Projective and direct limits of Banach tensor structures

Patrick Cabau and Fernand Pelletier

Abstract. We endow projective (resp. direct) limits of Banach G-stuctures and tensor structures with Fréchet (resp. convenient) structures. These situations are illustrated by many examples in the framework of tensor structures of type $(1,1)$ and $(2,0)$.

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1 Introduction

The concept of G-structure provides a unified framework for a lot of interesting geometric structures. This notion is defined in finite dimension as a reduction of the frame bundle. Different results are obtained in the finite case in [Kob] and in [Mol] (G-structures equivalent to a model, characteristic class of a G-structure, intrinsic geometry of G-structures, etc).

In order to extend this notion to the Banach framework, following Bourbaki ([Bou]), the frame bundle $\ell(TM)$ is defined as an open submanifold of the linear map bundle $L(M \times \mathbb{M}, TM)$ where the manifold M is modelled on the Banach space $\mathbb{M}$. In [Klo], Klotz shows that the automorphism group of some Banach manifolds can be turned into a Banach-Lie group acting smoothly on M.

The notion of tensor structure which corresponds to an intersection of G-structures, where the different G are isotropy groups for tensors, is relevant in a lot of domains in Differential Geometry: Krein metric (cf. [Bog]), almost tangent and almost cotangent structures (cf. [ClGo]), symplectic structures (cf. [Vai], [Wei]), complex structures (cf. [Die]), inner products and decomposable complex structures (cf.[ChMa]).

In this paper, we are interested in the study of projective (resp. direct) limits of Banach frame bundles sequences $(\ell(E_n))_{n\in\mathbb{N}}$, associated tensor structures and adapted connections to G-structures.

The main problem to endow the projective (resp. direct) limit $M = \varprojlim M_n$ (resp. $M = \varinjlim M_n$) of Banach manifolds M_n modelled on the Banach spaces $\mathbb{M}_n$ with a

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structure of Fréchet (resp. convenient) manifold is to find under which conditions we can build a chart around any point of M.

It is worth noting that the convenient setting considered by Frölicher, Kriegl and Michor in [FrKr] and [KrMi] is particularly adapted to the framework of direct limit of Banach manifolds because the direct limit topology τ_{DL} is c^∞-complete (cf. [CaPe1]). It appears as a generalisation of calculus *à la Leslie* on Banach spaces used in the projective limit framework.

Since the general linear group $\mathrm{GL}\,(\mathbb{M})$ does not admit a reasonable structure in both cases, it has to be replaced by the following groups:

– The Fréchet topological group $\mathcal{H}_0\,(\mathbb{M})$ (cf. [DGV]), recalled in § 5.1, in the case of projective sequences;

– The Fréchet topological group $\mathrm{G}(\mathbb{E})$ as introduced in [CaPe1], recalled in § 6.1, in the case of ascending sequences.

So it is possible to endow the projective (resp. direct) limit of a sequence of generalized linear frame bundles $(\ell\,(E_n))_{n\in\mathbb{N}}$, as defined in [DGV], (resp. linear frame bundles $(\ell\,(E_n))_{n\in\mathbb{N}}$) with a structure of Fréchet (resp. convenient) principal bundle.

We then obtain similar results for some projective (resp. direct) limits of sequences of Banach G-structures and tensor structures.

This paper is organized as follows.

In section 2, we recall or introduce some Banach structures. In section 3 (resp. 4), a lot of examples of tensor structures on a Banach vector bundle (resp. Banach manifold) is given. Section 5 is devoted to the study of projective limits of G-structures and tensor structures; we also give an application to sets of smooth maps. In section 6, direct limits of such structures are studied and an application to Sobolev loop spaces is given.

2 Banach structures

We give a brief account of different Banach structures which will be used in this paper. The main references for this section are [Bou], [DGV], [Lan], [Pay] and [Pel1].

2.1 Submanifolds

Among the different notions of submanifolds, in this paper, we are interested in the weak submanifolds as used in [Pel1] in the framework of integrability of distributions.

Definition 2.1. Let M be a Banach manifold modelled on the Banach space $\mathbb{M}$.
A weak submanifold of M is a pair (S,φ) where S is a connected Banach manifold modelled on a Banach space $\mathbb{S}$ and $\varphi : S \longrightarrow M$ is a smooth map such that:

(WSM 1) There exists a continuous injective linear map $i : \mathbb{S} \longrightarrow \mathbb{M}$;

(WSM 2) φ is an injective smooth map and the tangent map $T_x\varphi : T_xN \longrightarrow T_{\varphi(x)}M$ is an injective continuous linear map with closed range for all $x \in N$.

Note that for a weak submanifold $\varphi : S \longrightarrow M$, on the subset $\varphi(S)$ of M, we have two topologies:

— The induced topology from M;

— The topology for which φ is a homeomorphism from S to $\varphi(S)$.

With this last topology, via φ, we get a Banach manifold structure on $\varphi(S)$ modelled on $\mathbb{S}$. Moreover, the inclusion from $\varphi(S)$ into M is continuous as a map from the manifold $\varphi(S)$ to M. In particular, if U is an open set of M, then $\varphi(S) \cap U$ is an open set for the topology of the manifold on $\varphi(S)$.

2.2 G-structures and tensor structures on a Banach space

According to [Bel2], Corollary 3.7, we have the following result.

Theorem 2.1. *Let H be a Banach-Lie group and $\mathfrak{h}$ be its Lie algebra $\mathfrak{h}$. Consider a closed subgroup G of H and set*

$$\mathfrak{g} = \{X \in \mathfrak{h} \mid \forall t \in \mathbb{R},\ \exp_H(tX) \in G\}.$$

Then $\mathfrak{g}$ is a closed subalgebra of $\mathfrak{h}$ and there exist on G a uniquely determined topology τ and a manifold structure making G into a Banach-Lie group such that $\mathfrak{g}$ is the Lie algebra of G, the inclusion of G into H is smooth and the following diagram:

$$\begin{array}{ccc} \mathfrak{g} & \rightarrow & \mathfrak{h} \\ \exp_G \downarrow & & \downarrow \exp_H \\ G & \rightarrow & H \end{array}$$

is commutative, where the horizontal arrows stand for inclusion maps.

Definition 2.2. A closed topological subgroup G of H which satisfies the assumption of Theorem 2.1 is called a weak Lie subgroup of H.

We fix a Banach space $\mathbb{E}_0$. For any Banach space $\mathbb{E}$, the Banach space of linear bounded maps from $\mathbb{E}_0$ to $\mathbb{E}$ is denoted by $\mathcal{L}(\mathbb{E}_0, \mathbb{E})$ and, if it is non empty, the open subset of $\mathcal{L}(\mathbb{E}_0, \mathbb{E})$ of linear isomorphisms from $\mathbb{E}_0$ onto $\mathbb{E}$ will denoted by $\mathcal{L}is(\mathbb{E}_0, \mathbb{E})$. *We always assume that $\mathcal{L}is(\mathbb{E}_0, \mathbb{E}) \neq \varnothing$.*

We then have a natural right transitive action of $\mathrm{GL}(\mathbb{E}_0)$ on $\mathcal{L}is(\mathbb{E}_0, \mathbb{E})$ given by $(\phi, g) \mapsto \phi \circ g$.

According to [PiTa], we define the notion of G-structure on the Banach space $\mathbb{E}$.

Definition 2.3. Let G be a weak Banach Lie subgroup of $\mathrm{GL}(\mathbb{E}_0)$.

A G-structure on $\mathbb{E}$ is a subset $\mathcal{S}$ of $\mathcal{L}is(\mathbb{E}_0, \mathbb{E})$ such that:

(GStrB 1) $\forall\, (\phi, \psi) \in \mathcal{S}^2,\ \phi \circ \psi^{-1} \in G$;

(GStrB 2) $\forall\, (\phi, g) \in \mathcal{S} \times G,\ \phi \circ g \in \mathcal{S}$.

The set $L_r^s(\mathbb{E}_0)$ of tensors of type (r, s) on $\mathbb{E}_0$ is the Banach space of $(r + s)$-multilinear maps from $(\mathbb{E}_0)^r \times (\mathbb{E}_0^*)^s$ into $\mathbb{R}$.

Each $g \in \mathrm{GL}(\mathbb{E}_0)$ induces an automorphism g_r^s of $L_r^s(\mathbb{E}_0)$ which gives rise to a right action of $\mathrm{GL}(E_0)$ on $L_r^s(\mathbb{E}_0)$ which is $(\mathbb{T}, g) \mapsto g_r^s(\mathbb{T})$ where, for any $(u_1, \ldots, u_r, \alpha_1, \ldots, \alpha_s) \in (\mathbb{E}_0)^r \times (\mathbb{E}_0^*)^s$,

$$g_r^s(\mathbb{T})\,(u_1, \ldots, u_r, \alpha_1, \ldots, \alpha_s) = \mathbb{T}\left(g^{-1}.u_1, \ldots, g^{-1}.u_r, \alpha_1 \circ g, \ldots, \alpha_s \circ g\right).$$

Definition 2.4. The isotropy group of a tensor $\mathbb{T}_0 \in L_r^s(\mathbb{E}_0)$ is the set

$$G(\mathbb{T}_0) = \{g \in \mathrm{GL}(\mathbb{E}_0) \;:\; g_r^s(\mathbb{T}_0) = \mathbb{T}_0\}.$$

Definition 2.5. Let $\mathbb{T}$ be a k-uple $(\mathbb{T}_1, \ldots, \mathbb{T}_k)$ of tensors on $\mathbb{E}_0$.
A tensor structure on $\mathbb{E}_0$ of type $\mathbb{T}$ or a $\mathbb{T}$-structure on $\mathbb{E}_0$ is a $G = \bigcap\limits_{i=1}^{k} G(\mathbb{T}_i)$-structure on $\mathbb{E}_0$.

It is clear that such a group $G = \bigcap\limits_{i=1}^{k} G(\mathbb{T}_i)$ is a closed topological subgroup of the Banach Lie subgroup of $\mathrm{GL}(\mathbb{E}_0)$. Therefore, from Theorem 2.1, it is a weak Lie subgroup of $\mathrm{GL}(\mathbb{E}_0)$.

2.3 The frame bundle of a Banach vector bundle

Let (E, π_E, M) be a vector bundle of typical fibre the Banach space $\mathbb{E}$, with total space E, projection π_E and base M (modelled on the Banach space $\mathbb{M}$). Because $\mathbb{E}$ has not necessarily a Schauder basis, it is not possible to define the frame bundle of this vector bundle as it is done in finite dimension.

An extension of the notion of frame bundle in finite dimension to the Banach framework can be found in [Bou], § 7.10.1 and in [DGV], § 1.6.5.

The set of linear bicontinuous isomorphisms from $\mathbb{E}$ to E_x (where E_x is the fibre over $x \in M$) is denoted by $\mathcal{L}is\,(\mathbb{E}, E_x)$.

The set

$$P\,(E) = \{(x, f) : x \in M,\ f \in \mathcal{L}is\,(\mathbb{E}, E_x)\}$$

is an open submanifold of the linear map bundle $L\,(M \times \mathbb{E}, E) = \bigcup\limits_{x \in M} \mathcal{L}\,(\mathbb{E}, E_x)$. The Lie group $\mathrm{GL}\,(\mathbb{E})$ of the continuous linear automorphisms of $\mathbb{E}$ acts on the right of $P\,(E)$ as follows:

$$\begin{array}{cccc} \widehat{R}: & P\,(E) \times \mathrm{GL}\,(\mathbb{E}) & \longrightarrow & P\,(E) \\ & ((x, f)\,, g) & \longmapsto & (x, f)\,.g = (x, f \circ g) \end{array}.$$

Definition 2.6. The quadruple $\ell\,(E) = (P\,(E)\,, \pi, M, \mathrm{GL}\,(\mathbb{E}))$, where

$$\begin{array}{cccc} \pi: & P\,(E) & \longrightarrow & M \\ & (x, f) & \mapsto & x \end{array}$$

is the projection on the base, is a principal bundle, called the frame bundle of E.

We can write

$$P\,(E) = \bigcup_{x \in M} \mathcal{L}is\,(\mathbb{E}, E_x)\,.$$

Let us describe the local structure of $P(E)$.
Let $\{(U_\alpha,\tau_\alpha)\}_{\alpha\in A}$ be a local trivialization of E with $\tau_\alpha:\pi_E^{-1}(U_\alpha)\longrightarrow U_\alpha\times\mathbb{E}$. This trivialization gives rise to a local section $s_\alpha:U_\alpha\longrightarrow P(E)$ of $P(E)$ as follows:

$$\forall x\in U_\alpha,\ s_\alpha(x)=\left(x,(\overline{\tau}_{\alpha,x})^{-1}\right)$$

where $\overline{\tau}_{\alpha,x}\in\mathcal{L}is(E_x,\mathbb{E})$ is defined by $\overline{\tau}_{\alpha,x}=\mathrm{pr}_2\circ\tau_\alpha|_{E_x}$.
We then get a local trivialization of $P(E)$:

$$\begin{array}{cccc}\Psi_\alpha: & \pi^{-1}(U_\alpha) & \longrightarrow & U_\alpha\times\mathrm{GL}(\mathbb{E})\\ & (x,f) & \mapsto & (x,\overline{\tau}_{\alpha,x}\circ f)\end{array}.$$

In particular, we have

$$\Psi_\alpha(s_\alpha(x))=\left(x,\overline{\tau}_{\alpha,x}\circ(\overline{\tau}_{\alpha,x})^{-1}\right)=(x,\mathrm{Id}_{\mathbb{E}}).$$

Ψ_α gives rise to $\overline{\Psi}_{\alpha,x}$ defined by

$$\overline{\Psi}_{\alpha,x}(x,f)=\overline{\tau}_{\alpha,x}\circ f.$$

Moreover, for $g\in\mathrm{GL}(\mathbb{E})$, we have

$$\widehat{R}_g(s_\alpha(x))=\left(x,(\overline{\tau}_{\alpha,x})^{-1}\circ g\right).$$

Because the local structure of $P(E)$ is derived from the local structure of the vector bundle E, we get

$$\overline{\Psi}_{\alpha,x}\circ\left(\overline{\Psi}_{\beta,x}\right)^{-1}(\overline{\tau}_{\beta,x}\circ f)=\overline{\Psi}_{\alpha,x}(x,f)=\overline{\tau}_{\alpha,x}\circ f.$$

We then have the following result.

Proposition 2.2. *The transition functions*

$$\begin{array}{cccc}\mathrm{T}_{\alpha\beta}: & U_\alpha\cap U_\beta & \longrightarrow & \mathrm{GL}(\mathbb{E})\\ & x & \longmapsto & \overline{\tau}_{\alpha,x}\circ(\overline{\tau}_{\beta,x})^{-1}\end{array}$$

of E coincide with the transition functions of $P(E)$.

The transition functions form a cocycle, that is

$$\forall x\in U_\alpha\cap U_\beta\cap U_\gamma,\ \mathrm{T}_{\alpha\gamma}(x)=\mathrm{T}_{\alpha\beta}(x)\circ\mathrm{T}_{\beta\gamma}(x)$$

Definition 2.7. The tangent frame bundle $\ell(TM)$ of a Banach manifold M is the frame bundle of TM.

Proposition 2.3. *Let $\pi_{E_1}:E_1\longrightarrow M$ and $\pi_{E_2}:E_2\longrightarrow M$ be two Banach vector bundles with the same fibre type $\mathbb{E}$ and let $\Phi:E_1\longrightarrow E_2$ be a bundle morphism above* Id_M. *Then Φ induces a unique bundle morphism $\ell(\Phi):\ell(E_1)\longrightarrow\ell(E_2)$ that is injective (resp. surjective) if and only if Φ is injective (resp. surjective).*

2.4 G-structures and tensor structures on a Banach vector bundle

A reduction of the frame bundle $\ell(E)$ of a Banach fibre bundle $\pi_E : E \longrightarrow M$ of fibre type $\mathbb{E}$ corresponds to the data of a weak Banach Lie subgroup G of $\mathrm{GL}(\mathbb{E})$ and a topological principal subbundle $\left(F, \pi_{|F}, M, G\right)$ of $\ell(E)$ such that $\left(F, \pi_{|F}, M, G\right)$ has its own smooth principal structure, and the inclusion is smooth. In fact, such a reduction can be obtained in the following way.

Assume that there exists a bundle atlas $\{(U_\alpha, \tau_a)\}_{\alpha \in A}$ whose transition functions

$$\mathrm{T}_{\alpha\beta} : x \longmapsto \overline{\tau}_{\alpha,x} \circ \left(\overline{\tau}_{\beta,x}\right)^{-1}$$

belong to a weak Banach-Lie subgroup G of $\mathrm{GL}(\mathbb{E})$.

To any local trivialization τ_α of E, is associated the following map:

$$\begin{array}{cccc} \phi_\alpha : & U_\alpha \times G & \longrightarrow & P(E) \\ & (x,g) & \mapsto & s_\alpha(x).g \end{array}$$

where

$$s_\alpha(x).g = \left(x, \left(\overline{\tau}_{\alpha,x}\right)^{-1}\right).g = \left(x, \left(\overline{\tau}_{\alpha,x}\right)^{-1} \circ g\right).$$

If $U_\alpha \cap U_\beta \neq \phi$, we have

$$\forall x \in U_\alpha \cap U_\beta,\ s_\beta(x) = s_\alpha(x).\mathrm{T}_{\alpha\beta}(x).$$

Let $F = \bigcup\limits_{\alpha \in A} V_\alpha$ be the subset of $P(E)$ where $V_\alpha = \phi_\alpha(U_\alpha \times G)$. Because a principal bundle is determined by its cocycles (cf., for example, [DGV], § 1.6.3), the quadruple $\left(F, \pi_{|F}, M, G\right)$ can be endowed with a structure of Banach principal bundle and is a topological principal subbundle of $\ell(E)$. Moreover, we have the following lemma.

Lemma 2.4. *F is closed in $\ell(E)$ and F is a weak submanifold of $\ell(E)$.*

Proof. Consider a sequence $(x_n, g_n) \in F$ which converges to some $(x,g) \in \ell(E)$. Let $\phi_\alpha : U_\alpha \times \mathrm{GL}(\mathbb{E}) \longrightarrow P(E)$ be the mapping associated to a local trivialization of $P(E)$ where $(x,g) \in P(E)$. Now, for n large enough, $\phi_\alpha^{-1}(x_n, g_n)$ belongs to $U_\alpha \times G \subset U_\alpha \times \mathrm{GL}(\mathbb{E})$. But $U_\alpha \times G$ is closed in $U_\alpha \times \mathrm{GL}(\mathbb{E})$; This implies that (x,g) belongs to F. Since the inclusion of G into $\mathrm{GL}(\mathbb{E})$ is smooth, this implies that the inclusion of F into $\ell(E)$ is also smooth. $\square$

Definition 2.8. The weak subbundle $\left(F, \pi_{|F}, M, G\right)$ of the frame bundle $\ell(E) = (P(E), \pi, M, \mathrm{GL}(\mathbb{E}))$ is called a G-structure on E.
When $E = TM$, a G-stucture on TM is called a G-structure on M.

Intuitively, a G-structure F on a Banach bundle $\pi_E : E \longrightarrow M$ may be considered as a family $\{F_x\}_{x \in M}$ which varies smoothly with x in M, in the sense that $F = \bigcup\limits_{x \in M} F_x$ has a principal bundle structure over M with structural group G.

Definition 2.9. Let $\pi_{E_1} : E_1 \longrightarrow M$ and $\pi_{E_2} : E_2 \longrightarrow M$ be two Banach vector bundles with the same fibre $\mathbb{E}$ provided with the G-structures $\left(F_1, \pi_{E_1|F_1}, M, G\right)$ and $\left(F_2, \pi_{E_2|F_2}, M, G\right)$ respectively. A morphism $\Phi : E_1 \longrightarrow E_2$ is said to be G-structure preserving if for any $x \in M$, $\Phi\left((F_1)_x\right) = (F_2)_x$.

Definition 2.10. Let $\mathbb{T}$ be a k-uple $(\mathbb{T}_1, \dots, \mathbb{T}_k)$ of tensors on the fibre type $\mathbb{E}$ of a Banach vector bundle $\pi_E : E \longrightarrow M$. A tensor structure on E of type $\mathbb{T}$ is a $G(\mathbb{T}) = \bigcap_{i=1}^{k} G(\mathbb{T}_i)$ structure on E where $G(\mathbb{T}_i)$ is the isotropy group of $\mathbb{T}_i$ in $\mathrm{GL}(\mathbb{E})$.

Definition 2.11. The Banach fibre bundle $\pi_{\widehat{L_r^s}} : \widehat{L_r^s}(E) \longrightarrow M$ whose typical fibre is $L_r^s(\mathbb{E})$ is the tensor Banach bundle of type (r, s).

Definition 2.12. Let $\pi_{\widehat{L_r^s}} : \widehat{L_r^s}(E) \longrightarrow M$ be the tensor Banach bundle of type (r, s).

1. A smooth section $\mathcal{T}$ of this bundle defined on an open set $U \subset M$ is called a local tensor of type (r,s) on E. When $U = M$ we simply say that $\mathcal{T}$ is a tensor of type (r,s) on E.

2. Let $\Phi : E \longrightarrow E'$ be an isomorphism from a Banach vector bundle $\pi_E : E \longrightarrow M$ to another Banach vector bundle $\pi_{E'} : E' \longrightarrow M'$ over a map $\phi : M \longrightarrow M'$. Given a tensor $\mathcal{T}$ of type (r,s) on E', defined on an open set U' of M', the pullback of $\mathcal{T}$ is the tensor $\Phi^*\mathcal{T}$ of the same type defined (on $\phi^{-1}(U')$) by
$$\begin{aligned}(\Phi^*\mathcal{T})_{\phi^{-1}(x')} &\left(\Phi^{-1}(u'_1), \dots, \Phi^{-1}(u'_r), \Phi^*(\alpha'_1), \dots, \Phi^*(\alpha'_s)\right) \\ &= \mathcal{T}_{x'}(u'_1, \dots, u'_r, \alpha'_1, \dots, \alpha'_s)\end{aligned}$$

3. A tensor $\mathcal{T}$ is called locally modelled on $\mathbb{T} \in L_r^s(\mathbb{E})$ if, for any $x_0 \in M$, there exists around x_0, a trivialization $\tau : E_{|U} \longrightarrow U \times \mathbb{E}$ such that
$$\mathcal{T}_{|U}(x) = \overline{\tau}_x^*(\mathbb{T})$$
for any $x \in U$.

