_1 is an Einstein manifold.*

Theorem 3.2. *The scaler curvature of a D-homothetically deformed (LCS)$_n$-manifolds belonging to the class G_1 satisfies (3.8).*

3.2 D-homothetically deformed $(LCS)_n$-manifolds belonging to the class G_2

Here, we cosider D-homothetically deformed $(LCS)_n$-manifolds admitting the condition

$$(R^d(X,Y) \cdot R^d)(Z,U)V = 0,$$

which means

$$g^d(R^d(\xi^d, Y)R^d(Z,U)V, \xi^d) = g^d(R^d(R^d(\xi^d, Y)Z, U)V, \xi^d)$$
$$+g^d(R^d(Z, R^d(\xi^d, Y)U)V, \xi^d) + g^d(R^d(Z,U)R^d(\xi^d, Y)V, \xi^d). \tag{3.9}$$

Putting $Y = Z = e_i$ in (3.9), where $\{e_1, e_2, e_3, ..., e_{n-1}, e_n = \xi^d\}$ is an orthonormal basis of the tangent space at each point of the manifold M^n and taking the summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned}&\sum_{i=1}^{n} g^d(R^d(\xi^d, e_i)R^d(e_i, U)V, \xi^d)\\ =& \sum_{i=1}^{n} g^d(R^d(R^d(\xi^d, e_i)e_i, U)V, \xi^d) + \sum_{i=1}^{n} g^d(R^d(e_i, R^d(\xi^d, e_i)U)V, \xi^d)\\ &+ \sum_{i=1}^{n} g^d(R^d(e_i, U)R^d(\xi^d, e_i)V, \xi^d).\end{aligned} \tag{3.10}$$

Using (2.16)- (2.26), we obtain

$$\begin{aligned}&\sum_{i=1}^{n} g^d(R^d(\xi^d, e_i)R^d(e_i, U)V, \xi^d)\\ =& \frac{(\alpha^2 - b\rho)}{b^4}[-S^d(U, V) + \frac{(\alpha^2 - b\rho)}{b^4}\{g^d(U, V) + \eta^d(U)\eta^d(V)\}],\end{aligned} \tag{3.11}$$

$$\begin{aligned}&\sum_{i=1}^{n} g^d(R^d(R^d(\xi^d, e_i)e_i, U)V, \xi^d)\\ =& -\frac{(\alpha^2 - b\rho)^2}{b^8}\{b(b + n - 1) - 1\}[g^d(U, V) + \eta^d(U)\eta^d(V)],\end{aligned} \tag{3.12}$$

$$\sum_{i=1}^{n} g^d(R^d(e_i, R^d(\xi^d, e_i)U)V, \xi^d) = \frac{(\alpha^2 - b\rho)^2}{b^8}[g^d(U, V) + \eta^d(U)\eta^d(V)], \tag{3.13}$$

$$\begin{aligned}&\sum_{i=1}^{n} g^d(R^d(e_i, U)R^d(\xi^d, e_i)V, \xi^d)\\ =& \frac{(\alpha^2 - b\rho)^2}{b^8}\{b(b + n - 1) - 1\}\eta^d(U)\eta^d(V).\end{aligned} \tag{3.14}$$

By virtue of (3.11), (3.12), (3.13) and (3.14), the equation (3.10) yields

$$S^d(U, V) = \frac{(\alpha^2 - b\rho)}{b^4}\{b(b + n - 1) - 1\}g^d(U, V). \tag{3.15}$$

Contracting over U and V in (3.15) we obtain

$$r^d = \frac{(\alpha^2 - b\rho)}{b^4}\{b(b + n - 1) - 1\}b(b + n - 1). \tag{3.16}$$

ThIs leads to the following theorem:

Theorem 3.3. *A D-homothetically deformed $(LCS)_n$-manifold belonging to the class G_2 is an Einstein manifold.*

