On a type of quarter-symmetric non-metric ξ-connection on 3-dimensional quasi-Sasakian manifolds

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Abstract. The object of the present paper is to study a type of quarter-symmetric non-metric ξ-connection on a 3-dimensional non-cosymplectic quasi-Sasakian manifold. We investigate the curvature tensor and the Ricci tensor of a 3-dimensional quasi-Sasakian manifold with respect to the quarter-symmetric non-metric ξ-connection. We characterize ξ-projectively flat and ϕ-projectively flat 3-dimensional quasi-Sasakian manifolds with constant structure function admitting the quarter-symmetric non-metric ξ-connection. Also we study second order parallel tensor and Ricci semi-symmetric 3-dimensional quasi-Sasakian manifolds with constant structure function with respect to the quarter-symmetric non-metric ξ-connection. Finally, we give an example to verify our result.

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1 Introduction

On a 3-dimensional quasi-Sasakian manifold, the structure function β was defined by Olszak [21] and with the help of this function he has obtained necessary and sufficient conditions for the manifold to be conformally flat [22]. Next he has proved that if the manifold is additionally conformally flat with $\beta =$ constant, then (a) the manifold is locally a product of $\mathbb{R}$ and a 2-dimensional Kählerian space of constant Gauss curvature (the cosymplectic case), or, (b) the manifold is of constant positive curvature (the non-cosymplectic case, here the quasi-Sasakian structure is homothetic to a Sasakian structure). An example of a 3-dimensional quasi-Sasakian structure being conformally flat with non-constant structure function is also described in [22].

In 1924, Friedmann and Schouten [13] introduced the idea of semi-symmetric connection on a differentiable manifold. A linear connection $\widetilde{\nabla}$ on a differentiable

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manifold M is said to be a semi-symmetric connection if the torsion tensor T of the connection $\widetilde{\nabla}$ satisfies

$$T(X,Y) = u(Y)X - u(X)Y, \tag{1.1}$$

where u is a 1-form. $\chi(M)$ is the set of all differentiable vector fields on M, for all vector fields $X, Y \in \chi(M)$.

In 1932, Hayden [15] introduced the idea of semi-symmetric metric connection on a Riemannian manifold (M, g). A semi-symmetric connection $\widetilde{\nabla}$ is said to be a semi-symmetric metric connection if

$$\widetilde{\nabla} g = 0. \tag{1.2}$$

A relation between a semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇ of (M, g) is given by Yano [26]: $\tilde{\nabla}_X Y = \nabla_X Y + u(Y)X - g(X,Y)\rho_1$, where $u(X) = g(X, \rho_1)$

The study of semi-symmetric metric connections were further developed by Amur and Pujar [1], Binh [9], De [10], Singh et al. [24], Ozgur et al ([18],[19]), Ozen, Uysal Demirbag [20], Zhao ([27, 28]) and many others. After a long gap the study of a semi-symmetric connection $\bar{\nabla}$ satisfying

$$\bar{\nabla} g \neq 0. \tag{1.3}$$

was initiated by Prvanović [23] with the name of pseudo-metric semi-symmetric connection and was just followed by Andonie [2].

A semi-symmetric connection $\bar{\nabla}$ is said to be a semi-symmetric non-metric connection if it satisfies the condition (1.3).

In 1975, Golab [14] defined and studied quarter-symmetric connections in differentiable manifolds with affine connections. A linear connection $\bar{\nabla}$ on a Riemannian manifold M is called a quarter-symmetric connection [14] if its torsion tensor T satisfies $T(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y$, where η is a 1-form and ϕ is a (1,1)-tensor field. In particular, if $\phi X = X$, then the quarter-symmetric connection reduces to the semi-symmetric connection [13]. Thus the notion of quarter-symmetric connection generalizes the notion of semi-symmetric connection.

A quarter-symmetric connection $\check{\nabla}$ is said to be a quarter-symmetric metric connection if $\check{\nabla} g = 0$. If moreover, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition $(\bar{\nabla}_X g)(Y,Z) \neq 0$; for all $X, Y, Z \in \chi(M)$, then $\bar{\nabla}$ is said to be a quarter-symmetric non-metric connection.

The quarter-symmetric non-metric connections have been studied by Mishra and Pandey [16], Singh and Pandey [25], De and Mondal [12], Barman ([4], [3], [5]) Mondal and De [17] and many others.

In this paper we study 3-dimensional quasi-Sasakian manifolds with respect to a type of quarter-symmetric non-metric ξ-connection.