Proposition 2.5. *A tensor $\mathcal{T}$ of type (r, s) on a Banach bundle $\pi_E : E \longrightarrow M$, where M is a connected manifold, defines a tensor structure on E if and only if there exists a tensor $\mathbb{T} \in L_r^s(\mathbb{E})$ and a bundle atlas $\{(U_\alpha, \tau_a)\}_{\alpha \in A}$ such that $\mathcal{T}_\alpha = \mathcal{T}_{|U_\alpha}$ is localy modelled on $\mathbb{T}$ and the transition functions $T_{\alpha\beta}(x)$ belong to the isotropy group $G(\mathbb{T})$ for all $x \in U_\alpha \cap U_\beta$ (where $U_\alpha \cap U_\beta \neq \varnothing$) and all $(\alpha, \beta) \in A^2$.*

Proof. Fix some tensor $\mathbb{T}$ on $\mathbb{E}$ and denote the isotropy group of $\mathbb{T}$ by G. Assume that we have a G-structure on E and let $\{(U_\alpha, \tau_\alpha)\}_{\alpha \in A}$ be a bundle atlas such that each transition function $T_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(\mathbb{E})$ takes values in G. Fix some point $x_0 \in M$ and denote an open set of the previous atlas which contains x_0 by U_α. Since we have a trivialization $\tau_\alpha : E_{U_\alpha} \to U_\alpha \times \mathbb{E}$, we consider on U_α the section of $\widehat{L_r^s}(E)$ defined by $\mathcal{T}(x) = \bar{\tau}_{\alpha,x}^*\mathbb{T}$. If there exists U_β that contains x_0, then, on $U_\alpha \cap U_\beta$, the transition function $T_{\alpha\beta}$ takes values in G; it follows that the restriction of $\mathcal{T}$ on $U_\alpha \cap U_\beta$ is well defined. Therefore, there exists an open set U in M on which is defined a smooth section $\mathcal{T}$ of $\widehat{L_r^s}(E)$ such that $\mathcal{T}_{|U_\alpha \cap U} = \bar{\tau}_{\alpha,x}^*\mathcal{T}$, for all α such that $U_\alpha \cap U \neq \varnothing$. If x belongs to the closure of U, let U_β be an open set of the previous atlas which contains x. Then using the previous argument, we can extend $\mathcal{T}$ to $U \cup U_\beta$. Since M is connected, we can defined such a tensor $\mathcal{T}$ on M.
Conversely, if $\mathcal{T}$ is a tensor such that there exists a bundle atlas $\{(U_\alpha, \tau_\alpha)\}_{\alpha \in A}$ such that $\mathcal{T}_{|U_\alpha}$ is locally modeled on $\mathbb{T}$; this clearly implies that each transition function $T_{\alpha\beta}$ must take values in G. $\square$

2.5 Integrable tensor structures on a Banach manifold

Definition 2.13. A tensor $\mathcal{T}$ on a Banach manifold M modelled on the Banach space $\mathbb{M}$ is called an integrable tensor structure if there exists a tensor $\mathbb{T} \in L_r^s(\mathbb{M})$ and a bundle atlas $\{(U_\alpha, \tau_a)\}_{\alpha \in A}$ such that $\mathcal{T}_\alpha = \mathcal{T}_{|U_a}$ is locally modelled on $\mathbb{T}$ and the transition functions $T_{\alpha\beta}(x)$ belong to the isotropy group $G(\mathbb{T})$ for all $x \in U_\alpha \cap U_\beta$ (where $U_\alpha \cap U_\beta \neq \varnothing$) and all $(\alpha, \beta) \in A^2$.

3 Examples of tensor structures on a Banach bundle

Let (E, π_E, M) be a vector bundle whose fibre is the Banach space $\mathbb{E}$ and whose base is modelled on the Banach space $\mathbb{M}$.

3.1 Krein metrics

Definition 3.1. A pseudo-Riemannian metric on (E, π_E, M) is a smooth field of weak non degenerate symmetric forms g on E. Moreover g is called

- a weak Riemannian metric if each g_x is a pre-Hilbert inner product on the fibre E_x
- a Krein metric if there exists a decomposition $E = E^+ \oplus E^-$ in a Whitney sum of Banach bundles such that for each fibre E_x, g_x is a Krein inner product associated to the decomposition $E_x = E_x^+ \oplus E_x^-$
- a neutral metric if it is a Krein metric such that there exists a decomposition $E = E^+ \oplus E^-$ where E^+ and E^- are isomorphic sub-bundles of E
- a Krein indefinite metric if there exists a decomposition $E = E_1 \oplus E_2$ in a Whitney sum of Banach bundles such that for each fibre E_x, g_x is a Krein indefinite inner product associated to the decomposition $E_x = (E_1)_x \oplus (E_2)_x$.

Given a Krein metric g on E, according to [CaPe2], Proposition B.8, we get the following result.

Theorem 3.1.

1. *Let g be a Krein metric on a Banach bundle (E, π_E, M). Consider a decomposition $E = E^+ \oplus E^-$ in a Whitney sum such that the restriction g^+ (resp. $-g^-$) of g (resp. $-g$) to E^+ (resp. E^-) is a (weak) Riemannian metric. Let $\widehat{E}_x$ be the Hilbert space which is the completion of the pre-Hilbert space (E_x, g_x). Assume that $\widehat{E} = \bigcup\limits_{x \in M} \widehat{E}_x$ is a Banach bundle over M such that the inclusion of E into $\widehat{E}$ is a bundle morphism. Then g can be extended to a strong Krein metric $\widehat{g}$ on $\widehat{E}$ and we have a decomposition $\widehat{E} = \widehat{E}^+ \oplus \widehat{E}^-$ such that the restriction $\widehat{g}^+$ (resp. $\widehat{g}^-$) of $\widehat{g}$ (resp. $-\widehat{g}$) to $\widehat{E}^+$ (resp. $\widehat{E}^-$) is a strong Riemannian metric. In fact $\widehat{E}^+$ (resp. $\widehat{E}^-$) is the closure of E^+ (resp. E^-) in $\widehat{E}$.*
   Given any point x_0 in M, we identify the typical fibre of $\widehat{E}$ (resp. E) with $\widehat{E}_{x_0}$

(resp. E_{x_0}). Then $\widehat{\gamma}=\widehat{g}^+-\widehat{g}^-$ defines a $\widehat{\gamma}_{x_0}$-structure and a $\widehat{g}_{x_0}$-structure on $\widehat{E}$. Moreover, g defines a g_{x_0}-structure on E.

2. *Let $(\widehat{E},\pi_{\widehat{E}},M)$ be a Banach bundle whose typical fibre is reflexive and let $\widehat{g}$ be a strong Krein metric on $\widehat{E}$. If $\widehat{g}_{x_0}$ is the Krein inner product on $\widehat{E}_{x_0}$, then $\widehat{g}$ is a $\widehat{g}_{x_0}$-structure on $\widehat{E}$.*
Let (E,π_E,M) be any Banach bundle such that we have an injective morphism of Banach bundles $\iota : E \longrightarrow \widehat{E}$. Then the restriction $g=\iota^*\widehat{g}$ of $\widehat{g}$ is a Krein metric on E and g induces a g_{x_0}-structure on E.

Proof. 1. Consider a decomposition $E=E^+\oplus E^-$ in a Whitney sum such that the restriction g^+ (resp. $-g^-$) of g (resp. $-g$) to E^+ (resp. E^-) is a (weak) Riemannian metric.

Assume that $\widehat{E}=\bigcup\limits_{x\in M}\widehat{E}_x$ is a Banach bundle over M, such that the inclusion ι of E in $\widehat{E}$ is a bundle morphism.

This implies that we have a bundle atlas $\{(U_\alpha,\tau_\alpha)\}_{\alpha\in A}$ (resp. $\{(U_\alpha,\sigma_\alpha)\}_{\alpha\in A}$) for E (resp. for $\widehat{E}$) such that, for each $\alpha\in A$, we have $\sigma_\alpha\circ\iota=\tau_\alpha$, and, on $U_\alpha\cap U_\beta$, the transition function $T_{\alpha\beta}$ is the restriction to E of the transition function $S_{\alpha\beta}$ associated $\{(U_\alpha,\tau_\alpha)\}_{\alpha\in A}$ to $\mathbb{E}$.

Assume that g is a weak Riemannian metric. Then $\widehat{g}$ is a strong Riemannian metric on $\widehat{E}$.

Now given any $x\in M$, each fibre $\widehat{E}_x$ can be provided with the Hilbert inner product $\widehat{g}_x$. Therefore from [Lan], Theorem 3.1, there exists a bundle atlas $\{(U'_\alpha,\sigma'_\alpha)\}_{\alpha\in A}$ such that the transition functions $S'_{\alpha\beta}$ take values in the group of isometries of the typical fibre. Now, since ι is a bundle morphism, $\{(U'_\alpha,\tau'_\alpha=\sigma'_\alpha\circ\iota)\}_{\alpha\in A}$ is a bundle atlas for E and the transition function $T'_{\alpha\beta}$ for this atlas is nothing but the restriction to $\mathbb{E}$ of $S'_{\alpha\beta}$ and so is an isometry of $\mathbb{E}$ provided with the pre-Hilbert inner product induced on E from $\widehat{E}$. Now, for a given $x_0\in M$, the typical fibre of $\widehat{E}$ and E can be identified with $\widehat{E}_{x_0}$ and E_{x_0} respectively, which ends the proof when E^- is reduced to $\{0\}$.
Now, we can apply the previous result to E^+ and E^- and for the weak Riemannian metric g^+ and $-g^-$ respectively. The result is then a consequence of [CaPe2], Proposition 151.

2. According to the assumption of Point 2, let $\mathbb{E}$ and $\widehat{\mathbb{E}}$ be the typical fibres of E and $\widehat{E}$, respectively. Since $\iota : E\longrightarrow\widehat{E}$ is a bundle morphism, as for the proof of Point 1, we have a bundle atlas $\{(U_\alpha,\tau_\alpha)\}_{\alpha\in A}$ for E and $\{(U_\alpha,\sigma_\alpha)\}_{\alpha\in A}$ for $\widehat{E}$ such that, for all $\alpha\in A$, we have $\sigma_\alpha\circ\iota=\tau_\alpha$ and, on $U_\alpha\cap U_\beta$ the transition functions $T_{\alpha\beta}$ are the restrictions to $\mathbb{E}$ of the transition functions $S_{\alpha\beta}$ associated to $\{(U_\alpha,\tau_\alpha)\}_{\alpha\in A}$. Let $\bar{\mathbb{E}}$ be the closure of $\mathbb{E}$ in $\widehat{E}$. For each $\alpha\in A$ we set $\bar{E}_{U_\alpha}=\bigcup\limits_{x\in U_\alpha}\bar{E}_x$ where $\bar{E}_x$ is the closure of E_x in $\widehat{E}_x$. Since $\sigma_\alpha\circ\iota(x,u)=\tau_\alpha(x,u)$, then we can extend the trivialization

$\tau_\alpha : E_{U_\alpha} \longrightarrow U_\alpha \times \mathbb{E}$ to a trivialization $\bar{\tau}_\alpha : \bar{E}_{U_\alpha} \longrightarrow U_\alpha \times \bar{\mathbb{E}}$ and if $\bar{\iota} : \bar{E}_{U_\alpha} \longrightarrow \widehat{E}_{U_\alpha}$ is the canonical extension of $\iota : E_{U_\alpha} \longrightarrow \widehat{E}_{U_\alpha}$ then we also have $\sigma_\alpha \circ \bar{\iota} = \bar{\tau}_\alpha$. In the same way, the transition function $\bar{T}_{\alpha,\beta}$ associated to the bundle atlas $\{(U_\alpha, \bar{\tau}_\alpha)\}_{\alpha \in A}$ is the restriction of $S_{\alpha\beta}$ to $\bar{E}$ which is a Hilbert space. The result is a consequence of Point 1. □

3.2 Almost tangent, decomposable complex and para-complex structures

Let (E, π_E, M) be a Banach bundle such that M is connected. As in finite dimension, we introduce the notions of almost tangent and almost complex structures.

Definition 3.2. An endomorphism J of E is called an almost tangent structure on E if $\mathrm{im} J = \ker J$ and $\ker J$ is a supplemented sub-bundle of E.

Definition 3.3. An endomorphism $\mathcal{I}$ (resp. $\mathcal{J}$) is called an almost complex (resp. paracompact) structure on E if $\mathcal{I}^2 = -\operatorname{Id}_E$ (resp. $\mathcal{J}^2 = \operatorname{Id}_E$).

According to [CaPe2], § B.5 and § B.6, if J is an almost tangent structure, there exists a decomposition $E = \ker J \oplus K$ in a Whitney sum, the restriction J_K of J to K is a bundle isomorphism from K to $\ker J$. Moreover, we can associate to J an almost complex structure $\mathcal{I}$ (resp. $\mathcal{J}$) on E given by $\mathcal{I}(u) = -J_K u$ (resp. $\mathcal{J}(u) = J_K u$) for $u \in K$ and $\mathcal{I}(u) = J_K^{-1}(u)$ (resp. $\mathcal{I}(u) = J_K^{-1}(u)$) for $u \in \ker J$.

Definition 3.4. An almost complex structure is called decomposable if there exists a decomposition in a Whitney sum $E = E_1 \oplus E_2$ and an isomorphism $I : E_2 \longrightarrow E_1$ such that $\mathcal{I}$ can be written

$$\begin{pmatrix} 0 & -I \\ I^{-1} & 0 \end{pmatrix}.$$

Given a Withney decomposition $E = E_1 \oplus E_2$ associated to a decomposable almost complex structure $\mathcal{I}$, let $\mathcal{S}$ be a the isomorphism of E defined by the matrix

$$\begin{pmatrix} -\operatorname{Id}_{E_1} & 0 \\ 0 & \operatorname{Id}_{E_2} \end{pmatrix}.$$

According to [CaPe2], § B.6.2, $\mathcal{J} = \mathcal{S}\mathcal{I}$ is an almost para-complex structure. Conversely, if $\mathcal{J}$ is an almost para-complex structure there exists a decomposition in a Whitney sum $E = E_1 \oplus E_2$ and an isomorphism $I : E_2 \longrightarrow E_1$ such that $\mathcal{I}$ can be written

$$\begin{pmatrix} 0 & -I \\ I^{-1} & 0 \end{pmatrix}$$

and in this way, $\mathcal{I} = \mathcal{S}\mathcal{J}$ is decomposable almost complex structure.

From [CaPe2], Propositions B.23 and B.29, we obtain the following result.

Theorem 3.2. *Let (E, π_E, M) be a Banach vector bundle.*

1. *Let J be an almost tangent structure on (E, π_E, M).*
   For a fixed x_0 in M, we identify the fibre E_{x_0} with the typical fibre of E. If J_{x_0} is the induced tangent structure on E_{x_0}, then J defines a J_{x_0}-structure on E.

2. *Let $\mathcal{I}$ be an almost decomposable complex structure on (E,π_E,M).*
   For a fixed x_0 in M, we identify the fibre E_{x_0} with the typical fibre of E. If $\mathcal{I}_{x_0}$ is the induced complex structure on E_{x_0}, then $\mathcal{I}$ defines a $\mathcal{I}_{x_0}$-structure on E.

3. *Let $\mathcal{J}$ be an almost para-complex structure on (E,π_E,M).*
   For a fixed x_0 in M, we identify the fibre E_{x_0} with the typical fibre of E. If $\mathcal{J}_{x_0}$ is the induced para-complex structure on E_{x_0}, then $\mathcal{J}$ defines a $\mathcal{J}_{x_0}$-structure on E.

Proof. Consider a decomposition $E = \ker J \oplus K$ in Whitney sum. Let $\{(U_\alpha,\tau_\alpha)\}_{\alpha\in A}$ be a bundle atlas of E which induces an atlas on each sub-bundle $\ker J$ and K. For any $x \in U_\alpha$, $\tau_\alpha(x) : E_x \longrightarrow E_{x_0}$ is a an isomorphism and so $\tau_\alpha(x)^* J_{x_0}$ is a tangent structure on E_x. Moreover, if τ^1_α and τ^2_α are the restrictions of τ_α to $\ker J_{|U_\alpha}$ and $K_{|U_\alpha}$ respectively, then $\tau^1_\alpha(x)$ and $\tau^2_\alpha(x)$ are isomorphisms of $\ker J_x$ and K_x onto $\ker J_{x_0}$ and K_{x_0} respectively. Therefore $(\tau_\alpha(x)^{-1})^*(J_x)$ is a (linear) tangent structure on E_{x_0} whose kernel is also $\ker J_{x_0}$ with $K_{x_0} = (\tau^2_\alpha(x))^{-1}(K_x)$ and $(\tau_\alpha(x)^{-1})^* J_{|K_x}$ is an isomorphism from K_{x_0} to $\ker J_{x_0}$. From [CaPe2], Proposition B.23, $T_x : E_{x_0} \longrightarrow E_{x_0}$ defined by $T_x(u,v) = (u, (\tau_\alpha(x)^{-1})^* J_{|K_x} \circ J^{-1}_{x_0}$ is an automorphism of E_{x_0} such that $T^*_x(J_{x_0}) = (\tau_\alpha(x)^{-1})J_x$ and so $(T_x \circ \tau_\alpha(x))^*(J_{x_0}) = J_x$. Since τ_α is smooth on U, this implies that $\tau'_\alpha(x) = T_x \circ \tau_\alpha(x)$ defines a trivialization $\tau'_\alpha : E_{U_\alpha} \longrightarrow U_\alpha \times E_{x_0}$, so $J_{|U_\alpha}$ is locally modelled on J_{x_0}. Now it is clear that each transition map $T_{\alpha\beta}$ belongs to the isotropy group of J_{x_0}.
The proofs of Point 2 and Point 3 use, step by step, the same type of arguments as in the previous one and is left to the reader. □

3.3 Compatible almost tangent and almost cotangent structures

Let (E,π_E,M) be a Banach bundle such that M is connected.

Definition 3.5. A weak non degenerated 2-form Ω on E is called an almost cotangent structure on E if there exists a decomposition of E in a Whitney sum $L \oplus K$ of Banach sub-bundles of E such that each fibre L_x is a weak Lagrangian subspace of Ω_x in E_x for all $x \in M$. In this case, L is called a weak Lagrangian bundle.

Definition 3.6. An almost tangent structure J on E is compatible with an almost cotangent structure Ω if Ω_x is compatible with J_x[1] for all $x \in M$.

Assume that E is a Whitney sum $E_1 \oplus E_2$ of two sub-bundles of E and there exists a bundle isomorphism $J : E_2 \longrightarrow E_1$, then as we have already seen, we can associate to J an almost complex structure $\mathcal{I}$ on E. On the other hand, if g is a pseudo-Riemannian metric on E_1, we can extend g to a canonical pseudo-Riemannian metric $\bar{g}$ on E such that E_1 and E_2 are $\bar{g}$ orthogonal and the restriction of $\bar{g}$ on E_2 is $\bar{g}(u,v) = g(Ju,Jv)$. Now, according to [CaPe2], § B.7.1, by application of Proposition B.31 on each fibre, we get the following result.

Proposition 3.3.

[1] i.e. $\ker J_x$ is Lagrangian and we have $\Omega_x(J_x\ ,\ ) + \Omega_x(\ , J_x\ ) = 0$

1. *Assume that there exists an almost tangent structure J on E and let $E = \ker J \oplus K$ be an associated decomposition in a Whitney sum. If there exists a pseudo-Riemannian metric g on* $\ker J$*, then there exists a cotangent structure Ω on E compatible with J such that* $\ker J$ *is a weak Lagrangian bundle of E where*

$$\forall (u,v) \in E^2, \Omega(u,v) = \bar{g}(\mathcal{I}u, v) - \bar{g}(u, \mathcal{I}v)$$

if $\bar{g}$ is the canonical extension of g and $\mathcal{I}$ is the almost complex structure on E defined by J.

2. *Assume that there exists an almost cotangent structure Ω on E and let $E = L \oplus K$ be an associated Whitney decomposition. Then there exists a tangent structure J on E such that* $\ker J = L$.

3. *Assume that there exists a tangent structure J on E which is compatible with a cotangent structure Ω. Then there exists a decomposition $E = \ker J \oplus K$ where* $\ker J$ *is a weak Lagrangian bundle. Moreover $g(u,v) = \Omega(Ju, v)$ is a pseudo-Riemannian metric on* $\ker J$ *and Ω satisfies the relation (3.3).*

In the context of the previous Proposition, if g is a weak Riemannian metric, so is its extension $\bar{g}$. In this case, as we have already seen in § 3.1, each fibre E_x can be continuously and densely embedded in a Hilbert space which will be denoted $\widehat{E}_x$. According to Theorem 3.1, with these notations, we get the following theorem.

Theorem 3.4.