Theorem 3.4. *The scaler curvature of a D-homothetically deformed $(LCS)_n$-manifolds belonging to the class G_2 satisfies (3.16).*

3.3 D-homothetically deformed $(LCS)_n$-manifolds belonging to the class G_3

Choose a D-homothetically deformed $(LCS)_n$-manifolds admitting the condition

$$(R^d(X,Y)\cdot K^d)(Z,U)V=0,$$

that is,

$$\begin{aligned} & g^d(R^d(\xi^d,Y)K^d(Z,U)V,\xi^d)=g^d(K^d(R^d(\xi^d,Y)Z,U)V,\xi^d) \\ & +g^d(K^d(Z,R^d(\xi^d,Y)U)V,\xi^d)+g^d(K^d(Z,U)R^d(\xi^d,Y)V,\xi^d). \end{aligned} \tag{3.17}$$

Putting $Y=Z=e_i$ in (3.17) and taking the summation over i, $1\leq i\leq n$, we get

$$\begin{aligned} & \sum_{i=1}^{n} g^d(R^d(\xi^d,e_i)K^d(e_i,U)V,\xi^d) \\ = & \sum_{i=1}^{n} g^d(K^d(R^d(\xi^d,e_i)e_i,U)V,\xi^d)+\sum_{i=1}^{n} g^d(K^d(e_i,R^d(\xi^d,e_i)U)V,\xi^d) \\ & +\sum_{i=1}^{n} g^d(K^d(e_i,U)R^d(\xi^d,e_i)V,\xi^d). \end{aligned} \tag{3.18}$$

In view of (2.16)- (2.26) and (2.28), we obtain

$$\begin{aligned} & \sum_{i=1}^{n} g^d(R^d(\xi^d,e_i)K^d(e_i,U)V,\xi^d) \\ = & -\frac{(\alpha^2-b\rho)}{b^4}[S^d(U,V)-\frac{1}{n-2}\{S^d(U,V)\{b(b+n-1)-2\}+r^dg^d(U,V)\} \\ & -(\frac{(\alpha^2-b\rho)}{b^4}-\frac{(\alpha^2-b\rho)[b(b+n-1)-1]}{b^4(n-2)})\{g^d(U,V)+\eta^d(U)\eta^d(V)\} \\ & +\frac{1}{n-2}\{S^d(U,V)+\frac{(\alpha^2-b\rho)}{b^4}\{b(b+n-1)-1\}\eta^d(U)\eta^d(V)\}], \end{aligned} \tag{3.19}$$

$$\begin{aligned} & \sum_{i=1}^{n} g^d(K^d(R^d(\xi^d,e_i)e_i,U)V,\xi^d) \\ = & -\frac{(\alpha^2-b\rho)\{b(b+n-1)-1\}}{b^4}(\frac{(\alpha^2-b\rho)}{b^4} \\ & -\frac{(\alpha^2-b\rho)[b(b+n-1)-1]}{b^4(n-2)})[g^d(U,V) \\ & +\eta^d(U)\eta^d(V)]+\frac{\{b(b+n-1)-1\}}{n-2}\frac{(\alpha^2-b\rho)}{b^4}[S^d(U,V) \\ & +\frac{(\alpha^2-b\rho)}{b^4}\{b(b+n-1)-1\}\eta^d(U)\eta^d(V)\}], \end{aligned} \tag{3.20}$$

$$\begin{aligned}
&\sum_{i=1}^{n} g^d(K^d(e_i, R^d(\xi^d, e_i)U)V, \xi^d) \\
= \;& \frac{(\alpha^2 - b\rho)}{b^4}\Big(\frac{(\alpha^2 - b\rho)}{b^4} - \frac{(\alpha^2 - b\rho)[b(b+n-1)-1]}{b^4(n-2)}\Big)[g^d(U,V) \\
&+\eta^d(U)\eta^d(V)] - \frac{1}{n-2}\frac{(\alpha^2 - b\rho)}{b^4}[S^d(U,V) \\
&+\frac{(\alpha^2 - b\rho)}{b^4}\{b(b+n-1)-1\}\eta^d(U)\eta^d(V)], \qquad (3.21)
\end{aligned}$$