The paper is organized as follows:

After introduction, in section 2, we give a brief account of the 3-dimensional quasi-Sasakian manifolds. In section 3, we define a type of quarter-symmetric non-metric ξ-connection on 3-dimensional quasi-Sasakian manifolds. Section 4 is devoted to establish the relation between the curvature tensors with respect to a type of quarter-symmetric non-metric ξ-connection and the Levi-Civita connection. Next we study the projective curvature tensor with respect to the type of quarter-symmetric non-metric ξ-connection. Among others we characterize ϕ-projectively flat 3-dimensional quasi-Sasakian manifolds with constant structure function admitting a type of quarter-symmetric non-metric ξ-connection. Also we study a second order parallel tensor on a Ricci semi-symmetric 3-dimensional quasi-Sasakian manifolds with constant structure function with respect to a type of quarter-symmetric non-metric ξ-connection. Finally, we give an example to verify our result.

2 Preliminaries

Let M be a $(2n+1)$-dimensional connected differentiable manifold endowed with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field, η is a 1-form and g is the Riemannian metric on M such that ([6], [7])

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(X) = g(X,\xi), \quad \eta(\xi) = 1. \tag{2.3}$$

Let Φ be the fundamental 2-form defined by

$$\Phi(X,Y) = g(X, \phi Y) = -g(\phi X, Y). \tag{2.4}$$

M is said to be quasi-Sasakian if the almost contact structure (ϕ, η, ξ, g) is normal and the fundamental 2-form Φ is closed $(d\Phi = 0)$, which was first introduced by Blair [8]. The normality condition gives that the induced almost contact structure of $M \otimes \mathbb{R}$ is integrable or equivalently, the torsion tensor field $N = [\phi, \phi] + 2\xi \otimes d\eta$ vanishes identically on M. The rank of the quasi-Sasakian structure is always odd [8], it is equal to 1 if the structure is cosymplectic and it is equal to $2n+1$ if the structure is Sasakian.

An almost contact metric manifold M is a 3-dimensional quasi-Sasakian manifold if and only if [21]

$$\nabla_X \xi = -\beta\phi X, X \in \chi(M), \tag{2.5}$$

for a certain function β on M, such that $\xi\beta = 0$, ∇ being the operator of covariant differentiation with respect to the Levi-Civita connection of M. Clearly such a quasi-Sasakian manifold is cosymplectic if and only if $\beta = 0$. From the equation (2.5) we obtain [21]

$$(\nabla_X \phi)(Y) = \beta(g(X,Y)\xi - \eta(Y)X), \tag{2.6}$$

$$(\nabla_X \eta)(Y) = g(\nabla_X \xi, Y) = -\beta g(\phi X, Y) = \beta g(X, \phi Y). \tag{2.7}$$

Let M be a 3-dimensional quasi-Sasakian manifold. The Ricci tensor S of M is given by [22]

$$S(Y,Z) = (\frac{r}{2} - \beta^2)g(Y,Z) + (3\beta^2 - \frac{r}{2})\eta(Y)\eta(Z) - \eta(Y)d\beta(\phi Z) \tag{2.8}$$
$$-\eta(Z)d\beta(\phi Y),$$

where r is the scalar curvature of M.

In a 3-dimensional Riemannian manifold we always have

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y$$
$$-\frac{r}{2}(g(Y,Z)X - g(X,Z)Y),$$

where Q is the Ricci operator, i.e., $g(QX,Y) = S(X,Y)$. Now as a consequence of (2.8), we get for the Ricci operator Q

$$QY = (\frac{r}{2} - \beta^2)Y + (3\beta^2 - \frac{r}{2})\eta(Y)\xi + \eta(Y)(\phi grad\beta) - d\beta(\phi Y)\xi,$$

where the gradient of a function f is related to the exterior derivative df by the formula $df(X) = g(gradf, X)$.

Also from (2.8) it follows that

$$S(\phi Y, \phi Z) = S(Y,Z) - 2\beta^2\eta(Y)\eta(Z). \tag{2.9}$$

3 Quarter-symmetric non-metric ξ-connection on 3-dimensional quasi-Sasakian manifolds

This section deals with the quarter-symmetric non-metric ξ-connection on 3-dimensional non-cosymplectic quasi-Sasakian manifold. Let (M,g) be a quasi-Sasakian manifold with the Levi-Civita connection ∇ and we define a linear connection $\bar{\nabla}$ on M by

$$\bar{\nabla}_X Y = \nabla_X Y + (\beta - 1)\eta(X)\phi Y + \beta\eta(Y)\phi X, \tag{3.1}$$

where β is a certain function on M.