1. *Consider a tangent structure J on E, a weak Riemannian metric g on* $\ker J$ *and Ω the cotangent structure compatible with J as defined in (3.3). Assume that $\widehat{E} = \bigcup\limits_{x \in M} \widehat{E}_x$ is a Banach bundle over M such that the inclusion of E in $\widehat{E}$ is a bundle morphism. For any x_0 in M, the triple (J, g, Ω) defines a $(J_{x_0}, \bar{g}_{x_0}, \Omega_{x_0})$-structure on E where $\bar{g}$ is the natural extension of g to E.*

2. *Let $(\widehat{E}, \pi_{\widehat{E}}, M)$ be a Hilbert bundle over a connected Banach manifold M. Consider an almost tangent structure $\widehat{J}$ on $\widehat{E}$, a Riemannian metric $\widehat{g}$ on* $\ker \widehat{J}$ *and an almost cotangent structure $\widehat{\Omega}$ compatible with $\widehat{J}$ which satisfies (3.3) on $\widehat{E}$. Then for any Banach bundle (E, π_E, M) such that there exists an injective bundle morphism $\iota : E \longrightarrow \widehat{E}$ with dense range, the restriction J of $\widehat{J}$ to E is an almost tangent structure on E and the restriction Ω of $\widehat{\Omega}$ to E is an almost cotangent structure compatible with J and Ω satisfies the relation (3.3) on E. In particular, for any x_0 in M, the triple $(J, g, \Omega))$ defines a $(J_{x_0}, g_{x_0}, \Omega_{x_0})$-structure on E.*

Remark 3.7. According to Proposition 3.3, Point 3, if there exists a tangent structure J on E which is compatible with a cotangent structure Ω such that $g(u,v) = \Omega(Ju, v)$ is a Riemannian metric on $\ker J$, then Ω satisfies the relation (3.3). Therefore we have a corresponding version of Theorem 3.3 with the previous assumption.

Proof. 1. According to our assumption, by density of $\iota(E)$ (identified with E) in $\widehat{E}$ and using compatible bundle atlases for $\widehat{E}$ and E (cf. proof of Theorem 3.1) it is easy to see that we can extend J and g, to an almost tangent structure $\widehat{J}$ and a strong Riemannian metric $\widehat{g}$ on $\widehat{E}$. By the way, if $\mathcal{I}$ is the almost complex structure on E associated to J, we can also extend $\mathcal{I}$ to an almost complex structure $\widehat{\mathcal{I}}$ on $\widehat{E}$ which is exactly the almost complex structure associated to $\widehat{J}$. Thus this implies, by relation (3.3), that we can extend Ω into a 2-form $\widehat{\Omega}$ on $\widehat{E}$ which will satisfy also relation (3.3) on $\widehat{E}$, using $\widehat{g}$ and $\widehat{\mathcal{I}}$. Then $\widehat{\Omega}$ is a strong non-degenerate 2-form on $\widehat{E}$. Note that $\ker \widehat{J}$ is the closure in $\widehat{E}$ of $\ker J$ and, since $J : K \longrightarrow \ker J$ is an isomorphism, this implies that $\widehat{K} = \widehat{J}(\ker \widehat{J})$ is a Banach sub-bundle of $\widehat{E}$ such that $\widehat{E} = \ker \widehat{J} \oplus \widehat{K}$. Moreover, each fibre $\ker \widehat{J}_x$ of $\ker J$ is exactly the Hilbert space which is the closure of $\ker J_x$ provided with the pre-Hilbert product g_x. Now, from the construction of the extension $\bar{g}$ of g to E, the restriction of J to K is an isometry and so the restriction of $\widehat{J}$ to $\widehat{K}$ is also an isometry. This implies that $\mathcal{I}$ (resp. $\widehat{\mathcal{I}}$) is an isometry of $\bar{g}$ (resp. $\widehat{g}$). Fix a point x_0 in M and identify the fibres of E and $\widehat{E}$ with E_{x_0} and $\widehat{E}_{x_0}$. From Theorem 3.1, there exists an atlas $\{(U_\alpha, \tau_\alpha)\}_{\alpha \in A}$ of E and $\{(U_\alpha, \sigma_\alpha)\}_{\alpha \in A}$ such that $\tau_\alpha = \sigma_\alpha \circ \iota$ and the respective transition functions $T_{\alpha\beta}$ and $S_{\alpha\beta}$ are isometries of E relatively to $\bar{g}_{x_0}$ and $\widehat{g}_{x_0}$. Since $\mathcal{I}$ and $\widehat{\mathcal{I}}$ are isometries for $\bar{g}$ and $\widehat{g}$, this implies that $\mathcal{I}$ is also a $\mathcal{I}_{x_0}$-structure on E and $\widehat{\mathcal{I}}$ is a $\widehat{\mathcal{I}}_{x_0}$-structure on $\widehat{E}$. But since J and $\widehat{J}$ are canonically defined from $\mathcal{I}$ and $\widehat{\mathcal{I}}$ respectively, we obtain a similar result for J and $\widehat{J}$. Finally, according to relation (3.3), we also obtain a similar result for Ω and $\widehat{\Omega}$, which ends the proof.

2. Under the assumptions of Point 2, we can apply all the results of Point 1 to $(\widehat{E}, \pi_{\widehat{E}}, M)$. In particular, for a decomposition $\widehat{E} = \ker \widehat{J} \oplus \widehat{K}$, $\widehat{J}$ is an isometry from $\widehat{K}$ to $\ker \widehat{J}$ and the almost complex structure $\widehat{\mathcal{I}}$ associated to $\widehat{J}$ is an isometry of $\widehat{g}$. We can identify E with $\iota(E)$ in $\widehat{E}$. We set $K = E \cap \widehat{K}$. Then the range of the restriction of $\widehat{J}$ to K is included in $\ker \widehat{J}$. Using compatible bundle atlases for E and $\widehat{E}$, we can show that if J is the restriction of $\widehat{J}$ then $\ker J = \ker \widehat{J} \cap E$ and $E = \ker J \oplus K$ so that J is an almost tangent structure on E. Now $\widehat{g}$ induces a weak Riemannian metric $\bar{g}$ on E and J is again an isometry from K to $\ker J$. This implies that the almost complex structure $\mathcal{I}$ associated to J is nothing but than the restriction of $\widehat{\mathcal{I}}$ to E. Of course, we obtain that the restriction Ω of $\widehat{\Omega}$ satisfies the relation (3.3) relatively to $\bar{g}$ and $\mathcal{I}$. So the assumptions of Point 1 are satisfied which ends the proof. $\square$

3.4 Compatible weak symplectic form, weak Riemannian metric and almost complex structures

If g (resp. Ω) is a weak neutral metric (resp. a weak symplectic form) on a Banach bundle (E, π_E, M), as in the linear context, we denote by $g^\flat$ (resp. $\Omega^\flat$) the associated morphism from E to E^* where (E^*, π_{E^*}, M) is the dual bundle of (E, π_E, M). Moreover, if $\mathcal{I}$ is an almost complex structure on E, following [CaPe2], § B.7, we introduce the following notions.

Definition 3.8.

1. We say that a weak symplectic form Ω and an almost complex structure $\mathcal{I}$ on E are compatible if $(u, v) \mapsto \Omega(u, \mathcal{I}v)$ is a weak Riemannian metric on E and $\Omega(\mathcal{I}u, \mathcal{I}v) = \Omega(u, v)$ for all u and v in E.

2. We say that a weak Riemannian metric g and an almost complex structure $\mathcal{I}$ on E are compatible if $g(\mathcal{I}u, \mathcal{I}v) = g(u, v)$ for all u and v in E.

3. We say that a weak Riemannian metric g and a weak symplectic structure Ω on E are compatible if $\mathcal{I} = (g^\flat)^{-1} \circ \Omega^\flat$ is well defined and is a complex structure on E.

4. A weak symplectic form Ω on E will be called a Darboux form if there exists a decomposition $E = E_1 \oplus E_2$ such that for each $x \in M$ each fibre $(E_1)_x$ and $(E_2)_x$ is Lagrangian.

Now by application of [CaPe2], Proposition B.34 and Corollary B.35 we obtain:

Theorem 3.5. *Consider a Darboux form Ω, a weak Riemannian g and a decomposable complex structure $\mathcal{I}$ on a Banach space $\mathbb{E}$. Assume that any pair among such a triple exists on E and is compatible. Then the third one also exists and is compatible with any element of the given pair. Denote $\widehat{E}_x$ the Hilbert space defined by g_x for each $x \in M$. If $\widehat{E} = \bigcup\limits_{x \in M} \widehat{E}_x$ is a Banach bundle over M and there exists an injective morphism $\iota : E \longrightarrow \widehat{E}$, then we can extend Ω, g and $\mathcal{I}$ to a strong symplectic form $\widehat{\Omega}$, a strong Riemannian metric $\widehat{g}$ and an almost complex structure $\widehat{\mathcal{I}}$ on $\widehat{E}$ respectively. Moreover, for any $x_0 \in M$, if we identify the typical fibre of E and of $\widehat{E}$ with E_{x_0} and $\widehat{E}_{x_0}$ respectively, then the triple $(\Omega, g, \mathcal{I})$ (resp. $(\widehat{\Omega}, \widehat{g}, \widehat{\mathcal{I}})$) defines a $(\Omega_{x_0}, g_{x_0}, \mathcal{I}_{x_0})$-structure (resp. $(\widehat{\Omega}_{x_0} \widehat{g}_{x_0}, \widehat{\mathcal{I}}_{x_0})$-structure) on E (resp. $\widehat{E}$).*
Conversely, assume that we have such a triple $(\widehat{\Omega}, \widehat{g}, \widehat{\mathcal{I}})$ which is compatible on a Hilbert bundle $(\widehat{E}, \pi_{\widehat{E}}, M)$. For any Banach bundle which can be continuously and densely embedded in $\widehat{E}$, then this triple induces, by restriction to E, a triple $(\Omega, g, \mathcal{I})$ which is compatible. Therefore, we obtain a $(\Omega_{x_0}, g_{x_0}, \mathcal{I}_{x_0})$-structure (resp. $(\widehat{\Omega}_{x_0}, \widehat{g}_{x_0}, \widehat{\mathcal{I}}_{x_0})$-structure) on E (resp. $\widehat{E}$).

Proof. The first property is a direct application of [CaPe2], Proposition B.34 and Corollary B.35, the other ones are obtained as in the proof of the corresponding parts of Theorem 3.4. □

Definition 3.9. A Banach bundle $\pi_E : E \longrightarrow M$ has a weak (resp. strong) almost Kähler structure if there exists on E a weak (resp. strong) Darboux form Ω, a weak (resp. strong) Riemanian metric g and a decomposable almost complex structure $\mathcal{I}$ such that there exists a compatible pair among these three data.

From Remark [CaPe2], B.36, to a weak or strong almost Kähler structure on E is associated a decomposition $E = E_1 \oplus E_2$ of isomorphic sub-bundles which are lagrangian and orthogonal. Note that Theorem 3.5 can be seen as sufficient conditions under which a weak Kähler structure on a Banach bundle E is a tensor structure on E.

3.5 Compatible weak symplectic forms, weak neutral metrics and almost para-complex structures

If g (resp. Ω) is a weak Riemannian metric (resp. weak symplectic form) on a Banach bundle (E, π_E, M), as in the linear context, we denote by $g^\flat$ (resp. $\Omega^\flat$)

the associated morphism from E to E^* where (E^*, π_{E^*}, M) is the dual bundle of (E, π_E, M). Moreover, if $\mathcal{J}$ is an almost para-complex structure on E, following [CaPe2], § B.7, we introduce the following notions.

Definition 3.10.

1. We say that a weak symplectic form Ω and an almost para-complex structure $\mathcal{J}$ on E are compatible if $(u, v) \mapsto \Omega(u, \mathcal{J}v)$ is a weak neutral metric on E and $\Omega(\mathcal{J}u, \mathcal{J}v) = -\Omega(u, v)$ for all u and v in E.

2. We say that a weak neutral metric g and an almost para-complex structure $\mathcal{J}$ on E are compatible if $g(\mathcal{J}u, \mathcal{J}v) = -g(u, v)$ for all u and v in E.

3. We say that a weak neutral metric g and a weak symplectic structure Ω on E are compatible, if $\mathcal{J} = (g^\flat)^{-1} \circ \Omega^\flat$ is well defined and is an almost para-complex structure on E.

According to [CaPe2], Remark B.39 and Theorem 3.5, we have

Theorem 3.6. *Consider a Darboux form Ω, a weak neutral metric g and a para-complex structure $\mathcal{J}$ on a Banach space $\mathbb{E}$. Assume that any pair among such a triple exists on E and is compatible. Then the third one also exists and is compatible with any element of the given pair. Let h be a Riemaniann metric canonically associated to some decomposition $E = E^+ \oplus E^-$ relatively to g and let $\widehat{E}_x$ be the Hilbert space defined by the inner product associated to h_x for each $x \in M$. If $\widehat{E} = \bigcup\limits_{x \in M} \widehat{E}_x$ is a Banach bundle over M and there exists an injective morphism $\iota : E \longrightarrow \widehat{E}$, then we can extend Ω, g and $\mathcal{J}$ to a strong symplectic form $\widehat{\Omega}$, a strong neutral metric $\widehat{g}$ and an almost para-complex structure $\widehat{\mathcal{J}}$ on $\widehat{E}$ respectively. Moreover, for any $x_0 \in M$, if we identify the typical fibre of E and of $\widehat{E}$ with E_{x_0} and $\widehat{E}_{x_0}$ respectively, then the triple $(\Omega, g, \mathcal{J})$ (resp. $(\widehat{\Omega}, \widehat{g}, \widehat{\mathcal{J}})$) defines a $(\Omega_{x_0}, g_{x_0}, \mathcal{J}_{x_0})$-structure (resp. $(\widehat{\Omega}_{x_0} \widehat{g}_{x_0}, \widehat{\mathcal{J}}_{x_0})$-structure) on E (resp. $\widehat{E}$). Conversely, assume that we have such a triple $(\widehat{\Omega}, \widehat{g}, \widehat{\mathcal{J}})$ which is compatible on a Hilbert bundle $(\widehat{E}, \pi_{\widehat{E}}, M)$. For any Banach bundle which can be continuously and densely embedded in $\widehat{E}$, then this triple induces, by restriction to E, a triple $(\Omega, g, \mathcal{J})$ which is compatible. Therefore, we obtain a $(\Omega_{x_0}, g_{x_0}, \mathcal{J}_{x_0})$-structure (resp. $(\widehat{\Omega}_{x_0}, \widehat{g}_{x_0}, \widehat{\mathcal{J}}_{x_0})$-structure) on E (resp. $\widehat{E}$).*

As in finite dimension (cf. [Lib] for instance), we introduce the para-Kähler structures.

Definition 3.11. A Banach bundle $\pi_E : E \longrightarrow M$ has a weak (resp. strong) almost para-Kähler structure if there exists on E a weak (resp. strong) Darboux form Ω, a weak (resp. strong) neutral metric g and an almost para-complex structure $\mathcal{J}$ such that there exists a compatible pair among these three data.

As for weak or strong almost para-Kähler bundles, from [CaPe2], Remark B.42, to a weak or strong para-Kähler structure on E is associated a decomposition $E = E_1 \oplus E_2$ of isomorphic sub-bundles which are Lagrangian and the restriction of g to E_1 (resp. E_2) is positive definite (resp. negative definite).

Note also that Theorem 3.6 can be seen as sufficient conditions under which a weak almost para-Kähler structure on a Banach bundle E is a tensor structure on E.

4 Examples of integrable tensor structures on a Banach manifold

4.1 The Darboux Theorem on a Banach manifold

In finite dimension, from the Darboux theorem, a symplectic form on a manifold defines an integrable tensor structure on M. The extension of such a result to the Banach framework is given in [Bam] for weak symplectic Banach manifolds.

Definition 4.1. A weak symplectic form on a Banach manifold M modelled on a Banach space $\mathbb{M}$ is a closed 2-form Ω on M which is non-degenerate.

If $\Omega^\flat : TM \to T^*M$ is the associated morphism, then $\Omega^\flat$ is an injective bundle morphism. The symplectic form Ω is weak if $\Omega^\flat$ is not surjective. Assume that $\mathbb{M}$ is reflexive. We denote by $\widehat{T_xM}$ the Banach space which is the completion of T_xM provided with the norm $|| \; ||_{\Omega_x}$ defined by

$$||u||_{\Omega_x} = ||\Omega_x^\flat(u)||^*$$

where $|| \; ||^*$ is the norm on T_x^*M associated to a choice of norm $|| \; ||$ on T_xM (see [CaPe2] for more precisions). Recall that $\widehat{T_xM}$ does not depend on the choice of the norm on T_xM (cf. [CaPe2], Remark B.2). Then Ω can be extended to a continuous bilinear map $\widehat{\Omega}_x$ on $T_xM \times \widehat{T_xM}$ and $\Omega^\flat$ becomes an isomorphism from T_xM to $(\widehat{T_xM})^*$. We set

$$\widehat{TM} = \bigcup_{x\in M} \widehat{T_xM} \quad \text{and} \quad (\widehat{TM})^* = \bigcup_{x\in M} (\widehat{T_xM})^*.$$

We have the following Darboux Theorem ([Bam]) (see also [Pel2]):

Theorem 4.1 (Local Darboux theorem)**.** *Let Ω be a weak symplectic form on a Banach manifold modelled on a reflexive Banach space $\mathbb{M}$. Assume that we have the following assumptions:*

(i) There exists a neighbourhood U of $x_0 \in M$ such that $\widehat{TM}_{|U}$ is a trivial Banach bundle whose typical fibre is the Banach space $(\widehat{T_{x_0}M}, || \; ||_{\Omega_{x_0}})$;

(ii) Ω can be extended to a smooth field of continuous bilinear forms on $TM_{|U} \times \widehat{TM}_{|U}$.

Then there exists a chart (V,F) around x_0 such that $F^*\Omega_0 = \Omega$ where Ω_0 is the constant form on $F(V)$ defined by $(F^{-1})^*\Omega_{x_0}$.

Definition 4.2. The chart (V, F) in Theorem 4.1 will be called a Darboux chart around x_0.

Remark 4.3. The assumptions of Darboux theorem in [Bam] (Theorem 2.1) are formulated in a different way. In fact, the assumption on all those norms $|| \; ||_{\Omega_x}$ in this Theorem 2.1 on the typical fibre $\widehat{\mathbb{M}}$ is a consequence of assumptions (i) and (ii) of Theorem 4.1 after shrinking U if necessary (cf. [Pel2]).

Remark 4.4. If Ω is a strong symplectic form on M, then $\mathbb{M}$ is reflexive and $\Omega^\flat$ is a bundle isomorphism from TM to T^*M. In particular, the norm $|| \; ||_{\Omega_x}$ is equivalent to any norm $|| \; ||$ on T_xM which defines its Banach structure and so all the assumptions (i) and (ii) of Theorem 4.1 are always locally satisfied. Thus Theorem 4.1 recovers the Darboux Theorem which is proved in [Mar] or [Wei].

Weinstein gives an example of a weak symplectic form Ω on a neighbourhood of 0 of a Hilbert space $\mathbb{H}$ for which the Darboux Theorem is not true. The essential reason is that the operator $\Omega^\flat$ is an isomorphism from T_xU onto T_x^*U on $U \setminus \{0\}$, but $\Omega_0^\flat$ is not surjective.

Finally from Theorem 4.1, we also obtain the following global version of a Darboux Theorem:

Theorem 4.2 (Global Darboux Theorem)**.** *Let Ω be a weak symplectic form on a Banach manifold modelled on a reflexive Banach space $\mathbb{M}$. Assume that we have the following assumptions:*

(i) $\widehat{TM} \to M$ *is a Banach bundle whose typical fibre is* $\widehat{\mathbb{M}}$*;*

(ii) Ω *can be extended to a smooth field of continuous bilinear forms on* $TM \times \widehat{TM}_{|U}$.

Then, for any $x_0 \in M$, there exists a Darboux chart (V, F) around x_0.
In particular Ω defines an integrable tensor structure on M if and only if the assumptions (i) and (ii) are satisfied.

Note that, from Remark 4.4, a strong symplectic form on a Banach manifold is always an integrable tensor structure. The reader will find in [Pel2] an example of a weak symplectic form on a Banach manifold for which the assumptions of Theorem 4.2 are satisfied.

4.2 Flat pseudo-Riemannian metrics on a Banach manifold

In finite dimension, it is well known that a pseudo-riemannian metric on a manifold M defines an integrable tensor structure on M if and only if its Levi-Civita connection is flat. In this section, we give a generalization of this result to the Banach framework.

Definition 4.5. A pseudo-Riemannian metric (resp. a Krein metric) g on a Banach manifold M is a pseudo-Riemannian metric (resp. a Krein metric) on the tangent bundle TM.

When g is a Krein metric, we have a decomposition $TM = TM^+ \oplus TM^-$ in a Whitney sum such that the restriction g^+ (resp. $-g^-$) of g (resp. $-g$). to TM^+ (resp. TM^-) is a (weak) Riemannian metric. To g is associated a canonical Riemannian metric $\gamma = g^+ - g^-$. According to [CaPe2], § B.10, on each fibre T_xM, we have a norm $|| \ ||_{g_x}$ which is associated to the inner product γ_x. We denote by $\widehat{T_xM}$ the Banach Hilbert space associated to the normed space $(T_xM, || \ ||_{g_x})$.

By application of Theorem 3.1 to TM we have the following theorem.

Theorem 4.3.

1. *Let g be a Krein metric on a Banach manifold M. Consider a decomposition $TM = TM^+ \oplus TM^-$ in a Whitney sum such that the restriction g^+ (resp. $-g^-$) of g (resp. $-g$) to TM^+ (resp. TM^-) is a (weak) Riemannian metric. Moreover, assume that $\widehat{TM} = \bigcup_{x \in M} \widehat{T_xM}$ is a Banach bundle over M such that the inclusion of TM in $\widehat{TM}$ is a bundle morphism. Then g can be extended to a strong Krein metric $\widehat{g}$ on the bundle $\widehat{TM}$ and we have a decomposition $\widehat{TM} = \widehat{TM}^+ \oplus \widehat{TM}^-$ such that the restriction $\widehat{g}^+$ (resp. $\widehat{g}^-$) of $\widehat{g}$ (resp. $-\widehat{g}$) to $\widehat{TM}^+$ (resp. $\widehat{TM}^-$) is a strong Riemannian metric. In fact $\widehat{TM}^+$ (resp. $\widehat{TM}^-$) is the closure of TM^+ (resp. TM^+) in $\widehat{TM}$.*

2. *Let $(\widehat{TM}, \widehat{\pi}, M)$ be a Banach bundle whose typical fibre is a reflexive Banach space and let $\widehat{g}$ be a strong Krein metric on $\widehat{TM}$. Assume that there exists an injective morphism of Banach bundles $\iota : TM \longrightarrow \widehat{TM}$ whose range is dense. Then the restriction $g = \iota^* \widehat{g}$ of $\widehat{g}$ is a Krein metric on M.*

Now, as for a strong Riemannian metric on a Banach manifold, to a strong pseudo-Riemannian metric g is associated a Levi-Civita connection which is a Koszul connection ∇ characterized by

$$\begin{aligned} 2g(\nabla_X Y, Z) = \quad & X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ & + g([X,Y],Z) - g([Y,Z],X) - g([X,Z],Y) \end{aligned}$$

for all local vector fields X, Y, Z on M.