$$\begin{aligned}
&\sum_{i=1}^{n} g^d(K^d(e_i, U)R^d(\xi^d, e_i)V, \xi^d) \\
= \;& \frac{(\alpha^2 - b\rho)\{b(b+n-1)-1\}}{b^4} \\
&\Big(\frac{(\alpha^2 - b\rho)}{b^4} - \frac{(\alpha^2 - b\rho)[b(b+n-1)-1]}{b^4(n-2)}\Big)\eta^d(U)\eta^d(V) \\
&-\frac{\{b(b+n-1)-1\}^2}{n-2}\Big[\frac{(\alpha^2 - b\rho)}{b^4}\Big]^2\eta^d(U)\eta^d(V). \qquad (3.22)
\end{aligned}$$

In view of (3.19), (3.20), (3.21) and (3.22), the equation (3.18) yields

$$\begin{aligned}
&S^d(U,V) \\
= \;& \Big[\frac{r^d}{(n-1)} - \{b(b+n-1)-1\} \\
&\frac{(\alpha^2 - b\rho)[b^2 + n(b-1) - b + 3]}{b^4(n-1)}\Big]g^d(U,V). \qquad (3.23)
\end{aligned}$$

Setting $U = V = \xi$ in (3.23) we obtain

$$r^d = -\frac{(\alpha^2 - b\rho)}{b^4}\{b(b+n-1)-1\}\{b^2 + b(n-1) + 2\}. \qquad (3.24)$$

Thus we can state the following theorem:

Theorem 3.5. *A D-homothetically deformed $(LCS)_n$-manifold belonging to the class G_3 is an Einstein manifold.*

Theorem 3.6. *The scaler curvature of a D-homothetically deformed $(LCS)_n$-manifolds belonging to the class G_3 satisfies (3.24).*

3.4 D-homothetically deformed $(LCS)_n$manifolds belonging to the class G_4

Here, we consider D-homothetically deformed $(LCS)_n$-manifolds admitting the condition

$$(E^d(X,Y) \cdot K^d)(Z,U)V = 0,$$

which implies

$$g^d(E^d(\xi^d,Y)K^d(Z,U)V,\xi^d) = g^d(K^d(E^d(\xi^d,Y)Z,U)V,\xi^d) \\ +g^d(K^d(Z,E^d(\xi^d,Y)U)V,\xi^d) + g^d(K^d(Z,U)E^d(\xi^d,Y)V,\xi^d). \tag{3.25}$$

Putting $Y = Z = e_i$ in (3.25) and taking the summation over i, $1 \leq i \leq n$, we get

$$\begin{aligned} &\sum_{i=1}^{n} g^d(E^d(\xi^d,e_i)K^d(e_i,U)V,\xi^d) \\ = &\sum_{i=1}^{n} g^d(K^d(E^d(\xi^d,e_i)e_i,U)V,\xi^d) + \sum_{i=1}^{n} g^d(K^d(e_i,E^d(\xi^d,e_i)U)V,\xi^d) \\ &+\sum_{i=1}^{n} g^d(K^d(e_i,U)E^d(\xi^d,e_i)V,\xi^d). \end{aligned} \tag{3.26}$$