Using (3.1), the torsion tensor T of the connection $\bar{\nabla}$ is given by

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y] = \eta(Y)\phi X - \eta(X)\phi Y, \tag{3.2}$$

Thus the linear connection $\bar{\nabla}$ is a quarter-symmetric connection.

So the equation (3.1) with the help of (2.4) turns into

$$(\bar{\nabla}_X g)(Y,Z) = \bar{\nabla}_X g(Y,Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) = -\beta\eta(Y)g(\phi X, Z) \tag{3.3}$$
$$-\beta\eta(Z)g(Y,\phi X) \neq 0.$$

The linear connection $\bar{\nabla}$ satisfying (3.2) and (3.3) is called a quarter-symmetric non-metric connection on 3-dimensional non-cosymplectic quasi-Sasakian manifold.

By making use of (2.3), (2.5) and (3.1), it is obvious that

$$\bar{\nabla}_X\xi = \nabla_X\xi + (\beta - 1)\eta(X)\phi\xi + \beta\eta(\xi)\phi X = 0. \tag{3.4}$$

The linear connection $\bar{\nabla}$ defined by (3.1) satisfying (3.2), (3.3) and (3.4) is the quarter-symmetric non-metric ξ-connection on 3-dimensional quasi-Sasakian manifold.

Conversely, we show that a linear connection $\bar{\nabla}$ defined on M satisfying (3.2), (3.3) and (3.4) is given by (3.1). Let H be a tensor field of type $(1, 2)$ and

$$\bar{\nabla}_X Y = \nabla_X Y + H(X, Y). \tag{3.5}$$

Then we conclude that

$$T(X, Y) = H(X, Y) - H(Y, X). \tag{3.6}$$

Further using (3.5), it follows that

$$\begin{aligned}(\bar{\nabla}_X g)(Y, Z) = \bar{\nabla}_X g(Y, Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z) = &-g(H(X, Y), Z)\\ &-g(Y, H(X, Z)).\end{aligned} \tag{3.7}$$

In view of (3.3) and (3.7) yields

$$\begin{aligned}g(H(X, Y), Z) + g(Y, H(X, Z)) = &\beta\eta(Y)g(\phi X, Z)\\ &+\beta\eta(Z)g(Y, \phi X).\end{aligned} \tag{3.8}$$

Also using (3.8) and (3.6), we derive that

$$\begin{aligned}g(T(X, Y), Z) + g(T(Z, X), Y) + g(T(Z, Y), X) = 2g(H(X, Y), Z)\\ +2\beta\eta(X)g(Y, \phi Z) + 2\beta\eta(Y)g(X, \phi Z).\end{aligned}$$

The above equation yields

$$\begin{aligned}g(H(X, Y), Z) = \frac{1}{2}[&g(T(X, Y), Z) + g(T(Z, X), Y) + g(T(Z, Y), X)]\\ &-2\beta\eta(X)g(Y, \phi Z) - 2\beta\eta(Y)g(X, \phi Z).\end{aligned} \tag{3.9}$$

Let T' be a tensor field of type $(1, 2)$ given by

$$g(T'(X, Y), Z) = g(T(Z, X), Y). \tag{3.10}$$

Adding (2.4), (3.2) and (3.10), we obtain

$$T'(X, Y) = -\eta(X)\phi Y - g(\phi X, Y)\xi. \tag{3.11}$$

From (3.9) we have, by using (3.10) and (3.11):

$$\begin{aligned} g(H(X,Y),Z) = \frac{1}{2}[g(T(X,Y),Z) + g(T'(X,Y),Z) + g(T'(Y,X),Z)] \\ -2\beta\eta(X)g(Y,\phi Z) - 2\beta\eta(Y)g(X,\phi Z) = (\beta-1)\eta(X)g(\phi Y,Z) \\ +\beta\eta(Y)g(\phi X,Z). \end{aligned} \tag{3.12}$$

Now contracting Z in (3.12) and using (2.4), implies that

$$H(X,Y) = (\beta-1)\eta(X)\phi Y + \beta\eta(Y)\phi X. \tag{3.13}$$

Combining (3.5) and (3.13), it follows that

$$\bar{\nabla}_X Y = \nabla_X Y + (\beta-1)\eta(X)\phi Y + \beta\eta(Y)\phi X.$$

Now, we are in a position to state the following theorem:

Theorem 3.1. *On a 3-dimensional non-cosymplectic quasi-Sasakian manifold with structure function β there exists a unique linear connection $\bar{\nabla}$, satisfies (3.2), (3.3) and (3.4).*

If $\beta = 0$ in the equation (3.1), then we get

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y.$$

Hence, $\bar{\nabla}$ defines a quarter-symmetric metric connection. Such a case has been studied by De and Mandal [11].