When g is a weak pseudo-Riemannian metric, a Levi-Civita connection need not exist, but if it exists, this connection is unique.

For a weak Riemannian metric, there exist many examples of weak Riemannian metrics whose Levi-Civita connection is well defined. When the model $\mathbb{M}$ of M is a *reflexive* Banach space we will give sufficient conditions for a weak pseudo-Riemannian metric under which the Levi-Civita connection exists. Before, we need to introduce some preliminaries.

According to [CaPe2], § B.1, for each $x \in M$, from a given norm $|| \ ||$ on T_xM, we can define a norm $|| \ ||_{g_x}$ on T_xM and the completion T_xM_g of the normed space $(T_xM, || \ ||_{g_x})$ does not depend on the choice of the norm $|| \ ||$ (see [CaPe2], Remark B.2). In this way, $g_x^\flat$ can be extended to an isomorphism from T_xM_g to T_x^*M and $g_x^\flat$ becomes an isomorphism from T_xM to $(T_xM_g)^*$. Now we set:

$$TM_g = \bigcup_{x \in M} T_xM_g \quad \text{and} \quad T^*M_g = \bigcup_{x \in M} (T_xM_g)^*.$$

Note that TM is a subset of TM_g and each fibre T_xM is dense in T_xM_g and $g_x^\flat$ can be extended to an isometry from T_xM_g to T_x^*M. With these notations, we have:

Proposition 4.4. *Let g be a pseudo-Riemannian metric on a Banach manifold modelled on a reflexive Banach space. Assume that the following assumptions are satisfied:*

(i) TM_g has a structure of Banach bundle over M;

(ii) The injective morphism $g^\flat : TM \longrightarrow T^*M$ can be extended to a bundle morphism from TM_g to T^*M.

Then the Levi-Civita connection of g is well defined.

Proof. Fix some $x_0 \in M$ and, for the sake of simplicity, denote by $\mathbb{E}$ the Banach space $T_{x_0}M_g$. From assumption (i), it follows that around x_0, there exists a trivialization $\Psi : (TM_g)_{|U} \longrightarrow U \times \mathbb{M}_g$ where $\mathbb{M}_g$ denotes the typical fibre of TM_g. After shrinking U, if necessary, we may assume that U is a chart domain associated to some $\phi : U \longrightarrow \mathbb{M}$ and so $T\pi : TM_{|U} \longrightarrow \phi(U) \times \mathbb{M}$ is a trivialization. We may assume that $\Psi : (TM_g)_{|U} \longrightarrow \phi(U) \times \mathbb{M}$ is a trivialization and so, without loss of generality, we can also assume that U is an open subset of $\mathbb{M}$ and $TM_{|U} = U \times \mathbb{M}$. Now since $\mathbb{M}$ is dense in $\mathbb{M}_g$, the inclusion of the trivial bundle $U \times \mathbb{M}$ into $U \times \mathbb{E}$ is an injective bundle morphism.

For fixed local vector fields X and Y defined on U, the map which, to any local vector field Z defined on U, associates the second member of (4.2) is a well defined local 1-form on U denoted $\alpha_{X,Y}$.
Note that from [CaPe2], since $\mathbb{M}$ is reflexive, $g_x^\flat$ can be extended to an isomorphism from T_xM to $(T_xM_g)^*$ and also gives rise to an isomorphism from T_xM to $(T_xM_g)^*$. Thus, from assumption (ii), the extension of $g^\flat$ gives rise to a bundle isomorphism from TM to T^*M_g again denoted $g^\flat$. But it is clear that any 1-form α on $U \times \mathbb{M}$ can be extended to a smooth 1-form (again denoted α) on $(TM_g)_{|U} = U \times \mathbb{M}$. This implies that $(g^\flat)^{-1}(\alpha)$ is a smooth vector field on U, which ends the proof.

□

Theorem 4.5. *Let g be a pseudo-Riemannian metric on a Banach manifold. Assume that the Levi-Civita connection ∇ of g is defined. Then g defines an integrable tensor structure if and only if the curvature of ∇ vanishes.*

This result is well known but we have no precise and accessible reference to a complete proof of such a result. We give a sketch of the proof.

Proof. On the principal frame bundle $\ell(TM)$, the Levi-Civita gives rise to a connection form ω with values in the Lie algebra $\mathfrak{gl}(\mathbb{M})$ of the Banach Lie group $\mathrm{GL}(\mathbb{M})$ (cf. [KrMi] 8). Let Ω be the curvature of ω. If X and Y are local vector fields on M, let X^h and Y^h be their horizontal lifts in $\ell(TM)$ and we have $\Omega(X^h, Y^h) = -\omega([X^h, Y^h])$. Therefore, the horizontal bundle is integrable if the curvature vanishes. Moreover, in this case, over any simply connected open set U of M, the bundle $\ell(TM)_{|U}$ is trivial (consequence of [KrMi], Theorem 39.2, for instance). Thus, around any point $x_0 \in M$, there exists a chart domain (U, ϕ) such that $\ell(TM)_{|U} \equiv \phi(U) \times \mathrm{GL}(\mathbb{M})$. Without loss of generality, we may assume that $U = \mathbb{M}$, $x_0 = 0 \in \mathbb{M}$ and so $\ell(TM)_{|U} = \mathbb{M} \times \mathrm{GL}(\mathbb{M})$.

The horizontal leaves are obtained from $\mathbb{M}\times \mathrm{Id}_{\mathbb{M}}$ by right translation in $\mathrm{GL}(\mathbb{M})$. Consider a horizontal section $\psi : \mathbb{M} \longrightarrow \mathrm{GL}(\mathbb{M})$ such that $\psi(0) = \mathrm{Id}_{\mathbb{M}}$, then $\psi(x) = \mathrm{Id}_{\mathbb{M}}$ for all $x \in \mathbb{M}$. Since the parallel transport from some point x to some point y does not depend on the curve which joins x to y, this implies that $\psi(x)$ is the parallel transport from $T_0\mathbb{M}$ to $T_x\mathbb{M}$. For the same reason, if ∇ is the Levi-Civita connection, then for any constant vector field X and Y on M, we have $\nabla_Y X = 0$. As ∇ is compatible with g, this implies that, for any $X \in T_x\mathbb{M}$, we have $X\{g_x(Y,Z)\} = 0$, for all $x \in \mathbb{M}$, all $X \in T_x\mathbb{M}$ and any local constant vector field Y, Z around x. Consider the diffeomorphism $\Psi(x) = x + \mathrm{Id}_{\mathbb{M}}$ of $\mathbb{M}$. The previous property implies that $\Psi^* g_0 = g$. Let (U_1, ϕ_1) and (U_2, ϕ_2) be two charts such that $U_1 \cap U_2 \neq \varnothing$ and if $x_0 \in U_1 \cap U_2$ then, for $i \in \{1,2\}$, $\phi_i^* g = g_{x_0}$ on $\phi_i(U_i) \subset \mathbb{M} \equiv T_{x_0}M$ where g_{x_0} is the constant pseudo-Riemannian metric on $\mathbb{M}$ defined by g_{x_0}. Then it is easy to see that the the transition function T_{12} on $U_1 \cap U_2$ associated to $T\phi_1$ and $T\phi_2$ belongs to the isotropy group of g_{x_0}.

Conversely, if g is an integrable tensor structure, locally there exists a chart (U,ϕ) such that $\phi^* g = g_{x_0}$ on $\phi(U) \subset \mathbb{M} \equiv T_{x_0}M$ where g_{x_0} is the constant pseudo-Riemannian defined on $\mathbb{M} \equiv T_{x_0}M$ by g_{x_0}. Then it is clear that the curvature of the Levi-Civita connection of the constant metric g_{x_0} vanishes. Now since ϕ is an isometry from $(U, g_{|U})$ onto $(\phi(U), (g_{x_0})_{|\phi(U)})$, the curvature of the Levi-Cevita connection of g must also vanish on U, which ends the proof. □

Corollary 4.6. *A Krein metric g on a Banach manifold M is an integrable tensor structure if and only if the curvature of the Levi-Civita connection of g vanishes.*

4.3 Integrability of almost tangent, para-complex and decomposable complex structures

Definition 4.6. An almost tangent structure J (resp. para-complex structure $\mathcal{J}$, resp. decomposable complex) structure on a Banach manifold M is an almost tangent (resp. para-complex structure $\mathcal{J}$, resp. decomposable complex) structure structure on the tangent bundle TM of M.

If J is an almost tangent structure, we have a decomposition $TM = \ker J \oplus L$, such that K and L are isomorphic sub-bundles of TM. Moreover, there exists an isomorphism $J_L : L \to \ker J$ such that J can be written as a matrix of type $\begin{pmatrix} 0 & J_L \\ 0 & 0 \end{pmatrix}$.

If $\mathcal{J}$ is an almost para-complex structure, there exists a Whitney decomposition $TM = E^+ \oplus E^-$ where E^+ and E^- are the eigen-bundles associated to the eigenvalues $+1$ and -1 of $\mathcal{J}$ respectively.

Now for a classical criterion of the integrability of such structures we need the notion of Nijenhuis tensor.

Definition 4.7. If A is an endomorphism of TM, the Nijenhuis tensor of A is defined, for all local vector fields X and Y on M, by:

$$N_A(X,Y) = [AX, AY] - A[AX, Y] - A[X, AY] + A^2[X,Y].$$

Theorem 4.7. *An almost tangent (resp. para-complex) structure J (resp. $\mathcal{J}$) structure on M is integrable if and only its Nijenhuis tensor is null.*

Proof. For any almost tangent structure J, since $J^2 = 0$, we have:

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY].$$

Moreover, if X or Y is a section of $\ker J$, we always have $N_J(X,Y) = 0$. Therefore we only have to consider N_J in restriction to L. In particular, we can note that if $N_J \equiv 0$, then we have

$$[JX, JY] = J[JX, Y] + J[X, JY].$$

Therefore, when we restrict this relation to a section of L, since J_L is an isomorphism, this implies that $\ker J$ is an involutive supplemented sub-bundle of TM.

Fix some $x_0 \in M$ and consider a chart (U, ϕ) around x_0 such that $(\ker J)_{|U}$ and $K_{|U}$ are trivial. Therefore, $T\phi(TM) = \phi(U) \times \mathbb{M}$, $T\phi(\ker J) = \phi(U) \times \mathbb{K}$ and $T\phi(L) = \phi(U) \times \mathbb{L}$.

If $\mathbb{K} \subset \mathbb{M}$ and $\mathbb{L} \subset \mathbb{M}$ are the typical fibres of $\ker J$ and L respectively, we have $\mathbb{M} = \mathbb{K} \oplus \mathbb{L}$. Therefore, without loss of generality, we may assume that U is an open subset of $\mathbb{M} \equiv \mathbb{K} \times \mathbb{L}$ and $TM = U \times \mathbb{M}$, $\ker J = U \times \mathbb{K}$ and $L = U \times \mathbb{L}$. Thus $x \mapsto (J_L)_x$ is a smooth field which takes values in the set $\mathrm{Iso}(\mathbb{L}, \mathbb{K})$ of isomorphisms from $\mathbb{L}$ to $\mathbb{K}$ and $x \mapsto J_x$ is a smooth field from M to $L(\mathbb{M})$ such that $\ker J_x = \mathbb{K}$.

As in [Bel1], Proposition 2.1, we obtain in our context:

Lemma 4.8. *With the previous notations, we have*

$$N_J(X,Y) = J'(Y, JX) - J'(JX, Y) + J'(X, JY) - J'(X, JY)$$

where J' stands for the differential of J as a map from M to $L(\mathbb{M})$.

At first, assume that J is an integrable almost tangent structure. This means that, for any $x \in M$, the value J_x in $L(\mathbb{M})$ (resp. $(J_L)_x$ in $\mathrm{Iso}(\mathbb{L}, \mathbb{K})$) is a constant. So from Lemma 4.8. it follows that $N_J = 0$.

Conversely, assume that $N_J = 0$ and as we have already seen $\ker J$ is involutive, if X and Y are sections of L, in the expression of N_J given in Lemma 4.8, we can replace J by J_L and this implies that J_L satisfies the assumption of Theorem 1.2 in [Lan]. By the same arguments as in the proof of the Frobenius Theorem in [Lan], we produce a local diffeomorphism Ψ from a neighbourhood $U_0 \times V_0 \in \mathbb{K} \times \mathbb{L}$ of x_0 such that $\Psi^* J_L$ is a smooth field from $U_0 \times V_0$ to $\mathrm{Iso}(\mathbb{K}, \mathbb{L})$ which is constant. Thus the same is true for J, which ends the proof in this case.

Now, let us consider a para-complex structure $\mathcal{J}$. Recall that we have a canonical Whitney decomposition $TM = E^+ \oplus E^-$ where E^+ (resp. E^-) is the eigen-bundle associated to the eigenvalue $+1$ (resp. -1) of $\mathcal{J}$. Note that if $P^{\pm} = \frac{1}{2}(\mathrm{Id} \pm \mathcal{J})$, then $E^{\pm} = \mathrm{im}\, P^{\pm}$, $\ker P^{\pm} = E^{\mp}$ and $TM = E^+ \oplus E^-$. Now, we have $N_{\mathcal{J}} = 0$ if and only if $N_{P^+} = N_{P^-} = 0$ and E^+ and E^- are integrable sub-bundles of TM. Note that since

$\mathcal{J}^2 = Id$, Lemma 4.8 is also valid for $\mathcal{J}$; So if $\mathcal{J}$ is integrable, this implies that $N_{\mathcal{J}} = 0$.

Conversely, assume that $N_{\mathcal{J}} = 0$. Fix some $x_0 \in M$ and denote by $\mathbb{E}^{\pm}$ the typical fibre of $E^{\pm}$. If the sub-bundles $E^{\pm}$ are integrable, from Frobenius Theorem, there exists a chart $(U^{\pm}, \phi_{\pm})$ around x_0 where $\phi_{\pm}$ is a diffeomorphism from $U^{\pm}$ onto an open neighbourhood $V_1^{\pm} \times V_1^{\pm}$ in $\mathbb{E}^+ \times \mathbb{E}^-$ such that $\phi_+^* P^+$ and $\phi_-^* P^-$ are the fields of constant matrices $\begin{pmatrix} 0 & \mathrm{Id}_{\mathbb{E}^+} \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & \mathrm{Id}_{\mathbb{E}^-} \end{pmatrix}$. Now the transition map $\phi_- \circ \phi_+^{-1}$ is necessarily of type $(\bar{x}, \bar{y}) \mapsto (\alpha(\bar{x}), \beta(\bar{y}))$. So the restriction of ϕ_+ to $U^+ \cap U^-$ is such that $\phi_+^* P^+$ and $\phi_+^* P^-$ are also the previous matrices, which ends the proof since $\mathcal{J} = P^+ + P^-$.

□

Unfortunately, the problem of integrability of an almost complex structure $\mathcal{I}$ on a Banach manifold M is not equivalent to the relation $N_{\mathcal{I}} \equiv 0$. The reader can find in [Pat] an example of an almost complex structure $\mathcal{I}$ on a smooth Banach manifold M such that $N_{\mathcal{I}} \equiv 0$ for which there exists *no holomorphic chart* in which $\mathcal{I}$ is isomorphic to a linear complex structure. However, if M is analytic and if $N_{\mathcal{I}} \equiv 0$ there exists a holomorphic chart in which $\mathcal{I}$ is isomorphic to a linear complex structure (Banach version of Newlander-Nirenberg theorem see [Bel1]). In particular, there exists a structure of holomorphic manifold on M.

Definition 4.8. An almost complex structure $\mathcal{I}$ on a Banach manifold M is called formally integrable if $N_{\mathcal{I}} \equiv 0$.
$\mathcal{I}$ is called integrable if there exists a structure of complex manifold on M.

4.4 Flat Kähler and para-Kähler Banach manifolds

Following [Tum1], we introduce the following notions.

Definition 4.9. A Kähler (resp. formal-Kähler) Banach structure on a Banach manifold M is an almost Kähler structure $(\Omega, g, \mathcal{I})$ on TM such that the almost complex structure $\mathcal{I}$ is integrable (resp. formally integrable).

Note that if M is analytic, $(\Omega, g, \mathcal{I})$ is an almost Kähler structure on TM if and only if $N_{\mathcal{I}} \equiv 0$, then we have a Kähler structure on M. The reader will find many examples of weak and strong formal-Kähler Banach manifolds in [Tum2].

In the same way, a *weak (resp. strong) para-Kähler Banach structure* on a Banach manifold M is an almost para-Kähler structure $(\Omega, g, \mathcal{J})$ on TM (cf. Definition 3.9) *such that the almost para-complex structure $\mathcal{J}$ is integrable.*

In what follows, we use the terminology Kähler or para-Kähler manifold instead of weak Kähler or weak para-Kähler manifold.

According to § 3.4, if we have a Kähler or para-Kähler manifold structure on M, we have a decomposition $TM = E_1 \oplus E_2$ where E_1 and E_2 are isomorphic, Lagrangian and orthogonal sub-bundles of TM.

Definition 4.10. A formal-Kähler (resp. para-Kähler) Banach structure $(\Omega, g, \mathcal{I})$ (resp. $(\Omega, g, \mathcal{J})$) is said to be flat if the Levi-Civita connection ∇ of g exists and is flat.

Lemma 4.9. *Let $(\omega, g, \mathcal{I})$ (resp. $(\Omega, g, \mathcal{J})$) be a Kähler (resp. para-Kähler) Banach structure such that the Levi-Civita connection ∇ of g exists. Then we have $\nabla\mathcal{I} \equiv 0$ (resp. $\nabla\mathcal{J} \equiv 0$)*

Proof. cf. proof of [Tum2], Proposition 91. □

Theorem 4.10. *If a formal-Kähler (resp. para-Kähler) Banach structure $(\Omega, g, \mathcal{I})$ (resp. $(\Omega, g, \mathcal{J})$) on M is flat, for each $x_0 \in M$, there exists a chart (U, ϕ) and a linear Darboux form Ω_0, a linear decomposable complex structure $\mathcal{I}_0$ and an inner product g_0 (resp. a linear Darboux form Ω_0, a linear para-complex structure $\mathcal{J}_0$ and a neutral inner product g_0) on $\mathbb{M}$ such that $\phi^*\Omega_0 = \Omega$, $\phi^*\mathcal{I}_0 = \mathcal{I}$ and $\phi^* g_0 = g$ (resp. $\phi^*\Omega_0 = \Omega$, $\phi^*\mathcal{J}_0 = \mathcal{J}$ and $\phi^* g_0 = g$).*

Proof. We only consider the case of an almost para-Kähler manifold, the case of an almost Kähler manifold is similar.

Consider the decomposition $TM = E_1 \oplus E_2$ as recalled previously. Since $\nabla g = 0$, and E_1 and E_2 are orthogonal, the connection ∇ induces a connection ∇^i on E_i which preserves the restriction g_i of g to E_i for $i = 1, 2$ and $\nabla = \nabla^1 + \nabla^2$. Thus, for $i = 1, 2$, if X and Y are sections of E_i we have

$$[X, Y] = \nabla_X Y - \nabla_Y X = \nabla^i_X Y - \nabla^i_Y X$$

and so E_i is integrable. Moreover the parallel transport from $T_{\gamma(0)}M$ into $T_{\gamma(1)}M$ along a curve $\gamma : [0, 1] \longrightarrow M$ induces an isometry of the fibre $(E_i)_{\gamma(0)}$ into $(E_i)_{\gamma(1)}$ for $i = 1, 2$. Fix some $x_0 \in M$. Since ∇ is flat, according to the proof of Theorem 4.5, we may assume that $M = \mathbb{M}$ and $x_0 = 0 \in \mathbb{M}$. Moreover, if $\mathbb{E}_1$ (resp. and $\mathbb{E}_2$) is the typical fibre of the integrable sub-bundle E_1 (resp. E_2), we may assume that $\mathbb{M} = \mathbb{E}_1 \oplus \mathbb{E}_2$. Recall that under the flatness assumption, the diffeomorphism $\Psi(x) = x + Id_{\mathbb{M}}$ is such that $T_0\Psi$ is the parallel transport from $T_0\mathbb{M}$ to $T_x\mathbb{M}$ and $\Psi^0 g_0 = g$. But since E_1 and E_2 are invariant by parallel transport, and $\mathcal{J}$ is invariant by parallel transport we must have $\mathcal{J} \circ T\Psi = T\Psi \circ \mathcal{J}$ and so $\Psi^*\mathcal{J}_0 = \mathcal{J}$. As $\Omega(u, v) = g(u, \mathcal{J}v)$, we also get $\Psi^*\Omega_0 = \Omega$. □

Remark 4.11. Under the assumption of flatness of ∇ in Theorem 4.10, we obtain the integrability of $\mathcal{I}$. But as we have already seen, in general, the condition of nullity of $N_{\mathcal{I}}$ is not sufficient to ensure the integrability of $\mathcal{I}$.

5 Projective limits of tensor structures

The reference for this section is the book [DGV] where the reader can find the notion of projective limit for different categories (sets, Banach spaces, manifolds and Lie groups). Notations for projective limits of Banach vector or principal bundles and results can be found in [CaPe2].