Using (2.16)- (2.26) and (2.28), (2.29), we obtain

$$\begin{aligned} &\sum_{i=1}^{n} g^d(E^d(\xi^d,e_i)K^d(e_i,U)V,\xi^d) \\ = &-[\frac{(\alpha^2-b\rho)}{b^4} - \frac{r^d}{n(n-1)}][S^d(U,V) \\ &-\frac{1}{n-2}\{S^d(U,V)\{b(b+n-1)-2\} + r^d g^d(U,V)\} \\ &-(\frac{(\alpha^2-b\rho)}{b^4} - \frac{(\alpha^2-b\rho)[b(b+n-1)-1]}{b^4(n-2)})\{g^d(U,V)+\eta^d(U)\eta^d(V)\} \\ &+\frac{1}{n-2}\{S^d(U,V) + \frac{(\alpha^2-b\rho)}{b^4}\{b(b+n-1)-1\}\eta^d(U)\eta^d(V)\}], \end{aligned} \tag{3.27}$$

$$\begin{aligned} &\sum_{i=1}^{n} g^d(K^d(E^d(\xi^d,e_i)e_i,U)V,\xi^d) \\ = &-[\frac{(\alpha^2-b\rho)}{b^4} - \frac{r^d}{n(n-1)}]\{b(b+n-1)-1\} \\ &(\frac{(\alpha^2-b\rho)}{b^4} - \frac{(\alpha^2-b\rho)[b(b+n-1)-1]}{b^4(n-2)})[g^d(U,V) \\ &+\eta^d(U)\eta^d(V)] + \frac{\{b(b+n-1)-1\}}{n-2}[\frac{(\alpha^2-b\rho)}{b^4} - \frac{r^d}{n(n-1)}][S^d(U,V) \\ &+\frac{(\alpha^2-b\rho)}{b^4}\{b(b+n-1)-1\}\eta^d(U)\eta^d(V)\}], \end{aligned} \tag{3.28}$$

$$\sum_{i=1}^{n} g^d(K^d(e_i, E^d(\xi^d, e_i)U)V, \xi^d)$$
$$= [\frac{(\alpha^2 - b\rho)}{b^4} - \frac{r^d}{n(n-1)}](\frac{(\alpha^2 - b\rho)}{b^4} - \frac{(\alpha^2 - b\rho)[b(b+n-1)-1]}{b^4(n-2)})[g^d(U,V)$$
$$+\eta^d(U)\eta^d(V)] - \frac{1}{n-2}[\frac{(\alpha^2 - b\rho)}{b^4} - \frac{r^d}{n(n-1)}][S^d(U,V)$$
$$+\frac{(\alpha^2 - b\rho)}{b^4}\{b(b+n-1)-1\}\eta^d(U)\eta^d(V)], \tag{3.29}$$

$$\sum_{i=1}^{n} g^d(K^d(e_i, U)E^d(\xi^d, e_i)V, \xi^d)$$
$$= -[\frac{(\alpha^2 - b\rho)}{b^4} - \frac{r^d}{n(n-1)}]\frac{\{b(b+n-1)-1\}^2}{n-2}\frac{(\alpha^2 - b\rho)}{b^4}\eta^d(U)\eta^d(V)$$
$$+[\frac{(\alpha^2 - b\rho)}{b^4} - \frac{r^d}{n(n-1)}]\{b(b+n-1)-1\}$$
$$(\frac{(\alpha^2 - b\rho)}{b^4} - \frac{(\alpha^2 - b\rho)[b(b+n-1)-1]}{b^4(n-2)})\eta^d(U)\eta^d(V). \tag{3.30}$$

By virtue of (3.27), (3.28), (3.29) and (3.30), the equation (3.26) yields

$$S^d(U,V)$$
$$= [\frac{r^d}{(n-1)} - \{b(b+n-1)-1\}$$
$$\frac{(\alpha^2 - b\rho)[b^2 + n(b-1) - b + 3]}{b^4(n-1)}]g^d(U,V). \tag{3.31}$$

Setting $U = V = \xi$ in (3.31) we obtain

$$r^d = -\frac{(\alpha^2 - b\rho)}{b^4}\{b(b+n-1)-1\}\{b^2 + b(n-1) + 2\}. \tag{3.32}$$

Thus we can state the following theorem:

Theorem 3.7. *A D-homothetically deformed $(LCS)_n$-manifold belonging to the class G_4 is an Einstein manifold.*