4 Curvature tensor of 3-dimensional quasi-Sasakian manifolds admitting quarter-symmetric non-metric ξ-connection

In this section, we obtain the expressions of the curvature tensor and Ricci tensor of M with respect to the quarter-symmetric non-metric ξ-connection on a 3-dimensional non-cosymplectic quasi-Sasakian manifold defined by (3.1).

Analogous to the definition of the curvature tensor of M with respect to the Levi-Civita connection ∇, we define the curvature tensor $\bar{R}$ of M with respect to the quarter-symmetric non-metric ξ-connection $\bar{\nabla}$ by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z, \tag{4.1}$$

where $X, Y, Z \in \chi(M)$.

Using (2.3) and (3.1) in (4.1), we obtain

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z - (\beta-1)\eta(X)(\nabla_Y\phi)(Z) + (\beta-1)\eta(Y)(\nabla_X\phi)(Z)\\ &-\beta(\nabla_Y\eta)(Z)\phi X + \beta(\nabla_X\eta)(Z)\phi Y - (\beta-1)(\nabla_Y\eta)(X)\phi Z\\ &+(\beta-1)(\nabla_X\eta)(Y)\phi Z - \beta\eta(Z)(\nabla_Y\phi)(X) + \beta\eta(Z)(\nabla_X\phi)(Y)\\ &-\beta(\beta-1)\eta(X)\eta(Z)Y + \beta(\beta-1)\eta(Y)\eta(Z)X + (X(\beta-1))\eta(Y)\phi Z\\ &+(X\beta)\eta(Z)\phi Y - (Y(\beta-1))\eta(X)\phi Z - (Y\beta)\eta(Z)\phi X. \end{aligned} \tag{4.2}$$

By making use of (2.3), (2.6) and (2.7) in (4.2), we have

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + \beta^2 g(\phi Y,Z)\phi X - \beta^2 g(\phi X,Z)\phi Y\\ &+2\beta(\beta-1)g(\phi Y,X)\phi Z + \beta(\beta-1)\eta(Y)g(X,Z)\xi\\ &-\beta(\beta-1)\eta(X)g(Y,Z)\xi - \beta^2\eta(Y)\eta(Z)X\\ &+\beta^2\eta(X)\eta(Z)Y + (X(\beta-1))\eta(Y)\phi Z\\ &+(X\beta)\eta(Z)\phi Y - (Y(\beta-1))\eta(X)\phi Z - (Y\beta)\eta(Z)\phi X. \end{aligned} \tag{4.3}$$

So the equation (4.3) turns into

$$\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z.$$

Moreover in view of (4.3) it follows that

$$\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0.$$

Let $\{e_1, e_2, e_3\}$ be a local orthonormal basis of the tangent space at a point of the manifold M. Then by putting $X = U = e_i$ in (4.3) and taking summation over i, $1 \leq i \leq 3$ and also using (2.3), we get

$$\begin{aligned}\bar{S}(Y,Z) &= S(Y,Z) + \beta(2\beta-1)g(Y,Z) - \beta(4\beta-1)\eta(Y)\eta(Z)\\ &+(\phi Z)(\beta)\eta(Y) + (\phi Y)(\beta)\eta(Z), \end{aligned} \tag{4.4}$$

where $\bar{S}$ and S denote the Ricci tensor of M with respect to $\bar{\nabla}$ and ∇ respectively.

Again contracting Y and Z in the above equation (4.4) we have

$$\bar{r} = r - 2\beta + 2\beta^2. \tag{4.5}$$

Summing up all of the above equations we can state the following proposition:

Proposition 4.1. *In a 3-dimensional quasi-Sasakian manifold M admitting the quarter-symmetric non-metric ξ-connection $\bar{\nabla}$:*

(i) The curvature tensor $\bar{R}$ is given by
$\bar{R}(X,Y)Z = R(X,Y)Z + \beta^2 g(\phi Y,Z)\phi X - \beta^2 g(\phi X,Z)\phi Y + 2\beta(\beta-1)g(\phi Y,X)\phi Z + \beta(\beta-1)\eta(Y)g(X,Z)\xi - \beta(\beta-1)\eta(X)g(Y,Z)\xi - \beta^2\eta(Y)\eta(Z)X + \beta^2\eta(X)\eta(Z)Y + (X(\beta-1))\eta(Y)\phi Z + (X\beta)\eta(Z)\phi Y - (Y(\beta-1))\eta(X)\phi Z - (Y\beta)\eta(Z)\phi X,$