5.1 The Fréchet topological group $\mathcal{H}^0(\mathbb{F})$

Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be Fréchet spaces. The space of continuous linear mappings $\mathcal{L}(\mathbb{F}_1,\mathbb{F}_2)$ between these spaces drops out the Fréchet category. However, using the realization of the Fréchet space $\mathbb{F}_1$ (resp. $\mathbb{F}_2$) as projective limit of Banach spaces, say $\mathbb{F}_1 = \varprojlim \mathbb{E}_1^n$ (resp. $\mathbb{F}_2 = \varprojlim \mathbb{E}_2^n$), where $\left(\overline{\mu}_{1,i}^j\right)_{j\geq i}$ (resp. $\left(\overline{\mu}_{2,i}^j\right)_{j\geq i}$) are the bonding maps, $\mathcal{L}(\mathbb{F}_1,\mathbb{F}_2)$ can be replaced by a new space within the Fréchet framework, as defined in [Gal2].

For each $n\in\mathbb{N}$, we define the set

$$\mathcal{H}^n(\mathbb{F}_1,\mathbb{F}_2) = \left\{ (f_i)_{0\leq i\leq n} \in \prod_{i=0}^n \mathcal{L}\left(\mathbb{E}_1^i,\mathbb{E}_2^i\right) : \forall j\in\{i,\dots,n\} : f_i\circ\overline{\mu}_{1,i}^j = \overline{\mu}_{2,i}^j \circ f_j \right\}.$$

$\mathcal{H}^n(\mathbb{F}_1,\mathbb{F}_2)$ is a Banach space as a closed space of the Banach space $\prod\limits_{i=0}^n \mathcal{L}\left(\mathbb{E}_1^i,\mathbb{E}_2^i\right)$.

Proposition 5.1. *The space $\mathcal{L}(\mathbb{F}_1,\mathbb{F}_2)$ can be represented as the Fréchet space*

$$\mathcal{H}(\mathbb{F}_1,\mathbb{F}_2) = \left\{ (f_n) \in \prod_{n\in\mathbb{N}} \mathcal{L}(\mathbb{E}_1^n,\mathbb{E}_2^n) : \varprojlim f_n \text{ exists} \right\}$$

isomorphic to the Fréchet space $\varprojlim \mathcal{H}^n(\mathbb{F}_1,\mathbb{F}_2)$.

Now we consider

$$\mathcal{H}_0^n(\mathbb{F}_1,\mathbb{F}_2) = \mathcal{H}^n(\mathbb{F}_1,\mathbb{F}_2) \bigcap \prod_{i=1}^n \mathcal{L}is\left(\mathbb{E}_1^i,\mathbb{E}_2^i\right).$$

For a Fréchet space $\mathbb{F}$, we denote $\mathcal{H}_0^n(\mathbb{F},\mathbb{F})$ (resp. $\mathcal{H}_0(\mathbb{F},\mathbb{F})$) by $\mathcal{H}_0^n(\mathbb{F})$ (resp. $\mathcal{H}_0(\mathbb{F})$).
We then have the following result ([DGV], Proposition 5.1.1)

Proposition 5.2. *Every $\mathcal{H}_0^n(\mathbb{F})$ is a Banach Lie group modelled on $\mathcal{H}^n(\mathbb{F})$.*
Moreover the projective limit $\varprojlim \mathcal{H}_0^n(\mathbb{F})$ exists and coincides, up to an isomorphism of topological groups, with $\mathcal{H}_0(\mathbb{F})$.
Thus $\mathcal{H}_0(\mathbb{F})$ is a Fréchet topological group.

In this situation, the maps $(h_0)_i^j : \mathcal{H}_0^j(\mathbb{F}) \longrightarrow \mathcal{H}_0^i(\mathbb{F})$ are morphisms of topological groups satisfying, for every $i\leq j\leq k$, the relations $(h_0)_i^k = (h_0)_i^j \circ (h_0)_j^k$. Thus $\left(\mathcal{H}_0^i(\mathbb{F}),(h_0)_i^j\right)$ is a projective system of Banach-Lie groups, but $\mathcal{H}_0(\mathbb{F}) = \varprojlim \mathcal{H}_0^n(\mathbb{F})$ is not necessarily a Fréchet-Lie group because the projective limit of the open sets $\mathcal{H}_0^n(\mathbb{F})$ is not necessarily open.

5.2 Some Fréchet topological subgroups of $\mathcal{H}^0(\mathbb{E})$

Let $\left(\mathbb{E}_i,\overline{\lambda_i^j}\right)_{(i,j)\in\mathbb{N}\times\mathbb{N},\ j\geq i}$ be a projective sequence of Banach spaces. Consider a sequence $(G_n)_{n\in\mathbb{N}}$ of Banach Lie groups such that, for all $n\in\mathbb{N}$, G_n is a weak

Banach Lie subgroup of $\mathrm{GL}(\mathbb{E}_n)$ and denote by $\mathcal{L}_{G_n}(\mathbb{E}_n) \subset \mathcal{L}(\mathbb{E}_n)$ the Lie algebra of G_n. We set

$$\mathcal{G}^n(\mathbb{E}) = \left\{ (g_i)_{1 \leq i \leq n} \in \prod_{j=0}^{n} \mathcal{L}_{G_j}(\mathbb{E}_j) \ : \ \forall j \in \{1, \dots, n\} \ : \ g_i \circ \overline{\lambda_i^j} = \overline{\lambda_i^j} \circ g_j \right\}.$$

Note that $\mathcal{G}^n(\mathbb{E})$ is a closed linear subspace of $\mathcal{H}^n(\mathbb{E})$ and so has a natural structure of Banach space.
For $0 \leq i \leq j$, let us consider the natural projection

$$\begin{array}{cccc} \gamma_i^j : & \mathcal{G}^j(\mathbb{E}) & \longrightarrow & \mathcal{G}^j(\mathbb{E}) \\ & (g_0, \dots, g_j) & & (g_0, \dots, g_i) \end{array}.$$

We also consider:

$$\mathcal{G}_0^n(\mathbb{E}) = \mathcal{G}^n(\mathbb{E}) \bigcap \prod_{i=1}^{n} \mathcal{L}is(\mathbb{E}_i).$$

By the same arguments used in the proof of [DGV], Proposition 5.1.1, we obtain

Proposition 5.3. *Every $\mathcal{G}_0^n(\mathbb{F})$ is a Banach Lie group modelled on $\mathcal{G}^n(\mathbb{F})$.*
Moreover the projective limit $\varprojlim \mathcal{G}_0^n(\mathbb{E})$ exists and coincides, up to an isomorphism of topological groups, with $\mathcal{G}_0(\mathbb{E})$.
Thus $\mathcal{G}_0(\mathbb{E})$ is a Fréchet topological group and a closed topological subgroup of $\mathcal{H}_0(\mathbb{E})$.

The group $\mathcal{G}_0(\mathbb{E})$ will play the rôle of structure group for the principal bundle of projective limit of G-structures (cf. § 5).

As in the previous section, the projections γ_i^j induce projections

$$(\gamma_0)_i^j : \mathcal{G}_0^j(\mathbb{F}) \longrightarrow \mathcal{G}_0^i(\mathbb{E})$$

which are morphisms of topological groups satisfying, for all $i \leq j \leq k$, the relations

$$(\gamma_0)_i^k = (\gamma_0)_i^j \circ (\gamma_0)_j^k.$$

Thus $\left(\mathcal{G}_0^i(\mathbb{E}), (\gamma_0)_i^j\right)$ is a projective system of Banach-Lie groups, but $\mathcal{G}_0(\mathbb{E}) = \varprojlim \mathcal{G}_0^n(\mathbb{E})$ is not necessarily a Fréchet-Lie group.

From now on, without ambiguity, these projections $(\gamma_0)_i^j$ will simply denoted γ_i^j for short.

5.3 Projective limits of generalized frame bundles

We extend the notion developed in [DGV], 6.5, to the framework of generalized frame bundles defined over a projective sequence of Banach manifolds.

Let us consider a projective sequence of Banach vector bundles $\mathcal{E} = (E_n, \pi_n, M_n)_{n \in \mathbb{N}}$ (cf. [CaPe2], Definition 5.15) where the manifold M_n is modelled on the Banach space $\mathbb{M}_n$ and where $\mathbb{E}_n$ is the fibre type of E_n. The projective limit of $\mathcal{E}$ is a Fréchet vector

bundle (E,π,M) (cf. [CaPe2], Theorem 5.16).
We then define the space

$$\mathbf{P}\left(E_n\right)=\bigcup_{x_n\in M_n}\mathcal{H}_0^n\left(\mathbb{E},\left(E_n\right)_{x_n}\right)$$

where $\mathcal{H}_0^n\left(\mathbb{E},\left(E_n\right)_{x_n}\right)$ is

$$\left\{(q_0,\dots,q_n)\in\prod_{i=0}^{n}\mathcal{L}is\left(\mathbb{E}_i,\left(E_i\right)_{\mu_i^n(x_n)}\right):\forall\left(i,j\right)\in\left\{0,\dots,n\right\}^2,j\geq i,\ \lambda_i^j\circ q_j=q_i\circ\overline{\lambda_i^j}\right\}.$$

We denote by $\mathbf{p}_n:\mathbf{P}\left(E_n\right)\longrightarrow M_n$ the natural projection. Then we have:

Proposition 5.4. *The quadruple* $\left(\mathbf{P}\left(E_n\right),\mathbf{p}_n,M_n,\mathcal{H}_0^n\left(\mathbb{E}\right)\right)$ *is a Banach principal bundle over* M_n *with structural group* $\mathcal{H}_0^n\left(\mathbb{E}\right)$.

Proof. Consider a Fréchet bundle atlas $\left\{\left(U^\alpha=\varprojlim U_i^\alpha,\tau^\alpha=\varprojlim\tau_i^\alpha\right)\right\}_{\alpha\in A}$ for the Fréchet bundle (E,π,M). For each $i\in\mathbb{N}$, this bundle atlas gives rise to a a bundle atlas $\left(U_i^\alpha,\Phi_i^\alpha\right)_{\alpha\in A}$ for the principal bundle $\ell\left(E_i\right)$ and the transition maps $T_i^{\alpha\beta}:U_i^\alpha\cap U_i^\beta\longrightarrow\mathrm{GL}(\mathbb{E}_i)$ are the same for both these bundles (cf. 2.2).
We consider the map

$$\begin{array}{cccc}\mathbf{\Phi}_n^\alpha: & \left(\mathbf{p}_n\right)^{-1}\left(U_n^\alpha\right) & \longrightarrow & U_n^\alpha\times\prod\limits_{i=0}^{n}\mathrm{GL}(\mathbb{E}_n)\\ & (x_n;q_0,\dots,q_n) & \mapsto & \left(x_n;(\bar{\tau}_1^\alpha)_{\mu_1^n(x_n)}\circ q_0,\dots,(\bar{\tau}_1^\alpha)_{\mu_n^n(x_n)}\circ q_n\right)\end{array}$$

where $(\tau_i^\alpha)_{x_i}=pr_2\circ(\tau_i^\alpha)_{|(E_n)_{x_i}}$ (for more details see § 2.3).

Since μ_i^j and τ_i^α are smooth maps, it follows that $\mathbf{\Phi}_n^\alpha$ is smooth. Now, according to property (PSBVB 5) and the definition of $\mathcal{H}_0^n\left(\mathbb{E},\left(E_n\right)_{x_n}\right)$, the map $\mathbf{\Phi}_n^\alpha$ takes values in $\mathcal{H}_0^n(\mathbb{E})$.

Now the map

$$(x_n;g_0,\dots g_n)\mapsto\left(x_n;\{(\bar{\tau}_1^\alpha)\}_{\mu_1^n(x_n)}^{-1}(g_0),\dots\{(\bar{\tau}_n^\alpha)\}_{\mu_n^n(x_n)}^{-1}(g_n)\right)$$

is the inverse of $\mathbf{\Phi}_n^\alpha$ and this map is also smooth. It follows that τ_n^α is a trivialization. The transition maps $\mathbf{T}_n^{\alpha\beta}$ are given by

$$\mathbf{T}_n^{\alpha\beta}(x_n)=\left(T_1^{\alpha\beta}(\mu_1^n(x_n)),\dots,T_n^{\alpha\beta}(\mu_n^n(x_n))\right).$$

Again, property (PSBVB 5) implies that, for all $j\geq i\geq 0$,

$$T_i^{\alpha\beta}(\mu_i^j(x_j);\overline{\lambda_i^j}(u))=\overline{\lambda_i^j}\circ T_j^{\alpha\beta}(x_j;u)$$

Thus $\mathbf{T}_n^{\alpha\beta}(x_n)$ takes values in $\mathcal{H}_0^n(\mathbb{E})$ which ends the proof. □

Definition 5.1. The quadruple $\left(\mathbf{P}\left(E_n\right),\mathbf{p}_n,M_n,\mathcal{H}_0^n\left(\mathbb{E}\right)\right)$ is called the generalized frame bundle of E_n and is denoted by $\ell\left(E_n\right)$.

For every $j \geq i$, we define the following projection

$$\begin{array}{cccc} \mathbf{r}_i^j : & \mathbf{P}\left(E_j\right) & \longrightarrow & \mathbf{P}\left(E_i\right) \\ & \left(q_0, \ldots, q_j\right) & \mapsto & \left(q_0, \ldots, q_i\right). \end{array}$$

Then, as for [DGV], Lemma 6.5.3, from the previous results and definitions, we obtain

Lemma 5.5. *The triple $\left(\mathbf{r}_i^j, \mu_i^j, (h_0)_i^j\right)$ is a principal bundle morphism of $\left(\mathbf{P}\left(E_j\right), \mathbf{p}_j, M_j, \mathcal{H}_0^j\left(\mathbb{E}\right)\right)$ into $\left(\mathbf{P}\left(E_j\right), \mathbf{p}_i, M_i, \mathcal{H}_0^i\left(\mathbb{E}\right)\right)$.*

According to the proof of Proposition 5.4 and Lemma 5.5 and adapting the proof of [DGV], Proposition 6.5.4, we obtain:

Theorem 5.6. *The sequence $\left(\ell\left(E_n\right)\right)_{n\in\mathbb{N}} = \left(\mathbf{P}\left(E_n\right), \mathbf{p}_n, M_n, \mathcal{H}_0^n\left(\mathbb{E}\right)\right)_{n\in\mathbb{N}}$ is a projective sequence of principal bundles. The projective limit $\underleftarrow{\lim}\mathbf{P}\left(E_n\right)$ can be endowed with a structure of smooth Fréchet principal bundle over $\underleftarrow{\lim}M_n$ whose structural group is $\mathcal{H}_0\left(\mathbb{E}\right)$.*

Remark 5.2. Adapting [DGV], Corollary 6.5.2 in the context of Theorem 5.6 and according to the proof of Proposition 5.4, we have a Fréchet principal bundle atlas $\left\{\left(U^\alpha = \underleftarrow{\lim}U_n^\alpha, \mathbf{\Phi}^\alpha = \underleftarrow{\lim}\mathbf{\Phi}_n^\alpha\right)\right\}_{\alpha\in A}$ such that, for each $n \in \mathbb{N}$, each $\ell\left(E_n\right)$ is trivializable over each U_n^α and the transition maps are

$$\mathbf{T}_n^{\alpha\beta}(x_n) = \left(T_1^{\alpha\beta}(\mu_0^n(x_n)), \ldots, T_n^{\alpha\beta}(\mu_n^n(x_n))\right)$$

where $T_j^{\alpha\beta}$ are the transition functions of the Banach bundle (E_i, π_n, M_i) for the atlas $\{(U_i^\alpha, \tau_i^\alpha)\}_{\alpha\in A}$ where $0 \leq i \leq n$.

5.4 Projective limits of G-structures

Let $\left(\mathcal{G}_0^i\left(\mathbb{E}\right), \gamma_i^j\right)_{(i,j)\in\mathbb{N}\times\mathbb{N},\ j\geq i}$ be a projective sequence of Banach Lie groups associated to a sequence $(G_n)_{n\in\mathbb{N}}$ of weak Banach Lie subgroups G_n of GL($\mathbb{E}_n$) (cf. § 5.2).

Definition 5.3. A sequence $\left(F_n, \mathbf{p}_n|F_n, M_n, \mathcal{G}_0^n\left(\mathbb{E}\right)\right)_{n\in\mathbb{N}}$ is called a projective sequence of G-reductions of a projective sequence of Banach principal bundles $\left(E_n, \mathbf{p}_n, M_n, \mathcal{H}_0^n\left(\mathbb{E}\right)\right)_{n\in\mathbb{N}}$ if, for each $n \in \mathbb{N}$, $\left(F_n, \mathbf{p}_{n|F_n}, M_n, \mathcal{G}_0^n\left(\mathbb{E}\right)\right)$ is a topological principal subbundle of $\left(\mathbf{P}\left(E_n\right), \mathbf{p}_n, M_n, \mathcal{H}_0^n\left(\mathbb{E}\right)\right)$ such that $\left(F, \pi_{|F}, M, G\right)$ has its own smooth principal structure, and the inclusion is smooth.

We then have the following result:

Theorem 5.7. *Consider a sequence $\left(\ell\left(E_n\right)\right)_{n\in\mathbb{N}} = \left(\mathbf{P}\left(E_n\right), \mathbf{p}_n, M_n, \mathcal{H}_0^n\left(\mathbb{E}\right)\right)_{n\in\mathbb{N}}$ of generalized frame bundles over a projective sequence of Banach manifolds.*
Let $\left(F_n, \mathbf{p}_{n|F_n}, M_n, \mathcal{G}_0^n\left(\mathbb{E}\right)\right)_{n\in\mathbb{N}}$ be a projective system of G-reductions of $\left(\ell\left(E_n\right)\right)_{n\in\mathbb{N}}$.

Then $\underleftarrow{\lim}\mathbf{P}_n\left(E_n\right)$ can be endowed with a structure of Fréchet principal bundle over $\underleftarrow{\lim}M_n$ whose structural group is $\mathcal{G}_0\left(\mathbb{E}\right)$.

Proof. (summarized proof) According to Remark 5.2 where we consider the atlas $\left\{ \left(U^\alpha = \varprojlim U_n^\alpha, \mathbf{\Phi}^\alpha = \varprojlim \mathbf{\Phi}_n^\alpha \right) \right\}_{\alpha \in A}$, since each transition function $T_i^{\alpha\beta}$ belongs to $\mathcal{G}_0^n(\mathbb{E})$, thus the transition maps $\mathbf{T}^{\alpha\beta} = \varprojlim \mathbf{T}_n^\alpha$ associated to the atlas $\{(U^\alpha, \mathbf{\Phi}^\alpha)\}_{\alpha \in A}$ belongs to $\varprojlim \mathcal{G}_0^n(\mathbb{E})$.
□

5.5 Projective limits of tensor structures

5.5.1 Projective limits of tensor structures of type $(1,1)$

Recall that a tensor of type $(1,1)$ on a Banach space $\mathbb{E}$ is also an endomorphism of $\mathbb{E}$.

Definition 5.4.

1. Let $\left(\mathbb{E}_i, \overline{\lambda_i^j} \right)_{(i,j) \in \mathbb{N} \times \mathbb{N},\ j \geq i}$ be a projective sequence of Banach spaces. For any $n \in \mathbb{N}$, let $A_n : \mathbb{E}_n \longrightarrow \mathbb{E}_n$ be an endomorphism of the Banach space $\mathbb{E}_n$. A sequence $(A_n)_{n \in \mathbb{N}}$ is called a projective sequence of endomorphisms, or is coherent for short, if, for any integer $j \geq i \geq 0$, it fulfils the coherence condition:
$$A_i \circ \overline{\lambda_i^j} = \overline{\lambda_i^j} \circ A_j$$
2. Let $(E_n, \pi_n, M_n)_{n \in \mathbb{N}}$ be a projective sequence of Banach vector bundles. A sequence $(\mathcal{A}_n)_{n \in \mathbb{N}}$ of endomorphisms $\mathcal{A}_n$ of E_n is called a projective sequence of endomorphisms, or a coherent sequence for short, if, for each $x = \varprojlim x_n$, the sequence $\left((\mathcal{A}_n)_{x_n} \right)_{n \in \mathbb{N}}$ is a coherent sequence of endomorphisms of $(E_n)_{x_n}$, that is
$$\forall (i,j) \in \mathbb{N}^2 : j \geq i,\ (\mathcal{A}_i)_{x_i} \circ \lambda_i^j = \lambda_i^j \circ (\mathcal{A}_j)_{x_j}.$$

We have the following properties:

Proposition 5.8. *Consider a sequence $(\mathcal{A}_n)_{n \in \mathbb{N}}$ of coherent endomorphisms $\mathcal{A}_n$ of E_n on a projective sequence of Banach vector bundles $(E_n, \pi_n, M_n)_{n \in \mathbb{N}}$. Then the projective limit $\mathcal{A} = \varprojlim (\mathcal{A}_n)$ is well defined and is a smooth endomorphism of the Fréchet bundle $E = \varprojlim E_n$.*

Proof. By definition, a coherent sequence of endomorphisms is nothing but a projective sequence of linear maps; so the projective limit $A = \varprojlim A_n$ is a well defined endomorphism of $\mathbb{E} = \varprojlim \mathbb{E}_n$. It follows that, for each $x = \varprojlim x_n \in M = \varprojlim M_n$, the projective limit $\mathcal{A}_x = \varprojlim (\mathcal{A}_n)_{x_n}$ is well defined. Now from property (PSBVB 5), there exists a local trivialization $\tau_n : \pi_n^{-1}(U_n) \longrightarrow U_n \times \mathbb{E}_n$ such that the following diagram is commutative:

$$\begin{tikzcd} (\pi_i)^{-1}(U_i) \arrow[d, "\tau_i"'] & (\pi_j)^{-1}(U_j) \arrow[l, "\lambda_i^j"'] \arrow[d, "\tau_j"] \\ U_i \times \mathbb{E}_i & U_j \times \mathbb{E}_j \arrow[l, "\mu_i^j \times \overline{\lambda_i^j}"] \end{tikzcd}$$

This implies that the restriction of $\mathcal{A}_n$ to $(\pi_n)^{-1}(U_n)$ is a projective sequence of smooth maps and so, from [DGV], Proposition 2.3.12, the restriction of $\mathcal{A}$ to $\pi^{-1}(U)$ is a smooth map where $U = \varprojlim U_n$, which ends the proof. □

Let $(E_n,\pi_n,M_n)_{n\in\mathbb{N}}$ be a projective sequence of Banach vector bundles. Consider a coherent sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ of endomorphisms $\mathcal{A}_n$ defined on the Banach bundle $\pi_n : E_n \longrightarrow M_n$.
Fix any $x^0 = \varprojlim x_n^0 \in M = \varprojlim M_n$ and identify each typical fibre of E_n with $(E_n)_{x_n^0}$. For all $n\in\mathbb{N}$, we denote by G_n the isotropy group of $(\mathcal{A}_n)_{x_n^0}$ and by $\mathcal{G}_0^n$ the associated weak Banach Lie subgroup of $\mathcal{H}_0^n$ (cf. § 5.2).