Theorem 3.8. *The scaler curvature of a D-homothetically deformed $(LCS)_n$-manifolds belonging to the class G_4 satisfies (3.32).*

4 Semisymmetric structures on D-homothetically deformed $(LCS)_n$-manifold and the case of the Riemannian soliton

Hirica and Udrisste [11] in 2016 introduced and studied Riemann soliton. A smooth manifold M with Riemannian metric g is called Riemann soliton if g satisfies

$$2R + \lambda(g \wedge g) + (g \wedge \pounds_\xi g) = 0 \tag{4.1}$$

where ξ is a potential vector field, $\pounds_\xi$ denotes the Lie-derivative and λ is a constant and for $(0,2)$-tensors π and ϖ, the Kulkarni-Nomizu product $(a \wedge b)$ is given by

$$\begin{aligned}(\pi \wedge \varpi)(Y,U,V,Z) &= \pi(Y,V)\varpi(U,Z) + \pi(U,Z)\varpi(Y,V)\\ &\quad -\pi(Y,Z)\varpi(U,V) - \pi(U,V)\varpi(Y,Z).\end{aligned} \tag{4.2}$$

A Riemann soliton is called expanding, steady and shrinking when $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively. The Riemann soliton are also studied in [7], [10], [9].

In this section, we study D-homothetically deformed $(LCS)_n$-manifold with semisymmetric structures when the metric g^d is deformed Riemann soliton. Thus from (4.1), we get

$$2R^d + \lambda(g^d \wedge g^d) + (g^d \wedge \pounds^d_{\xi^d} g^d) = 0. \tag{4.3}$$

Now, using (4.2) in (4.3) and then contracting U over V we obtain

$$\begin{aligned}&S^d(Y,Z)\\ =\ &[\{\lambda + \frac{2\alpha}{b}\}\{b(b+n-1)-1\} - \frac{\alpha}{b}]g^d(Y,Z)\\ &+\frac{\alpha}{b}\{b(b+n-1)-2\}\eta^d(Y)\eta^d(Z).\end{aligned} \tag{4.4}$$

Contracting over Y and Z in (4.4) we have

$$\begin{aligned}&r^d\\ =\ &[\{\lambda + \frac{2\alpha}{b}\}\{b(b+n-1)-1\} - \frac{\alpha}{b}]\\ &\{b(b+n-1)\} - \frac{\alpha}{b}\{b(b+n-1)-2\}\end{aligned} \tag{4.5}$$

Comparing (3.16) and (4.5) we get

$$\begin{aligned}&[\frac{(\alpha^2 - b\rho)}{b^4} - \lambda]\{b(b+n-1)-1\}b(b+n-1)\\ =\ &\frac{\alpha}{b} - \frac{2\alpha}{b}\{b(b+n-1)\}^2\end{aligned} \tag{4.6}$$

which yields

Theorem 4.1. *The Riemann soliton of D-homothetically deformed $(LCS)_n$-manifold belonging to the class G_1 and G_2 is expanding, steady and shrinking when $\frac{(\alpha^2-b\rho)}{b^4}k(k-1)+\frac{2\alpha}{b}k^2 >=< \frac{\alpha}{b}$, where $k = b(b+n-1)$.*

Again, comparing (3.24) and (4.5) we get

$$\lambda\{b(b+n-1)\} = -\frac{(\alpha^2 - b\rho)}{b^4}\{b^2 + b(n-1) + 2\} - \frac{2\alpha}{b}\{b(b+n-1)-1\}. \tag{4.7}$$

This leads to

Theorem 4.2. *The Riemann soliton of D-homothetically deformed $(LCS)_n$-manifold belonging to the class G_3 and G_4 is expanding, steady and shrinking when $\frac{2\alpha}{b}(k-1)+\frac{(\alpha^2-b\rho)}{b^4}(k+2) <=> 0$, where $k = b(b+n-1)$.*

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