(ii) The Ricci tensor $\bar{S}$ is given by
$\bar{S}(Y,Z) = S(Y,Z)+\beta(2\beta-1)g(Y,Z)-\beta(4\beta-1)\eta(Y)\eta(Z)+(\phi Z)(\beta)\eta(Y)+(\phi Y)(\beta)\eta(Z)$,

(iii) $\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z$,

(iv) $\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0$,

(v) $\bar{r} = r - 2\beta + 2\beta^2$.

Moreover, if the structure function β is constant, then we conclude the following:

Corollary 4.2. *In a 3-dimensional non-cosymplectic quasi-Sasakian manifold M with constant structure function admitting the quarter-symmetric non-metric ξ-connection $\bar{\nabla}$:*

(i) The curvature tensor $\bar{R}$ is given by
$\bar{R}(X,Y)Z = R(X,Y)Z + \beta^2 g(\phi Y,Z)\phi X - \beta^2 g(\phi X,Z)\phi Y + 2\beta(\beta-1)g(\phi Y,X)\phi Z + \beta(\beta-1)\eta(Y)g(X,Z)\xi - \beta(\beta-1)\eta(X)g(Y,Z)\xi - \beta^2\eta(Y)\eta(Z)X + \beta^2\eta(X)\eta(Z)Y$,

(ii) The Ricci tensor $\bar{S}$ is given by
$\bar{S}(Y,Z) = S(Y,Z) + \beta(2\beta-1)g(Y,Z) - \beta(4\beta-1)\eta(Y)\eta(Z)$,

(iii) $\bar{R}(X,Y)Z = -\bar{R}(Y,X)Z$,

(iv) $\bar{R}(X,Y)Z + \bar{R}(Y,Z)X + \bar{R}(Z,X)Y = 0$,

(v) $g(\bar{R}(X,Y)Z,U) = -g(\bar{R}(X,Y)U,Z)$,

(vi) $\bar{r} = r - 2\beta + 2\beta^2$.

5 Projective curvature tensor on 3-dimensional non-cosymplectic quasi-Sasakian manifolds with respect to the quarter-symmetric non-metric ξ-connection $\bar{\nabla}$

In this section we characterize ξ-projectively flat and ϕ-projectively flat 3-dimensional non-cosymplectic quasi-Sasakian manifold with respect to the quarter-symmetric non-metric ξ-connection $\bar{\nabla}$.

After the conformal curvature tensor, the projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a $(2n+1)$-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each coordinate neighbourhood of M and a domain in the Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well-known projective curvature tensor P

vanishes. The projective curvature tensor is defined by [13]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{5.1}$$

where S is the Ricci tensor of M.

Let M be an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). At each point $p \in M$, we decompose the tangent space T_pM into direct sum $T_pM = \phi(T_pM) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the 1-dimensional linear subspace of T_pM generated by $\{\xi_p\}$. Thus the conformal curvature tensor C is a map

$$C : T_pM \times T_pM \times T_pM \longrightarrow \phi(T_pM) \oplus \{\xi_p\}, \qquad p \in M.$$

It may be natural to consider the following particular cases:

(1) $C : T_pM \times T_pM \times T_pM \longrightarrow \{\xi_p\}$, i.e, the projection of the image of C in $\phi(T_pM)$ is zero.

(2) $C : T_pM \times T_pM \times T_pM \longrightarrow \phi(T_pM)$, i.e, the projection of the image of C in $\{\xi_p\}$ is zero. This condition is equivalent to

$$C(X,Y)\xi = 0. \tag{5.2}$$

(3) $C : \phi(T_pM) \times \phi(T_pM) \times \phi(T_pM) \longrightarrow \{\xi_p\}$, i.e, when C is restricted to $\phi(T_pM) \times \phi(T_pM) \times \phi(T_pM)$, the projection of the image of C in $\phi(T_pM)$ is zero. This condition is equivalent to

$$\phi^2 C(\phi X, \phi Y)\phi Z = 0. \tag{5.3}$$

A K-contact manifold satisfying (5.2) and (5.3) are called ξ-conformally flat and ϕ-conformally flat respectively. A K-contact manifold satisfying the cases (1), (2) and (3) are considered in [29], [30] and [31] respectively.