Remark 5.5. If each $\mathcal{A}_i$ is an isomorphism then so is the projective limit $\mathcal{A} = \varprojlim \mathcal{A}_n$. However the converse is not true in general. If for instance some or even all λ_i^j are not injective, then in general, some or all $\mathcal{A}_i$ could be not injective even if $\mathcal{A}$ is an isomorphism.

Now according to Theorem 5.7, we have:

Theorem 5.9. *Let $(E_n,\pi_n,M_n)_{n\in\mathbb{N}}$ be a projective sequence of Banach vector bundles. Consider a coherent sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ of endomorphisms $\mathcal{A}_n$ defined on the Banach bundle $\pi_n : E_n \longrightarrow M_n$.*
Then $\varprojlim \mathbf{P}_n(E_n)$ can be endowed with a structure of Fréchet principal bundle over $\varprojlim M_n$ whose structural group is $\mathcal{G}_0(\mathbb{E})$ if and only if there exists a Fréchet atlas bundle $\left\{\left(U^\alpha = \varprojlim U_n^\alpha, \tau^\alpha = \tau_n^\alpha\right)\right\}_{n\in\mathbb{N},\alpha\in A}$ such that $\mathcal{A}_n^\alpha = \mathcal{A}_{n|U_n^\alpha}$ is locally modelled on $(\mathcal{A}_n)_{x_n^0}$ and the transition maps $T_n^{\alpha\beta}(x_n)$ take values in the isotropy group G_n, for all $n\in\mathbb{N}$ and all $\alpha,\beta\in A$ such that $U_n^\alpha \cap U_n^\beta \neq \varnothing$.

Proof. According to Proposition 2.5, for each $n\in\mathbb{N}$, the Banach principal bundle $\ell(E_n)$ can be endowed with a structure of Banach principal bundle over M_n whose structural group is G_n if and only if there exists an atlas bundle $(U_n^\alpha,\tau_n^\alpha)_{\alpha\in A}$ such that $\mathcal{A}_n^\alpha$ is locally modelled $(\mathcal{A}_n)_{x_n^0}$ and the transition functions $T_{\alpha\beta}(x)$ belong to the isotropy group G_n for all $x_n \in U_\alpha \cap U_\beta$. Now, assume that we have Fréchet atlas $\{(U^\alpha,\tau^\alpha)\}_{\alpha\in A} = \left\{\varprojlim (U_n^\alpha,\tau_n^\alpha)\right\}_{n\in\mathbb{N},\alpha\in A}$ for which such a property is true for all $n\in\mathbb{N}$. For all $1\leq i\leq j$ we have $\tau_i^\alpha \circ \lambda_i^j = \mu_i^j \times \overline{\lambda_i^j} \circ \tau_j^\alpha$ for all $\alpha\in A$.
Thus we obtain

$$T_i^{\alpha\beta} \circ \overline{\lambda_i^j} = \overline{\lambda_i^j} \circ T_j^{\alpha\beta}$$

and so $(T_1^{\alpha\beta},\dots,T_n^{\alpha\beta})$ belongs to $\mathcal{G}_0^n(\mathbb{E})$, which implies that $\varprojlim \mathbf{P}_n(E_n)$ can be endowed with a structure of Fréchet principal bundle over $\varprojlim M_n$ whose structural group is $\mathcal{G}_0(\mathbb{E})$ according to section 5.4.
Conversely, under such an assumption, according to Theorem 5.7, consider a projective sequence $\left(F_n,\mathbf{p}_{n|F_n},M_n,\mathcal{G}_0^n(\mathbb{E})\right)_{n\in\mathbb{N}}$ of G-reductions for $(\ell(\mathbb{E}_n))_{n\in\mathbb{N}}$. From the proof of this theorem and Remark 5.2, each transition map $T_n^{\alpha\beta}$ takes values in G_n and so the the frame bundle of (E_n,π_n,M_n) can be endowed with a structure of Banach principal bundle over M_n whose structural group is G_n (cf. § 2.4), which ends the proof. $\square$

5.5.2 Projective limits of tensor structures of type $(2,0)$

This section is formally analogue to the previous one and we only give an adequate Definition and results without any proof.

Definition 5.6.

1. Let $\left(\mathbb{E}_i, \overline{\lambda_i^j}\right)_{(i,j)\in\mathbb{N}\times\mathbb{N},\ j\geq i}$ be a projective sequence of Banach spaces. For any $n\in\mathbb{N}$, let $\omega_n : \mathbb{E}_n \times \mathbb{E}_n \longrightarrow \mathbb{R}$ be a bilinear form of the Banach space $\mathbb{E}_n$.
$(\omega_n)_{n\in\mathbb{N}}$ is called a projective sequence of 2-forms if it fulfils the coherence condition
$$\forall\,(i,j)\in\mathbb{N}^2 : j\geq i,\ \omega_j = \omega_i \circ \left(\overline{\lambda_i^j}\times\overline{\lambda_i^j}\right).$$

2. Let $(E_n,\pi_n,M_n)_{n\in\mathbb{N}}$ be a projective sequence of Banach vector bundles. A sequence $(\Omega_n)_{n\in\mathbb{N}}$ of bilinear forms Ω_n on E_n is called a projective sequence of bilinear forms, or is a coherent sequence for short, if for each $x=\underleftarrow{\lim}x_n$, the sequence $\left((\Omega_n)_{x_n}\right)_{n\in\mathbb{N}}$ is a coherent sequence of bilinear forms of $(E_n)_{x_n}$ that is
$$\forall\,(i,j)\in\mathbb{N}^2 : j\geq i,\ (\Omega_j)_{x_j} = (\Omega_i)_{x_i}\circ(\lambda_i^j\times\lambda_i^j).$$

We have the following properties:

Proposition 5.10. *Consider a sequence $(\Omega_n)_{n\in\mathbb{N}}$ of coherent bilinear forms Ω_n of E_n on a projective sequence of Banach vector bundles $(E_n,\pi_n,M_n)_{n\in\mathbb{N}}$.*
Then the projective limit $\Omega=\underleftarrow{\lim}(\Omega_n)$ is well defined and is a smooth bilinear form on the Fréchet bundle $E=\underleftarrow{\lim}E_n$.

Remark 5.7. If each Ω_i is non degenerate then so is the projective limit $\Omega=\underleftarrow{\lim}\Omega_n$. However the converse is not true in general. For instance if some or each λ_i^j is not injective then in general, some or even all Ω_i could be degenerated even if Ω is non degenerate.

Let $(E_n,\pi_n,M_n)_{n\in\mathbb{N}}$ be a projective sequence of Banach vector bundles. Consider a coherent sequence $(\Omega_n)_{n\in\mathbb{N}}$ of bilinear forms Ω_n defined on the Banach bundle $\pi_n : E_n \longrightarrow M_n$.
Fix any $x^0=\underleftarrow{\lim}x_n^0\in M=\underleftarrow{\lim}M_n$ and identify each typical fibre of E_n with $(E_n)_{x_n^0}$.
For all $n\in\mathbb{N}$, we denote by G_n the isotropy group of $(\Omega_n)_{x_n^0}$ and by $\mathcal{G}_0^n$ the associated weak Banach Lie subgroup of $\mathcal{H}_0^n$ (cf. § 5.2).

Theorem 5.11. *Let $(E_n,\pi_n,M_n)_{n\in\mathbb{N}}$ be a projective sequence of Banach vector bundles. Consider a coherent sequence $(\Omega_n)_{n\in\mathbb{N}}$ of bilinear forms Ω_n defined on the Banach bundle $\pi_n : E_n\longrightarrow M_n$.*
Then $\underleftarrow{\lim}\mathbf{P}_n(E_n)$ can be endowed with a structure of Fréchet principal bundle over $\underleftarrow{\lim}M_n$ whose structural group is $\mathcal{G}_0(\mathbb{E})$ if and only if there exists a Fréchet bundle atlas $\left\{\left(U^\alpha=\underleftarrow{\lim}U_n^\alpha,\tau^\alpha=\tau_n^\alpha\right)\right\}_{n\in\mathbb{N},\alpha\in A}$ such that $\Omega_n^\alpha=\Omega_{n|U_n^\alpha}$ is locally modelled on $(\mathcal{A}_n)_{x_n^0}$ and the transition maps $T_n^{\alpha\beta}(x_n)$ take values in the isotropy group G_n, for all $n\in\mathbb{N}$ and all $(\alpha,\beta)\in A^2$ such that $U_n^\alpha\cap U_n^\beta\neq\varnothing$.

Let $(E_n, \pi_n, M_n)_{n\in\mathbb{N}}$ be a projective sequence of Banach vector bundles. Consider a coherent sequence $(\Omega_n)_{n\in\mathbb{N}}$ of bilinear forms Ω_n defined on the Banach bundle $\pi_n : E_n \longrightarrow M_n$.
Fix any $x^0 = \varprojlim x_n^0 \in M = \varprojlim M_n$ and identify each typical fibre of E_n with $(E_n)_{x_n^0}$. For all $n \in \mathbb{N}$, we denote by G_n the isotropy group of $(\Omega_n)_{x_n^0}$ and by $\mathcal{G}_0^n$ the associated weak Banach-Lie subgroup of $\mathcal{H}_0^n$.

5.6 Examples of projective limits of tensor structures

5.6.1 Projective limit of compatible weak almost Kähler and almost para-Kähler structures

We consider the context and notations of [CaPe2], § B.10.7.
If Ω (resp. g, resp. $\mathcal{I}$) is a weak symplectic form (resp. a weak Riemannian metric, resp. an almost complex structure) on a Fréchet bundle (E, π_E, M), the compatibility of any pair of the data $(\Omega, g, \mathcal{I})$ as given in Definition 3.8 can be easily adapted to this Fréchet context. We can also easily transpose the notion of "local tensor $\mathcal{T}$ locally modelled on a linear tensor $\mathbb{T}$" (cf. Definition 2.12) in the Fréchet framework. We will see that such situations can be obtained by "projective limit" on a sequence $(E_n, \pi_n, M_n)_{n\in\mathbb{N}}$ of Banach vector bundles.

Now, we introduce the following notations:

– Let $(\mathbb{T}_n)_{n\in\mathbb{N}}$ be a sequence of coherent tensors of type $(1,1)$ or $(2,0)$ on a projective sequence of Banach spaces $\left(\mathbb{E}_i, \overline{\lambda_i^j}\right)_{(i,j)\in\mathbb{N}\times\mathbb{N},\ j\geq i}$.
We denote by $G_n(\mathbb{T}_n)$ the isotropy group of $\mathbb{T}_n$. If $\mathbb{T} = \varprojlim \mathbb{T}_n$, then $\mathcal{G}_0^n(\mathbb{T})$ is the Banach Lie group associated to the sequence $(G_n(\mathbb{T}_n))_{n\in\mathbb{N}}$ (see section 5.2). We set $\mathcal{G}_0(\mathbb{T}) = \varprojlim \mathcal{G}_0^n(\mathbb{T})$.

– Given a sequence $(E_n, \pi_n, M_n)_{n\in\mathbb{N}}$ of Banach vector bundles, we provide each Banach bundle E_n with a symplectic form Ω_n and/or a weak Riemaniann metric g_n and/or an complex structure $\mathcal{I}_n$.

According to this context, let us consider the following assumption:

(H) Assume that we have a sequence of weak Riemaniann metrics $(g_n)_{n\in\mathbb{N}}$ on the sequence $(E_n)_{n\in\mathbb{N}}$. For each $x_n \in M_n$, we denote by $(\widehat{E}_n)_{x_n}$ the completion (according to g_{x_n}) of $(E_n)_{x_n}$ and we set $\widehat{E}_n = \bigcup_{x_n\in M_n} (\widehat{E}_n)_{x_n}$ and $\widehat{E} = \bigcup_{x_n\in M_n} (\widehat{E}_n)_{x_n}$.
Then $\widehat{\pi}_n : \widehat{E}_n \longrightarrow M_n$ has a Hilbert vector bundle structure and the inclusion $\iota_n : E_n \to \widehat{E}_n$ is a Banach bundle morphism.

From Theorem 3.5, we then obtain:

Theorem 5.12.

1. *Under the assumption (H), $\left(\widehat{E}_n, \widehat{\pi}_n, M_n\right)_{n\in\mathbb{N}}$ is a projective sequence of Banach bundles and there exists an injective Fréchet bundle morphism $\iota : E = \varprojlim E_n \to \widehat{E} = \varprojlim \widehat{E}_n$ with dense range.*

Let $\widehat{\Omega}_n$, (resp.$\widehat{g}_n$, resp. $\widehat{\mathcal{I}}_n$) be the extension of Ω_n (resp. g_n, resp. $\mathcal{I}_n$) to $\widehat{E}_n$ (cf. Theorem 3.5). If the sequence $(\Omega_n)_{n\in\mathbb{N}}$, (resp. $(g_n)_{n\in\mathbb{N}}$, resp. $(\mathcal{I}_n)_{n\in\mathbb{N}}$) is coherent, so is the sequence $(\widehat{\Omega}_n)_{n\in\mathbb{N}}$ (resp. $(\widehat{g}_n)_{n\in\mathbb{N}}$, resp.$(\widehat{\mathcal{I}}_n)_{n\in\mathbb{N}}$). We set $\widehat{\Omega}=\varprojlim\widehat{\Omega}_n$ (resp. $\widehat{g}=\varprojlim\widehat{g}_n$, resp. $\widehat{\mathcal{I}}=\varprojlim\widehat{\mathcal{I}}_n$). We then have $\Omega=\varprojlim\Omega_n=i^*\widehat{\Omega}$, $g=\varprojlim g_n=\iota^*\widehat{g}$ and $\mathcal{I}=\varprojlim\mathcal{I}_n=\iota^*\widehat{\mathcal{I}}$. Moreover, if any pair of the data $(\Omega_n,g_n,\mathcal{I}_n)$ are compatible for all $n\in\mathbb{N}$, then $(\widehat{\Omega},\widehat{g},\widehat{\mathcal{I}})$ and $(\Omega,g,\mathcal{I})$ are compatible structures on the Fréchet vector bundles $\widehat{E}$ and E respectively.

2. *Let $(\ell(E_n))_{n\in\mathbb{N}}=(\mathbf{P}(E_n),\mathbf{p}_n,M_n,\mathcal{H}_0^n(\mathbb{E}))_{n\in\mathbb{N}}$ be the associated sequence of generalized frame bundles. Denote by $\mathcal{T}_n$ any tensor of the triple $(\Omega_n,g_n,\mathcal{I}_n)$. Assume that $\mathcal{T}_n$ defines a $\mathbb{T}$-tensor structure on E_n for each $n\in\mathbb{N}$. Then $\varprojlim\mathbf{P}_n(E_n)$ is a Fréchet principal bundle over $\varprojlim M_n$ whose structural group is $\mathcal{G}_0(\mathbb{T})$.*
   Moreover, if the sequence $(\Omega_n,g_n,\mathcal{I}_n)_{n\in\mathbb{N}}$ is coherent and each triple is a tensor structure $(\Omega_n,g_n,\mathcal{I}_n)$ on E_n, then $(\Omega,g,\mathcal{I})$ is also a tensor structure on E which is compatible if $(\Omega_n,g_n,\mathcal{I}_n)$ fulfils this property.
3. *Under the assumption (H), all the properties of 2. are also valid for the sequence $(\widehat{\Omega}_n,\widehat{g}_n,\widehat{\mathcal{I}}_n)_{n\in\mathbb{N}}$ relatively to $\widehat{E}$.*

Proof. 1. According to our assumption (H), by density of $\iota_n(E_n)$ (identified with E_n) in $\widehat{E}_n$ and using compatible bundle atlases for $\widehat{E}$ and E (cf. proof of Theorem 3.1), for each $x_n\in M_n$ and each $n\in\mathbb{N}$, there exist local trivializations $\widehat{\tau}_n:\widehat{\pi}_n^{-1}(U_n)\longrightarrow U_n\times\widehat{\mathbb{E}}_n$ and $\tau_n:\pi_n^{-1}(U_n)\longrightarrow U_n\times\mathbb{E}_n$ around x_n such that $\widehat{\tau}_n\circ\iota_n=\tau_n$ where $\widehat{\tau}_n\circ\iota_n(\widehat{\pi}_n^{-1}(U_n))$ is dense in $U_n\times\widehat{\mathbb{E}}_n$. Now, we can choose such a trivialization so that that the property (PSBVB 5) is satisfied for the sequence $\{(E_n,\pi_n,M_n)\}_{n\in\mathbb{N}}$. But since, for each $y_n\in U_n$, the fibre $(E_n)_{y_n}$ is dense in $(\widehat{E}_n)_{y_n}$, and each fibre inclusion $(\iota_n)_{y_n}$ is bounded, the bounding map $(\bar{\lambda}_i^j)_{y_j}:(E_j)_{y_j}\longrightarrow(E_i)_{y_i}$ is closed and $(\bar{\lambda}_i^j)_{y_j}$ can be extended to a bounding map $(\widehat{\bar{\lambda}}_i^j)_{y_j}:(\widehat{E}_j)_{y_j}\longrightarrow(\widehat{E}_i)_{y_i}$. Thus (PSBVB 5) is satisfied for the sequence $\left(\widehat{E}_n,\widehat{\pi}_n,M_n\right)_{n\in\mathbb{N}}$. Thus $\left(\widehat{E}_n,\widehat{\pi}_n,M_n\right)_{n\in\mathbb{N}}$ is a projective sequence of Banach bundles and of course

$$\iota=\varprojlim\iota_n:E=\varprojlim E_n\longrightarrow\widehat{E}=\varprojlim\widehat{E}_n$$

is an injective Fréchet bundle morphism whose range is dense in $\widehat{E}$.

2 and 3. With the notations of Point 2, assume that the sequence $(\mathcal{T}_n)_{n\in\mathbb{N}}$ is coherent. Denote by $\widehat{\mathcal{T}}_n$ the extension of $\mathcal{T}_n$ to $\widehat{E}_n$ (which is well defined by Theorem 3.5). Using the same arguments as the ones used for the bonding maps, it follows that $\left(\widehat{\mathcal{T}}_n\right)_{n\in\mathbb{N}}$ is also a coherent sequence. Thus $\mathcal{T}=\varprojlim\mathcal{T}_n$ and $\widehat{\mathcal{T}}=\varprojlim\widehat{\mathcal{T}}_n$ are well defined and by construction we have $\mathcal{T}=\iota^*\widehat{\mathcal{T}}$. Moreover, assume that each $\mathcal{T}_n$ defines a $\mathbb{T}_n$-structure on E_n. Then from Theorem 3.5, $\widehat{\mathcal{T}}_n$ is a $\widehat{\mathbb{T}}$-structure on $\widehat{E}_n$. Thus from Theorem 5.9 or Theorem 5.11, each principal bundle $\mathbf{P}_n(E_n)$ (resp. $\mathbf{P}_n\left(\widehat{E}_n\right)$) is well defined and we can reduce its structural group $\mathcal{G}_0^n(\mathbb{T})$ (resp. $\mathcal{G}_0^n\left(\widehat{\mathbb{T}}\right)$). The last part of the proof is a direct consequence of the previous properties. $\square$

Corollary 5.13. *Consider an almost Kähler (resp. para-Kähler) structure* $(\Omega_n, g_n, \mathcal{I}_n)_{n\in\mathbb{N}}$ *on a projective* $(E_n, \pi_n, M_n)_{n\in\mathbb{N}}$ *of Banach vector bundles. If* $(\Omega_n, g_n, \mathcal{I}_n)_{n\in\mathbb{N}}$ *is coherent, then* $\left(\Omega = \varprojlim \Omega_n, g = \varprojlim g_n, \mathcal{I} = \varprojlim \mathcal{I}_n\right)$ *is an almost Kähler (resp. para-Kähler) structure on the Fréchet bundle* $E = \varprojlim E_n$.

5.6.2 Application to sets of smooth maps

Let M be a connected manifold of dimension m and N a connected compact manifold of dimension n. We denote by $\mathsf{C}^k(N, M)$ the set of maps $f : N \longrightarrow M$ of class C^k. It is well known that $\mathsf{C}^k(N, M)$ is a Banach manifold modelled on the Banach space $\mathsf{C}^k(N, \mathbb{R}^m)$ (cf. [Eli]). If $\lambda_i^j : \mathsf{C}^j(N, M) \longrightarrow \mathsf{C}^i(N, M)$ is the natural injection, the sequence $\left(\mathsf{C}^k(N, M)\right)_{k\in\mathbb{N}}$ is a projective sequence of Banach manifolds. By the way, the projective limit $\mathsf{C}^\infty(N, M) = \varprojlim \mathsf{C}^k(N, M)$ is provided with a Fréchet structure which is the usual Fréchet manifold structure on $\mathsf{C}^\infty(N, M)$.

Assume that M is a Kähler manifold. Let ω, $\mathfrak{g}$ and $\mathfrak{I}$ be the associated symplectic form, Riemannian metric and complex structure on M respectively.