In an analogous way we define projective curvature tensor $\bar{P}$ on 3−dimension non-cosymplectic quasi-Sasakian manifold with respect to a special type of quarter-symmetric non-metric ξ-connection $\bar{\nabla}$, by

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{2}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y]. \tag{5.4}$$

Definition 5.1. A 3−dimensional quasi-Sasakian manifold M with respect to the quarter-symmetric non-metric ξ-connection $\bar{\nabla}$ is said to be ξ-projectively flat if the condition $\bar{P}(X,Y)\xi = 0$ holds.

So using (4.3), (4.4) and (2.3) the equation (5.4) becomes

$$\begin{aligned}\bar{P}(X,Y)Z = P(X,Y)Z &- \beta(\beta-1)\eta(X)g(Y,Z)\xi + \beta(\beta-1)\eta(Y)g(X,Z)\xi \\ &+\beta^2 g(\phi Y,Z)\phi X - \beta^2 g(\phi X,Z)\phi Y + 2\beta(\beta-1)g(\phi Y,X)\phi Z \\ &+\beta^2\eta(X)\eta(Z)Y - \beta^2\eta(Y)\eta(Z)X - \frac{1}{2}[\beta(2\beta-1)g(Y,Z)X \\ &-\beta(2\beta-1)g(X,Z)Y + \beta(4\beta-1)\eta(X)\eta(Z)Y \\ &-\beta(4\beta-1)\eta(Y)\eta(Z)X],\end{aligned} \tag{5.5}$$

where $P(X,Y)Z = R(X,Y)Z - \frac{1}{2}[S(Y,Z)X - S(X,Z)Y]$ is the projective curvature tensor with respect to the Levi-Civita connection on a 3−dimensional quasi-Sasakian manifold.

Putting $Z = \xi$ in equation (5.5) and using (2.3), we conclude that

$$\bar{P}(X,Y)\xi = P(X,Y)\xi.$$

Thus, we can state the following theorem:

Theorem 5.1. *A 3−dimensional non-cosymplectic quasi-Sasakian manifold M with constant structure function admitting the quarter-symmetric non-metric ξ-connection $\bar{\nabla}$ is ξ-projectively flat if and only if the Levi-Civita connection ∇ is so.*

Definition 5.2. A 3−dimensional quasi-Sasakian manifold M with respect to the quarter-symmetric non-metric ξ-connection $\bar{\nabla}$ is said to be ϕ-projectively flat if it satisfies the condition

$$g(\bar{P}(\phi X, \phi Y)\phi Z, \phi U) = 0. \tag{5.6}$$

From (5.4), it follows that

$$\tilde{\bar{P}}(X,Y,Z,U) = \tilde{\bar{R}}(X,Y,Z,U) - \frac{1}{2}[\bar{S}(Y,Z)g(X,U) - \bar{S}(X,Z)g(Y,U)], \tag{5.7}$$

where $\tilde{\bar{P}}(X,Y,Z,U) = g(\bar{P}(X,Y)Z,U)$, for X, Y, Z, $U \in \chi(M)$.

Putting $X = \phi X$, $Y = \phi Y$, $Z = \phi Z$ and $U = \phi U$ in (5.7), it is obvious that

$$\tilde{\bar{P}}(\phi X,\phi Y,\phi Z,\phi U) = \tilde{\bar{R}}(\phi X,\phi Y,\phi Z,\phi U) - \frac{1}{2}[\bar{S}(\phi Y,\phi Z)g(\phi X,\phi U) - \bar{S}(\phi X,\phi Z)g(\phi Y,\phi U)]. \tag{5.8}$$

Let $\{e_1, e_2, \xi\}$ be a local orthonormal basis of vector fields in M. Then $\{\phi e_1, \phi e_2, \xi\}$ is also a local orthonormal basis. Putting $X = U = e_i$ in (5.8), taking summation over i, $1 \le i \le 2$ and also using (2.3), we get

$$\tilde{\bar{P}}(\phi e_i,\phi Y,\phi Z,\phi e_i) = \tilde{\bar{R}}(\phi e_i,\phi Y,\phi Z,\phi e_i) - \frac{1}{2}[\bar{S}(\phi Y,\phi Z)g(\phi e_i,\phi e_i) - \bar{S}(\phi e_i,\phi Z)g(\phi Y,\phi e_i)]. \tag{5.9}$$