If ν is a volume measure on N, on the tangent bundle $T\mathsf{C}^k(N, M)$ we introduce (cf. [Kum1] and [Kum2]):

– $(\Omega_k)_f(X, Y) = \displaystyle\int_N \omega\left(X_{f(t)}, Y_{f(t)}\right) d\nu(t)$;

– $(g_k)_f(X, Y) = \displaystyle\int_N \mathfrak{g}\left(X_{f(t)}, Y_{f(t)}\right) d\nu(t)$;

– $(\mathcal{I}_k)_f(X)(t) = \mathfrak{I}\left(X_{f(t)}\right)$.

It is clear that Ω_k (resp. g_k) is a weak bilinear form (resp. a weak Riemaniann metric) on $\mathsf{C}^k(N, M)$ and $\mathcal{I}_k$ is an almost complex structure on $T\mathsf{C}^k(N, M)$. Moreover, the compatibility of the triple $(\omega, \mathfrak{g}, \mathfrak{I})$ clearly implies the compatibility of $(\Omega_k, g_k, \mathcal{I}_k)$; In fact $(\Omega_k, g_k, \mathcal{I}_k)_{k\in\mathbb{N}}$ is a coherent sequence. Then, from Corollary 5.13, we get an almost Kähler structure on $\mathsf{C}^\infty(N, M)$. Note that this structure which can be directly defined on $\mathsf{C}^\infty(N, M)$ by the same formulae.

If M is a para-Kähler manifold, we can provide $\mathsf{C}^\infty(N, M)$ in the same way.

Remark 5.8. If M is a Kähler manifold, then the almost complex structure is integrable and, in particular, $N_\mathfrak{I} \equiv 0$. It is clear that that we also have $N_{\mathcal{I}_k} \equiv 0$ for all $k \in \mathbb{N}$ and $N_\mathcal{I} \equiv 0$. Thus $\mathcal{I}$ is formally integrable but not integrable in general (cf. [Kum1] in the case of sets of paths). On the opposite, if M is a para-Kähler manifold, then again the Nijenhuis tensor of the almost para-complex structure $\mathcal{J}_k$ is zero and, by Theorem 4.7, this implies that the almost para-complex structure $\mathcal{J}$ on $\mathsf{C}^\infty(N, M)$ is also integrable.

6 Direct limits of tensor structures

In order to endow the direct limit of an ascending sequence of frame bundles $(\ell(E_n))_{n\in\mathbb{N}} = (P_n(E_n), \mathbf{p}_n, M_n, G_n)_{n\in\mathbb{N}}$ with a structure of convenient principal bundle, we first consider the situation on the base $\varinjlim M_n$ where the models are supplemented Banach manifolds. In order to get a convenient structure on the direct limit $\varinjlim \ell\left(E_n\right)$ we have to replace the pathological general linear group $\mathrm{GL}\left(\mathbb{E}\right)$ where $\mathbb{E} = \varinjlim \mathbb{E}_n$ (where $\mathbb{E}_n$ is the typical fibre of the bundle E_n) by another convenient Lie group $\mathrm{G}\left(\mathbb{E}\right)$.

The main references for this section are the papers [CaPe1], [CaPe2] and [Pel2] where the reader can find the notion of direct limit for different categories and notations used through this section (see also [Bou], [Dah], [Glo], [Nee] and [RoRo]).

6.1 The Fréchet topological group $\mathrm{G}(\mathbb{E})$

Let $(\mathbb{E}_n, \iota_n^{n+1})_{n\in\mathbb{N}}$ be an ascending sequence of supplemented Banach spaces where $\mathbb{E} = \varinjlim \mathbb{E}_n$ can be endowed with a structure of convenient vector space. The group $\mathrm{GL}\left(\mathbb{E}\right)$ does not admit any reasonable topological structure. So we replace it by the convenient Lie group $\mathrm{G}(\mathbb{E})$ defined as follows.

Let $\mathbb{E}_0 \subset \mathbb{E}_1 \subset \cdots$ be the direct sequence of supplemented Banach spaces; so there exist Banach subspaces $\mathbb{E}'_0, \mathbb{E}'_1, \dots$ such that:

$$\left\{ \begin{array}{c} \mathbb{E}_0 = \mathbb{E}'_0, \\ \forall i \in \mathbb{N}, \mathbb{E}_{i+1} \backsimeq \mathbb{E}_i \times \mathbb{E}'_{i+1} \end{array} \right.$$

For $(i,j) \in \mathbb{N}^2,\ i \leq j$, we have the injection

$$\begin{array}{cccc} \iota_i^j: & \mathbb{E}_i \backsimeq \mathbb{E}'_0 \times \cdots \times \mathbb{E}'_i & \to & \mathbb{E}_j \backsimeq \mathbb{E}'_0 \times \cdots \times \mathbb{E}'_j \\ & (x'_0, \dots, x'_i) & \mapsto & (x'_0, \dots, x'_i, 0, \dots, 0) \end{array}$$

Any $A_{n+1} \in GL\left(\mathbb{E}_{n+1}\right)$ is represented by $\begin{pmatrix} A_n & B_{n+1} \\ A'_n & B'_{n+1} \end{pmatrix}$ where

$A_n \in \mathcal{L}\left(\mathbb{E}_n, \mathbb{E}_n\right),\ A'_n \in \mathcal{L}\left(\mathbb{E}_n, \mathbb{E}'_{n+1}\right),\ B_{n+1} \in \mathcal{L}\left(\mathbb{E}'_{n+1}, \mathbb{E}_n\right)$ and $B'_{n+1} \in \mathcal{L}\left(\mathbb{E}'_{n+1}, \mathbb{E}'_{n+1}\right)$.

The group

$$GL_0\left(\mathbb{E}_{n+1} | \mathbb{E}_n\right) = \left\{A \in GL\left(\mathbb{E}_{n+1}\right) : A\left(\mathbb{E}_n\right) = \mathbb{E}_n\right\}$$

can be identified with the Banach-Lie sub-group of operators of type $\begin{pmatrix} A_n & B_{n+1} \\ 0 & B'_{n+1} \end{pmatrix}$ (cf. [ChSt]). The set

$$G_n = \left\{A_n \in GL(\mathbb{E}_n) : \forall k \in \left\{0, \dots, n-1\right\}, A_n(\mathbb{E}_k) = \mathbb{E}_k\right\}$$

can be endowed with a structure of Banach-Lie subgroup.
An element A_n of G_n can be seen as

$$A_n = \begin{pmatrix} A_0 & B_1 & B_2 & B_3 & & B_i & & B_n \\ 0 & B'_1 & & & & & & \\ \multicolumn{2}{c}{0} & B'_2 & & & & & \\ \multicolumn{3}{c}{0} & B'_3 & & & & \\ & & & & \ddots & & & \\ \multicolumn{5}{c}{0} & B'_i & & \\ & & & & & & \ddots & \\ \multicolumn{7}{c}{0} & B'_n \end{pmatrix}$$

For $0 \leq i \leq j \leq k$, we consider the following diagram

$$\begin{array}{ccc}
\mathbb{E}_k & \underrightarrow{A_k} & \mathbb{E}_k \\
\iota_j^k \uparrow & & \downarrow P_j^k \\
\mathbb{E}_j & \underrightarrow{A_j} & \mathbb{E}_j \\
\iota_i^j \uparrow & & \downarrow P_i^j \\
\mathbb{E}_i & \underrightarrow{A_i} & \mathbb{E}_i
\end{array}$$

where $P_i^j : \mathbb{E}_j \to \mathbb{E}_i$ is the projection along the direction $\mathbb{E}'_{i+1} \oplus \cdots \oplus \mathbb{E}'_j$. The map

$$\begin{array}{cccc}
\theta_i^j : & G_j & \to & G_i \\
& A_j & \mapsto & P_i^j \circ A_j \circ \iota_i^j
\end{array}$$

is perfectly defined and we have

$$\left(\theta_i^j \circ \theta_j^k\right)(A_k) = \theta_i^j \left[\theta_j^k (A_k)\right] = \theta_i^j \left(P_j^k \circ A_j \circ \iota_j^k\right) = P_i^j \circ P_j^k \circ A_j \circ \iota_j^k \circ \iota_i^j.$$

Because $P_i^j \circ P_j^k = P_i^k$ (projective sequence) and $\iota_j^k \circ \iota_i^j = \iota_i^k$ (inductive sequence), we have

$$\left(\theta_i^j \circ \theta_j^k\right)(A_k) = P_i^k \circ A_j \circ \iota_i^k = \theta_i^k (A_k)$$

So $\left(G_i, \theta_i^j\right)_{j \geq i}$ is a projective sequence of Banach-Lie groups and the projective limit $\mathrm{G}(\mathbb{E}) = \varprojlim G_i$ can be endowed with a structure of Fréchet topological group.

6.2 Convenient Lie subgroups of $\mathrm{G}(\mathbb{E})$

Let $(\mathrm{H}_n)_{n \in \mathbb{N}}$ be a sequence where each H_n is a weak Banach-Lie subgroup of $\mathrm{G}(E_n)$ where

$$\forall (i,j) \in \mathbb{N}^2 : i \leq j,\ \theta_i^j (\mathrm{H}_j) = \mathrm{H}_i$$

Then $\left(\mathrm{H}_i, \theta_i^j\right)_{(i,j) \in \mathbb{N} \times \mathbb{N},\ j \geq i}$ is a projective sequence and the space $\mathrm{H}(\mathbb{E}) = \varprojlim \mathrm{H}_n$ can be endowed with a structure of Fréchet group.

6.3 Direct limits of frame bundles

Definition 6.1. A sequence $(\ell(E_n))_{n \in \mathbb{N}} = (P_n(E_n), \mathbf{p}_n, M_n, G_n)_{n \in \mathbb{N}}$ of tangent frame bundles is called an ascending sequence of tangent frame bundles if the following assumptions are satisfied:

(ASTFB 1) $\mathcal{M} = \left(M_n, \varepsilon_n^{n+1}\right)_{n \in \mathbb{N}}$ is an ascending sequence of Banach manifolds;

(ASTFB 2) For any $n \in \mathbb{N}$, $\left(\lambda_n^{n+1}, \varepsilon_n^{n+1}, \gamma_n^{n+1}\right)$ is a morphism of principal bundles from $(P_n(E_n), \mathbf{p}_n, M_n, G_n)$ to $(P_{n+1}(E_{n+1}), \mathbf{p}_{n+1}, M_{n+1}, G_n)$ where λ_n^{n+1} is defined *via* local sections s_α^n of $P_n(E_n)$ and fulfils the condition

$$\lambda_n^{n+1} \circ s_\alpha^n = s_\alpha^{n+1} \circ \varepsilon_n^{n+1};$$

(ASTFB 3) Any $x \in M = \varinjlim M_n$ has the direct limit chart property for $(U = \varinjlim U_n, \phi = \varinjlim \phi_n)$;

(ASTFB 4) For any $x \in M = \varinjlim M_n$, there exists a trivialization $\Psi_n : (\pi_n)^{-1}(U_n) \longrightarrow U_n \times \mathbb{E}_n$ such that the following diagram is commutative:

$$\begin{array}{ccc} (\pi_n)^{-1}(U_n) & \xrightarrow{\lambda_n^{n+1}} & (\pi_{n+1})^{-1}(U_{n+1}) \\ \Psi_n \downarrow & & \downarrow \Psi_{n+1} \\ U_n \times \mathbb{E}_n & \xrightarrow{\left(\varepsilon_n^{n+1} \times \iota_n^{n+1}\right)} & U_{n+1} \times \mathbb{E}_{n+1}. \end{array}$$

for each $n \in \mathbb{N}$.

Theorem 6.1. *If* $(\ell(E))_{n\in\mathbb{N}} = (P_n(E), \mathbf{p}_n, M_n, G_n)_{n\in\mathbb{N}}$ *is an ascending sequence of frame bundles, then the direct limit*

$$\left(\varinjlim P_n(E), \varinjlim \mathbf{p}_n, \varinjlim M_n, \mathrm{G}(\mathbb{E})\right)$$

can be endowed with a structure of convenient principal bundle.

Proof. According to (ASTFB 1) and (ASTFB 3), $M = \varinjlim M_n$ can be endowed with a structure of non necessary Hausdorff convenient manifold (cf. [CaPe1], § Direct limit of Banach manifolds). From (ASTFB 2), we deduce $\pi_{n+1} \circ \lambda_n^{n+1} = \varepsilon_n^{n+1} \circ \pi_n$. So the projection $\pi = \varinjlim \pi_n : \varinjlim P_n(E_n) \longrightarrow \varinjlim M_n$ is well defined. Let u be an element of $P = \varinjlim P_n(E_n)$; so u belongs to some $P_n(E_n)$. In particular, $x = \pi(u) \in M_n$ where $\pi = \varinjlim \pi_n$. According to (ASTFB 3) and (ASTFB 4), there exists a local chart (U, ϕ) of the convenient manifold M whose domain contains x. Moreover, there exists a local trivialization $\Psi_n : \pi_n^{-1}(U_n) \longrightarrow U_n \times G_n$ of $P_n(E_n)$.
For $i \geq n$, one can define the local trivilization $\Psi_i : \pi_i^{-1}(U_i) \longrightarrow U_i \times G_i$ of $P_i(E_i)$. According to (ASTFB 2), the map $\Psi : \pi^{-1}(U) \longrightarrow U \times$ G$(\mathbb{E})$ can be defined and it is one to one.
Let us study the transition functions. Using the commutativity of the diagram

$$\begin{array}{ccc} \left(U_i^\alpha \cap U_i^\beta\right) \times \mathrm{G}_i & \xrightarrow{\varepsilon_i^j \times J_i^j} & \left(U_j^\alpha \cap U_j^\beta\right) \times \mathrm{G}_j \\ \Psi_i^\alpha \circ \left(\Psi_i^\beta\right)^{-1} \downarrow & & \downarrow \Psi_j^\alpha \circ \left(\Psi_j^\beta\right)^{-1} \\ \left(U_i^\alpha \cap U_i^\beta\right) \times \mathrm{G}_i & \xrightarrow{\varepsilon_i^j \times J_i^j} & \left(U_j^\alpha \cap U_j^\beta\right) \times \mathrm{G}_j \end{array}$$

we obtain transition functions $\Psi^\alpha \circ \left(\Psi^\beta\right)^{-1}$which are diffeomorphisms of $\left(U^\alpha \cap U^\beta\right) \times \mathrm{G}(\mathbb{E})$.
We then have a set of local trivializations $\{(u^\alpha, \Psi^\alpha)\}_{\alpha\in A}$ of P where the projection $\pi : P \longrightarrow M$ is smooth.
This structure is endowed with a right action well defined according to (ASTFB 2). □

6.4 Direct limit of G-structures

Considering the reductions of the frame bundles, we then get the following result whose proof is rather analogous to the previous one: We have to check the different compatibility conditions and replace the structure group G$(\mathbb{E})$ by $H(\mathbb{E})$.

Theorem 6.2. *Let* $(\ell(E_n))_{n\in\mathbb{N}} = (P_n(E_n), \mathbf{p}_n, M_n, G_n)_{n\in\mathbb{N}}$ *be an ascending sequence of frame bundles.*
Let $\left(F_n, \mathbf{p}_{n|F_n}, M_n, H_n\right)_{n\in\mathbb{N}}$ *be a sequence of associated G-structures where, for any* $n \in \mathbb{N}$, $\left(\left(\lambda_n^{n+1}\right)_{|F_n}, \varepsilon_n^{n+1}, \theta_n^{n+1}\right)$ *is a morphism of principal bundles.*
Then $\left(\underrightarrow{\lim} F_n, \underrightarrow{\lim}\mathbf{p}_{n|F_n}, \underrightarrow{\lim} M_n, H(\mathbb{E})\right)$ *can be endowed with a convenient G-structure.*

6.5 Direct limits of tensor structures

6.5.1 Direct limits of tensor structures of type $(1,1)$

In a similar way to the projective system of tensor of type $(1,1)$, we also introduce the following definition and results.

Definition 6.2.

1. Let $\left(\mathbb{E}_i, \overline{\lambda_i^j}\right)_{(i,j)\in\mathbb{N}\times\mathbb{N},\ i\leq j}$ be an ascending sequence of supplemented Banach spaces.
   For any $n \in \mathbb{N}$, let $A_n : \mathbb{E}_n \to \mathbb{E}_n$ be an endomorphism of the Banach space $\mathbb{E}_n$. $(A_n)_{n\in\mathbb{N}}$ is called an ascending sequence of endomorphisms if it fulfils the coherence conditions
   $$\forall (i,j) \in \mathbb{N}^2 : i \leq j,\ \overline{\lambda_i^j} \circ A_i = A_j \circ \overline{\lambda_i^j}.$$
2. Let $(E_n, \pi_n, M_n)_{n\in\mathbb{N}}$ be an ascending sequence of supplemented Banach vector bundles. A sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ of endomorphisms $\mathcal{A}_n$ of E_n is called a coherent sequence of endomorphisms if, for each $x = \underrightarrow{\lim} x_n$, the sequence $\left((\mathcal{A}_n)_{x_n}\right)_{n\in\mathbb{N}}$ is a coherent sequence of endomorphisms of $\left((E_n)_{x_n}\right)_{n\in\mathbb{N}}$, that is:
   $$\forall (i,j) \in \mathbb{N}^2 : i \leq j,\ \lambda_i^j \circ (\mathcal{A}_i)_{x_i} = (\mathcal{A}_j)_{x_j} \circ \lambda_i^j.$$

We have the following properties:

Proposition 6.3. *1. Consider a sequence* $(A_n)_{n\in\mathbb{N}}$ *of coherent endomorphisms* A_n *of* E_n *on an ascending sequence of supplemented Banach spaces* $(\mathbb{E}_n)_{n\in\mathbb{N}}$. *If* $G(A_n)$ *is the isotropy group of* A_n, *then* $G(A_n)$ *is a convenient subgroup of* $\mathrm{G}(\mathbb{E})$.

2. Consider a sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ *of coherent endomorphisms* $\mathcal{A}_n$ *of* E_n *on an ascending sequence of supplemented Banach vector bundles* $(E_n, \pi_n, M_n)_{n\in\mathbb{N}}$. *Then the direct limit* $\mathcal{A} = \underleftarrow{\lim}(\mathcal{A}_n)$ *is a well defined and is a smooth endomorphism of the Fréchet bundle* $E = \underleftarrow{\lim} E_n$.

Proof.

1. Under our assumption, we have $\mathbb{E}_0 \subset \mathbb{E}_1 \subset \cdots$ and Banach spaces $\mathbb{E}'_0, \mathbb{E}'_1, \ldots$ such that:
$$\begin{cases} \mathbb{E}_0 = \mathbb{E}'_0, \\ \forall i \in \mathbb{N}, \mathbb{E}_{i+1} \backsimeq \mathbb{E}_i \times \mathbb{E}'_{i+1} \end{cases}$$

Of course $G(A_0) \subset \mathrm{GL}(\mathbb{E}) = \mathrm{G}(\mathbb{E})$. Assume that, for all $0 \leq i \leq n$, we have $G(A_i) \subset \mathrm{G}(\mathbb{E}_i)$. Since $\mathbb{E}_{n+1} \backsimeq \mathbb{E}_n \times \mathbb{E}'_{n+1}$ and the sequence $(A_n)_{n\in\mathbb{N}}$ is coherent, if T_{n+1} belongs to $G(A_{n+1})$, it follows that $T_{n+1} \circ A_n = T_{n+1} \circ A_n$. Thus if T_n is the restriction of T_{n+1} to E_n, then T_n belongs to $\mathrm{GL}(\mathbb{E}_n)$ and so T_n can be written as a matrix of type

$$\begin{pmatrix} T_n & T'_n \\ 0 & S_{n+1} \end{pmatrix}$$

Thus T_{n+1} belongs to $\mathrm{G}(\mathbb{E}_{n+1})$, which ends the proof of Point 1.

2. By definition, a coherent sequence of endomorphisms is nothing but an ascending system of linear maps, so the direct limit $A = \varinjlim A_n$ is a well defined endomorphism of $\mathbb{E} = \varinjlim \mathbb{E}_n$. It follows that, for each $x = \varinjlim x_n \in M = \varinjlim M_n$, the direct limit $\mathcal{A}_x = \varinjlim (\mathcal{A}_n)_{x_n}$ is well defined. Now, from property (ASBVB 5), for each $n \in \mathbb{N}$, there exists a trivialization $\Psi_n : (\pi_n)^{-1}(U_n) \longrightarrow U_n \times \mathbb{E}_n$ such that, for any $i \leq j$, the following diagram is commutative:

$$\begin{array}{ccc} (\pi_i)^{-1}(U_i) & \overrightarrow{\lambda_i^j} & (\pi_j)^{-1}(U_j) \\ \Psi_i \downarrow & & \downarrow \Psi_j \\ U_i \times \mathbb{E}_i & \xrightarrow{\varepsilon_i^j \times \iota_i^j} & U_j \times \mathbb{E}_j. \end{array}$$

This implies that the restriction of $\mathcal{A}_n$ to $(\pi_n)^{-1}(U_n)$ is an ascending system of smooth maps and so from [CaPe1], Lemma 3.9, the restriction of $\mathcal{A}$ to $\pi^{-1}(U)$ is a smooth map where $U = \varinjlim U_n$ which ends the proof. □

Let $(E_n, \pi_n, M_n)_{n\in\mathbb{N}}$ be an ascending sequence of supplemented Banach vector bundles. Consider a coherent sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ of endomorphisms $\mathcal{A}_n$ defined on the Banach bundle $\pi_n : E_n \longrightarrow M_n$.
Fix any $x^0 = \varinjlim x_n^0 \in M = \varinjlim M_n$ and identify each typical fibre of E_n with $(E_n)_{x_n^0}$.
For all $n \in \mathbb{N}$, we denote by $G((\mathcal{A}_n)_{x_n^0})$ the isotropy group of $(\mathcal{A}_n)_{x_n^0}$.
Now, by the same type of arguments as the ones used in the proof of Theorem 5.9 but replacing "projective sequence" by "ascending sequence" and "Fréchet atlas" by "convenient atlas", we obtain:

Theorem 6.4. *Let $(E_n, \pi_n, M_n)_{n\in\mathbb{N}}$ be an ascending sequence of supplemented Banach vector bundles. Consider a coherent sequence $(\mathcal{A}_n)_{n\in\mathbb{N}}$ of endomorphisms $\mathcal{A}_n$ defined on the Banach bundle $\pi_n : E_n \longrightarrow M_n$.*
Then $\varinjlim \mathbf{P}_n(E_n)$ can be endowed with a structure of convenient principal bundle over $\varinjlim M_n$ whose structural group is $G(\mathcal{A}_{x^0}) = \varinjlim G(\mathcal{A}_n)_{x_n^0}$ if and only if there exists a convenient atlas bundle $\left\{ \left(U^\alpha = \varprojlim U_n^\alpha, \tau^\alpha = \tau_n^\alpha \right) \right\}_{n\in\mathbb{N}, \alpha\in A}$ such that $\mathcal{A}_n^\alpha = \mathcal{A}_{n|U_n^\alpha}$ is locally modelled on $(\mathcal{A}_n)_{x_n^0}$ and the transition maps $T_n^{\alpha\beta}(x_n)$ take values in to the isotropy group $G((\mathcal{A}_n)_{x_n^0})$, for all $n \in \mathbb{N}$ and all $\alpha, \beta \in A$ such that $U_n^\alpha \cap U_n^\beta \neq \varnothing$.