In view of (5.6) and (5.9) and using (2.3), we obtain

$$\bar{S}(\phi Y, \phi Z) = 0. \tag{5.10}$$

Combining (2.9), (4.4) and (5.10) and using (2.3), we have

$$S(Y,Z) - 2\beta^2\eta(Y)\eta(Z) + \beta(2\beta - 1)g(\phi Y, \phi Z) = 0. \tag{5.11}$$

By making use of (2.1), (2.2) and (2.3) in (5.11), we can write

$$S(Y,Z) - \beta(4\beta - 1)\eta(Y)\eta(Z) + \beta(2\beta - 1)g(Y,Z) = 0. \tag{5.12}$$

Interchanging Y and Z in (5.12) and using (2.7), it follows that

$$S(Y,Z) - \beta(4\beta - 1)\eta(Y)\eta(Z) + \beta(2\beta - 1)g(Y,Z) = 0. \tag{5.13}$$

Adding the equations (5.12) and (5.13), we conclude that

$$S(Y,Z) = \beta(4\beta - 1)\eta(Y)\eta(Z) - \beta(2\beta - 1)g(Y,Z),$$

which means that a ϕ-projectively flat $3-$dimensional quasi-Sasakian manifold with constant structure function admitting the quarter-symmetric non-metric ξ-connection $\bar{\nabla}$ is an η-Einstein manifold with respect to the Levi-Civita connection.

In view of the above discussions we state the following theorem:

Theorem 5.2. *A ϕ-projectively flat $3-$dimensional non-cosymplectic quasi-Sasakian manifold M with constant structure function admitting the quarter-symmetric non-metric ξ-connection $\bar{\nabla}$ is an η-Einstein manifold with respect to the Levi-Civita connection ∇.*

6 The second order symmetric parallel tensor

Definition 6.1. A tensor α of second order is said to be a parallel tensor if $\nabla\alpha = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

Let $\bar{\alpha}$ be a $(0,2)$-symmetric tensor field on a $3-$dimensional non-cosymplectic quasi-Sasakian manifold with respect to the quarter-symmetric non-metric ξ-connection $\bar{\nabla}$ such that $\bar{\nabla}\alpha = 0$. Then it follows that

$$\bar{\alpha}(\bar{R}(W,X)Y,Z) + \bar{\alpha}(Y,\bar{R}(W,X)Z) = 0, \tag{6.1}$$

for all vector fields W, X, Y, Z.

Substitution of $W = Y = Z = \xi$ in (6.1) yields

$$\bar{\alpha}(R(\xi,X)\xi,\xi) = 0, \tag{6.2}$$

since $\bar{\alpha}$ is symmetric. Using (4.3) in (6.2) we have

$$\bar{\alpha}(\xi,\xi) = 0. \tag{6.3}$$

Differentiating (6.3) covariantly along X, we get

$$\bar{\alpha}(\phi X,\xi) = 0. \tag{6.4}$$

Substituting X by ϕX in (6.4) yields

$$\bar{\alpha}(X,\xi)=0. \tag{6.5}$$

Again differentiating (6.4) covariantly along Y and making use of (6.3) and (6.4), we get

$$\bar{\alpha}(\phi X,\phi Y)=0. \tag{6.6}$$

Substitution of $X=\phi X$ and $Y=\phi Y$ in (6.6) yields

$$\bar{\alpha}(X,Y)=0. \tag{6.7}$$

In view of (6.7) we can state the following:

Theorem 6.1. *On a 3−dimensional non-cosymplectic quasi-Sasakian manifold there does not exist a nonzero symmetric parallel tensor of second order with respect to a quarter-symmetric non-metric ξ-connection.*

7 Example

In this section, we give an example of a 3−dimensional quasi-Sasakian manifold M with constant structure function admitting the quarter-symmetric non-metric ξ-connection $\bar{\nabla}$.

We consider the 3−dimensional manifold $\{(x,y,z)\in\mathbb{R}^3 : (x,y,z)\neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$.

We choose the vector fields

$$e_1=\frac{\partial}{\partial z}-y\frac{\partial}{\partial x},\ e_2=\frac{\partial}{\partial y},\ e_3=2\frac{\partial}{\partial x},$$

which are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$g(e_i,e_j)=\begin{cases} 1 & \text{if} \quad i=j \\ 0 & \text{if} \quad i\neq j;\ i,j=1,2,3. \end{cases}$$

Let η be the 1-form defined by

$$\eta(Z)=g(Z,e_3),$$

for any $Z\in\chi(M)$.

Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(e_1)=-e_2,\ \phi(e_2)=e_1,\ \phi(e_3)=0.$$

Using the linearity of ϕ and g, we have

$$\eta(e_3)=1,\ \phi^2 Z=-Z+\eta(Z)e_3$$

and

$$g(\phi Z,\phi U)=g(Z,U)-\eta(Z)\eta(U),$$

for any vector fields $Z, U \in \chi(M)$. Thus for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Then we have

$$[e_1, e_2] = \frac{1}{2}e_3, [e_1, e_3] = 0, [e_2, e_3] = 0. \tag{7.1}$$

The Levi-Civita connection ∇ of the metric tensor g is given by Koszul's formula

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)$$
$$-g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]).$$

Using Koszul's formula, we get the following:

$$\nabla_{e_1} e_1 = 0,\ \nabla_{e_1} e_2 = -\frac{1}{4}e_3,\ \nabla_{e_1} e_3 = \frac{1}{4}e_2,$$
$$\nabla_{e_2} e_1 = \frac{1}{4}e_3,\ \nabla_{e_2} e_2 = 0,\ \nabla_{e_2} e_3 = -\frac{1}{4}e_1,$$
$$\nabla_{e_3} e_1 = \frac{1}{4}e_2,\ \nabla_{e_3} e_2 = -\frac{1}{4}e_1,\ \nabla_{e_3} e_3 = 0.$$

In view of the above relations, we see that

$$\nabla_X \xi = -\beta\phi X,\ (\nabla_X \phi)Y = \beta g(X, Y)\xi - \eta(Y)X.$$

Therefore, the manifold is a 3-dimensional quasi-Sasakian manifold M with constant structure function $\beta = \frac{1}{4}$.

Using (3.1) in above equations, we obtain

$$\bar{\nabla}_{e_1} e_1 = 0,\ \bar{\nabla}_{e_1} e_2 = -\frac{1}{4}e_3,\ \bar{\nabla}_{e_1} e_3 = 0,$$
$$\bar{\nabla}_{e_2} e_1 = \frac{1}{4}e_3,\ \bar{\nabla}_{e_2} e_2 = 0,\ \bar{\nabla}_{e_2} e_3 = 0,$$
$$\bar{\nabla}_{e_3} e_1 = e_2,\ \bar{\nabla}_{e_3} e_2 = -e_1,\ \bar{\nabla}_{e_3} e_3 = 0.$$

The above arguments tell us that M is a $3-$dimensional quasi-Sasakian manifold with constant structure function admitting a type of quarter-symmetric non-metric ξ-connection $\bar{\nabla}$.

The expressions of the curvature tensor with respect to ∇ are :

$$R(e_1, e_2)e_3 = 0,\ R(e_2, e_3)e_3 = \frac{1}{16}e_2,\ R(e_1, e_3)e_3 = \frac{1}{16}e_1,$$
$$R(e_1, e_2)e_2 = -\frac{3}{16}e_1,\ R(e_2, e_3)e_2 = -\frac{1}{16}e_3,\ R(e_1, e_3)e_2 = 0,$$
$$R(e_1, e_2)e_1 = \frac{3}{16}e_2,\ R(e_2, e_3)e_1 = 0,\ R(e_1, e_3)e_1 = -\frac{1}{16}e_3.$$

From the above expressions the non-zero components of the Ricci tensor with respect to ∇ are given by

$$S(e_1, e_1) = -\frac{1}{8},\ S(e_2, e_2) = -\frac{1}{8},\ S(e_3, e_3) = \frac{1}{8}.$$

Similarly, the expressions of the curvature tensor with respect to $\bar{\nabla}$ are :

$$\begin{aligned}
&\bar{R}(e_1, e_2)e_3 = 0,\ \bar{R}(e_1, e_3)e_3 = 0,\ \bar{R}(e_2, e_3)e_3 = 0,\\
&\bar{R}(e_1, e_2)e_1 = -\frac{1}{2}e_2,\ \bar{R}(e_1, e_2)e_2 = 0,\ \bar{R}(e_2, e_3)e_3 = 0,\\
&R(e_2, e_1)e_1 = 0.
\end{aligned}$$

From the above expressions the components of the Ricci tensor with respect to $\bar{\nabla}$ are given by

$$\bar{S}(e_1, e_1) = 0,\ \bar{S}(e_2, e_2) = 0,\ \bar{S}(e_3, e_3) = 0.$$

In view of the above equations we can easily obtain

$$P(e_i, e_j)e_3 = 0 = \bar{P}(e_i, e_j)e_3,$$

for all $1 \le i, j \le 3$. Therefore theorem 5.1 is verified.

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