6.5.2 Direct limits of tensor structures of type $(2, 0)$

We adapt the results obtained for coherent sequences tensor of type $(1, 1)$ to such sequences of $(2, 0)$-tensors.

Definition 6.3. 1. Let $\left(\mathbb{E}_i, \overline{\lambda_i^j}\right)_{(i,j)\in\mathbb{N}\times\mathbb{N},\ i\leq j}$ be an ascending sequence of supplemented of Banach spaces.
For any $n \in \mathbb{N}$, let $\omega_n : \mathbb{E}_n \times \mathbb{E}_n \longrightarrow \mathbb{R}$ be a bilinear form of the Banach space $\mathbb{E}_n$.
$(\omega_n)_{n\in\mathbb{N}}$ is called a projective sequence of 2-forms if it fulfils the coherence condition,

$$\forall\,(i,j) \in \mathbb{N}^2 : i \leq j,\ \omega_i = \omega_j\left(\overline{\lambda_i^j} \times \overline{\lambda_i^j}\right)$$

2. Let $(E_n, \pi_n, M_n)_{n\in\mathbb{N}}$ be a projective sequence of Banach vector bundles.
A sequence $(\Omega_n)_{n\in\mathbb{N}}$ of bilinear forms Ω_n on E_n is called a coherent sequence of bilinear forms if for each $x = \underrightarrow{\lim} x_n$, the sequence $\left((\Omega_n)_{x_n}\right)_{n\in\mathbb{N}}$ is a coherent sequence of bilinear forms of $(E_n)_{x_n}$, that is

$$\forall\,(i,j) \in \mathbb{N}^2 : i \leq j,\ (\Omega_i)_{x_i} = (\Omega_j)_{x_j}(\lambda_i^j \times \lambda_i^j)$$

We have the following properties:

Proposition 6.5.

1. *Consider a sequence $(\Omega_n)_{n\in\mathbb{N}}$ of coherent bilinear forms Ω_n of E_n on an ascending sequence of supplemented Banach spaces $(\mathbb{E}_n)_{n\in\mathbb{N}}$. If $G(\Omega_n)$ is the isotropy group of Ω_n, then $G(\Omega) = \underrightarrow{\lim} G(\Omega_n)$ is a convenient subgroup of* $\mathrm{G}(\mathbb{E})$.

2. *Consider a sequence $(\Omega_n)_{n\in\mathbb{N}}$ of coherent bilinear forms Ω_n of E_n on an ascending sequence of supplemented Banach vector bundles $(E_n, \pi_n, M_n)_{n\in\mathbb{N}}$. Then the projective limit $\Omega = \underrightarrow{\lim}(\Omega_n)$ is a well defined and is a smooth bilinear form of the convenient bundle $E = \underrightarrow{\lim} E_n$.*

Let $(E_n, \pi_n, M_n)_{n\in\mathbb{N}}$ be an ascending sequence of supplemented Banach vector bundles. Consider a coherent sequence $(\Omega_n)_{n\in\mathbb{N}}$ of bilinear forms Ω_n defined on the Banach bundle $\pi_n : E_n \longrightarrow M_n$.
Fix any $x^0 = \underrightarrow{\lim} x_n^0 \in M = \underrightarrow{\lim} M_n$ and identify each typical fibre of E_n with $(E_n)_{x_n^0}$.
For all $n \in \mathbb{N}$, we denote by $G((\Omega_n)_{x_n^0})$ the isotropy group of $(\Omega_n)_{x_n^0}$.

Theorem 6.6. *Let $(E_n, \pi_n, M_n)_{n\in\mathbb{N}}$ be an ascending sequence of supplemented Banach vector bundles.
Consider a coherent sequence $(\Omega_n)_{n\in\mathbb{N}}$ of bilinear forms Ω_n defined on the Banach bundle $\pi_n : E_n \longrightarrow M_n$.
Then $\underrightarrow{\lim}\mathbf{P}_n(E_n)$ can be endowed with a structure of Fréchet principal bundle over $\underleftarrow{\lim} M_n$ whose structural group can be reduced to $G((\Omega)_{x^0}) = \underleftarrow{\lim} G((\Omega_n)_{x_n^0})$ if and only if there exists a convenient atlas bundle $\left\{\left(U^\alpha = \underleftarrow{\lim} U_n^\alpha, \tau^\alpha = \tau_n^\alpha\right)\right\}_{n\in\mathbb{N},\alpha\in A}$ such that $\Omega_n^\alpha = \mathcal{A}_{n|U_n^\alpha}$ is locally modelled $(\Omega_n)_{x_n^0}$ and the transition maps $T_n^{\alpha\beta}(x_n)$ take values in to the isotropy group $G((\Omega_n)_{x_n^0})$, for all $n \in \mathbb{N}$ and all $(\alpha,\beta) \in A^2$ such that $U_n^\alpha \cap U_n^\beta \neq \varnothing$.*

6.6 Examples of direct limits of tensor structures

6.6.1 Direct limit of compatible weak almost Kähler and almost para-Kähler structures

We consider the context and notations of [CaPe2], § B.7.
If Ω (resp. g, resp. $\mathcal{I}$) is a weak symplectic form (resp. a weak Riemannian metric, resp. an almost complex structure) on a convenient bundle (E, π_E, M), the compatibility of any pair of the data $(\Omega, g, \mathcal{I})$ as given in Definition 3.8 can be easily adapted to this convenient context. We can also easily transpose the notion of "local tensor $\mathcal{T}$ locally modelled on a linear tensor $\mathbb{T}$" (cf. Definition 2.12) in the convenient framework. We will see that such situations can be obtained by some "direct limit" of a sequence $\{(E_n, \pi_n, M_n)\}_{n\in\mathbb{N}}$ of Banach vector bundles. Once again, this context is very similar to the context of § 5.6 and we will only write down the details without any proof.

First we introduce the following corresponding notations:

– Let $(\mathbb{T}_n)_{n\in\mathbb{N}}$ be a sequence of coherent tensor of type $(1,1)$ or $(2,0)$ on an ascending sequence of supplemented Banach spaces $\left(\mathbb{E}_i, \overline{\lambda_i^j}\right)_{(i,j)\in\mathbb{N}\times\mathbb{N},\ i\leq j}$. We denote by $G(\mathbb{T}_n)$ the isotropy group of $\mathbb{T}_n$. If $\mathbb{T} = \underrightarrow{\lim}\mathbb{T}_n$, then $G(\mathbb{T}) = \underrightarrow{\lim} G(\mathbb{T}_n)$ is convenient sub-group of G($\mathbb{E}$)

– Given an ascending sequence $(E_n, \pi_n, M_n)_{n\in\mathbb{N}}$ of supplemented Banach vector bundles, we provide each Banach bundle E_n with a symplectic form Ω_n,and/or a weak Riemaniann metric g_n and/or an almost complex structure $\mathcal{I}_n$.

According to this context, let us consider the following assumption:

(H) Assume that we have a sequence of weak Riemaniann metrics $(g_n)_{n\in\mathbb{N}}$ on the sequence $(E_n)_{n\in\mathbb{N}}$. For each $x_n \in M_n$, we denote by $(\widehat{E}_n)_{x_n}$ the completion of $(E_n)_{x_n}$ and we set $\widehat{E}_n = \bigcup\limits_{x_n\in M_n} (\widehat{E}_n)_{x_n}$. Then $\widehat{\pi}_n : \widehat{E}_n \longrightarrow M_n$ has a Hilbert vector bundle structure and the inclusion $\iota_n : E_n \to \widehat{E}_n$ is a Banach bundle morphism.

From Theorem 3.5, we then obtain

Theorem 6.7.

1. *Under the assumption (H), $\left(\widehat{E}_n, \widehat{\pi}_n, M_n\right)_{n\in\mathbb{N}}$ is an ascending sequence of supplemented Banach bundles and there exists an injective convenient bundle morphism $\iota : E = \underrightarrow{\lim} E_n \to \widehat{E} = \underrightarrow{\lim}\widehat{E}_n$ with dense range.*

   Let $\widehat{\Omega}_n$, (resp. $\widehat{g}_n$, resp. $\widehat{\mathcal{I}}_n$) the extension of Ω_n (resp. g_n, resp. $\mathcal{I}_n$) to $\widehat{E}_n$ (cf. Theorem 3.5). If $(\Omega_n)_{n\in\mathbb{N}}$, (resp. $(g_n)_{n\in\mathbb{N}}$, resp. $(\mathcal{I}_n)_{n\in\mathbb{N}}$) is a coherent sequence, so is the sequence $(\widehat{\Omega}_n)_{n\in\mathbb{N}}$ (resp. $(\widehat{g}_n)_{n\in\mathbb{N}}$, resp.$(\widehat{\mathcal{I}}_n)_{n\in\mathbb{N}}$). We set $\widehat{\Omega} = \underrightarrow{\lim}\widehat{\Omega}_n$ (resp. $\widehat{g} = \underrightarrow{\lim}\widehat{g}_n$, resp. $\widehat{\mathcal{I}} = \underrightarrow{\lim}\widehat{\mathcal{I}}_n$). We then have $\Omega = \underrightarrow{\lim}\Omega_n = i^*\widehat{\Omega}$, $g = \underrightarrow{\lim} g_n = \iota^*\widehat{g}$ and $\mathcal{I} = \underrightarrow{\lim}\mathcal{I}_n = \iota^*\widehat{\mathcal{I}}$. Moreover, if any *pair of the data $(\Omega_n, g_n, \mathcal{I}_n)$ is compatible for all $n \in \mathbb{N}$, then $(\widehat{\Omega}, \widehat{g}, \widehat{\mathcal{I}})$ (resp. $(\Omega, g, \mathcal{I})$) on the Fréchet vector bundle $\widehat{E}$ (resp. E) respectively is compatible.*

2. *Let $(\ell(E_n))_{n\in\mathbb{N}} = (\mathbf{P}(E_n), \pi_n, M_n, \mathcal{H}_0^n(\mathbb{E}))_{n\in\mathbb{N}}$ be the associated sequence of generalized frame bundles. Denote by $\mathcal{T}_n$ any tensor among $(\Omega_n, g_n, \mathcal{I}_n)$. Assume that $\mathcal{T}_n$ defines a $\mathbb{T}$-tensor structure on E_n for each $n \in \mathbb{N}$. Then $\varinjlim \mathbf{P}_n(E_n)$ is a convenient principal bundle over $\varinjlim M_n$ whose structural group can be reduced to $G(\mathbb{T})$. Moreover, if the sequence $(\Omega_n, g_n, \mathcal{I}_n)_{n\in\mathbb{N}}$ is coherent and each triple is a tensor structure on E_n, then $(\Omega, g, \mathcal{I})$ is also a tensor structure on E which is compatible if $(\Omega_n, g_n, \mathcal{I}_n)$ are so.*

3. *Under the assumption (H), all the properties of Point 2 are also valid for the sequence $(\widehat{\Omega}_n, \widehat{g}_n, \widehat{\mathcal{I}}_n)_{n\in\mathbb{N}}$ relatively to $\widehat{E}$.*

Corollary 6.8. *Consider an almost Kähler (resp. para-Kähler) structure $(\Omega_n, g_n, \mathcal{I}_n)_{n\in\mathbb{N}}$ on an ascending sequence $\{(E_n, \pi_n, M_n)\}_{n\in\mathbb{N}}$ of Banach vector bundles. If $(\Omega_n, g_n, \mathcal{I}_n)_{n\in\mathbb{N}}$ is coherent, then $(\Omega = \varinjlim \Omega_n, g = \varinjlim g_n, \mathcal{I} = \varinjlim \mathcal{I}_n)$ is an almost Kähler (resp. para-Kälher) structure on the Fréchet bundle $E = \varinjlim E_n$.*

6.6.2 Application to Sobolev loop spaces

It is well known that if M is a connected manifold of dimension m, the set $\mathsf{L}_k^p(\mathbb{S}^1, M)$ of loops $\gamma : \mathbb{S}^1 \longrightarrow M$ of Sobolev class L_k^p is a Banach manifold modelled on the Sobolev space $\mathsf{L}_k^p(\mathbb{S}^1, \mathbb{R}^m)$ (see [Pel2]).

Let us consider now an ascending sequence $\{(M_n, \omega_n)\}_{n\in\mathbb{N}}$ of finite dimensional manifolds. Then the direct limit $M = \varinjlim M_n$ is modelled on the convenient space $\mathbb{R}^\infty$. According to [Pel2], Proposition 34, the direct limit $\mathsf{L}_k^p(\mathbb{S}^1, M) = \varinjlim(\mathsf{L}_k^p(\mathbb{S}^1, M_n))$ has a Hausdorff convenient manifold structure modelled on the convenient space $\mathsf{L}_k^p(\mathbb{S}^1, \mathbb{R}^\infty) = \varinjlim(\mathsf{L}_k^p(\mathbb{S}^1, \mathbb{R}^m))$.

Assume now that each M_n is a Kähler manifold and let ω_n, $\mathfrak{g}_n$ and $\mathfrak{I}_n$ be the associated symplectic form, Riemannian metric and complex structure on M_n respectively. Moreover, we suppose that each sequence $(\omega_n)_{n\in\mathbb{N}}$, $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ and $(\mathfrak{I}_n)_{n\in\mathbb{N}}$ is a coherent sequence on the ascending sequence $(M_n)_{n\in\mathbb{N}}$.

For p and k fixed, on the tangent bundle $T\mathsf{L}_k^p(\mathbb{S}^1, M_n)$, we introduce (cf. [Kum1] and [Kum2]):

- $(\Omega_n)_\gamma(X, Y) = \displaystyle\int_{\mathbb{S}^1} \omega_n\left(X_{\gamma(t)}, Y_{\gamma(t)}\right) dt;$
- $(g_n)_\gamma(X, Y) = \displaystyle\int_{\mathbb{S}^1} \mathfrak{g}_n\left(X_{\gamma(t)}, Y_{\gamma(t)}\right) d\nu(t);$
- $(\mathcal{I}_n)_\gamma(X)(t) = \mathfrak{I}_n\left(X_{\gamma(t)}\right).$

It is clear that Ω_n (resp. g_n) is a weak bilinear form (resp. a Riemaniann metric) on $\mathsf{L}_k^p(\mathbb{S}^1, M)$ and $\mathcal{I}_n$ is an almost complex structure on $T\mathsf{L}_k^p(\mathbb{S}^1, M)$. The compatibility of the triple $(\omega, \mathfrak{g}, \mathfrak{I})$ implies clearly the compatibility of any $(\Omega_n, g_n, \mathcal{I}_n)$. Moreover,

$(\Omega_n, g_n, \mathcal{I}_n)_{n\in\mathbb{N}}$ is a coherent sequence. Then, from Corollary 6.8, we get an almost Kähler structure on $\mathsf{L}^p_k(\mathbb{S}^1, M)$.

If each M_n is a para-Kähler manifold, we can provide $\mathsf{L}^p_k(\mathbb{S}^1, M)$ with an almost para-Kähler structure in the same way.

Remark 6.4. As in the projective limit context, if each M_n is a Kähler manifold, then the almost complex structure is integrable and, in particular, the Nijenhuis tensor $N_{\mathfrak{I}_n}$ vanishes. It is clear that we also have $N_{\mathcal{I}_n} \equiv 0$ for all $n \in \mathbb{N}$ and also $N_{\mathcal{I}} \equiv 0$, by classical direct limit argument; Thus $\mathcal{I}$ is formally integrable but not integrable in general. On the opposite, if each M_n is a para-Kähler manifold, then, again, the Nijenhuis tensor of the almost para-complex structure $\mathcal{J}_n$ vanishes and, by Theorem 4.7, this implies that the almost para-complex structure $\mathcal{J}$ on $\mathsf{L}^p_k(\mathbb{S}^1, M)$ is also integrable.

References

[AbMa] M. Abbati, A. Manià, *On Differential Structure for Projective Limits of Manifolds,* J. Geom. Phys. **29** 1-2 (1999) 35–63.

[ADGS] M. Aghasi, C.T. Dodson, G.N. Galanis, A. Suri, *Conjugate connections and differential equations on infinite dimensional manifolds,* J. Geom. Phys. (2008).

[AgSu] M. Aghasi, A. Suri, *Splitting theorems for the double tangent bundles of Fréchet manifolds*, Balkan Journal of Geometry and Its Applications **15** 2 (2010) 1–13.

[Bam] D. Bambusi, *On the Darboux theorem for weak symplectic manifolds*, Procceedings AMS Vol 127-11(1999) 3383–3391.

[Bel1] D. Beltiţă, *Integrability of Analytic Almost Complex Structures on Banach Manifolds*, Annals of Global Analysis and Geometry 28(1):59–73 2005.

[Bel2] D. Beltiţă, *Smooth Homogeneous Structures in Operator Theory*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 137, Chapman & Hall/CRC Press, Boca Raton-London-New York-Singapore, 2006.

[Bog] J. Bognar, *Indefinite Inner Product Spaces*, Springer-Verlag, Berlin Heidelberg New York (1974).

[Bom] J. Boman, *Differentiability of a function and of its compositions with functions of one variable*, Mathematica Scandinavica 20 (1967) 249–268.

[Bou] N. Bourbaki, *Variétés différentiables et analytiques. Fascicule de résultats*, §§1–7, Hermann, Paris, 1967.

[Cab] P. Cabau, *Strong projective limits of Banach Lie algebroids*, Portugaliae Mathematica, Volume 69, Issue 1 (2012).

[CaPe1] P. Cabau, F. Pelletier, *Integrability of direct limits of Banach manifolds*, to appear in Annales de la Faculté des Sciences de Toulouse.

[CaPe2] P. Cabau, F. Pelletier, *Projective and Direct limits of Banach G and tensor structures*, arXiv:1901.09010 [math.DG].

[Can] Ana Cannas da Silva, *Lectures on Symplectic Geometry* Springer Lecture Notes in Mathematics 1764, second printing (2008).

[Car] W. C. Carrera, *A Banach space with a countable infinite number of complex structures*, Journal of Functional Analysis Vol 267, Issue 5, 1 September 2014.

[ChMa] P. Chernoff, J. Marsden, *Properties of Infinite Dimensional Hamiltonian Systems*, Springer-Verlag Lecture Notes in Mathematics 425 (1974).

[ChSt] D. Chillingworth, P. Stefan, *Integrability of singular distributions on Banach manifolds*, Math. Proc. Cambridge Phil. Soc. 79 (1976) 117–128.

[ClGo] R. S. Clark, D. S. Goel, *Almost cotangent manifolds,* J. Differential Geom. 9 (1974) 109–122.

[CrTh] M. Crampin, G. Thompson, *Affine bundles and integrable almost tangent structures*, Math. Proc. Cambridge Philos. Soc. 98 (1985).

[Dah] R. Dahmen, *Direct limit constructions in infinite dimensional Lie theory*, Dissertation Universität Paderborn. http://nbn-resolving.de/urn:nbn:de:hbz:466:2-239.

[Die] J. Dieudonné, *Complex structures on real Banach spaces*, Proc. Amer. Math. Soc. (3) 1 (1952).

[DGV] C.T.J. Dodson, G. Galanis, E. Vassiliou, *Geometry in a Fréchet Context: A projective Limit Approach*, London Mathematical Society, Lecture Notes Series 428. Cambridge University Press, 2015.

[EgWu] M. Egeileh, T. Wurzbacher, *Infinite-Dimensional Manifolds as Ring Spaces*, Publications of the Research Institute for Mathematical Sciences, Volume 53, Issue 1, 2017, 187–209.

[Eli] H.I. Eliasson, *Geometry of manifolds of maps*, J. Differential Geometry 1 (1967) 169–194.

[Fer] V. Ferenczi, *Infinite-Dimensional Manifolds as Ring Spaces*, Advances in Mathematics 213 (2007).

[FeGa] V. Ferenczi, E. Galego, *Countable groups of isometries on Banach spaces*, Trans. Amer. Math. Soc. 362-8 (2010).

[FiTe] D. Filipović, J. Teichmann, *Existence of invariant manifolds for stochastic equation in infinite dimension*, Journal of Functional Analysis 197 (2003) 398–432.

[FrKr] A. Frölicher, A. Kriegl, *Linear Spaces and Differentiation Theory*, Pure and Applied Mathematics, J. Wiley, Chichester 1988.

[Gal1] G.N. Galanis, *Projective Limits of Banach-Lie groups*, Periodica Mathematica Hungarica **32** (1996) 179–191.

[Gal2] G.N. Galanis, *Projective Limits of Banach Vector Bundles*, Portugaliae Mathematica **55** 1 (1998) 11–24.

[Gal3] G.N. Galanis, *Differential and geometric structure for the tangent bundle of a projective limit manifold*, Rend. Sem. Mat. Univ. Padova **112** (2004) 103–115.

[Glo] H. Glöckner, *Direct limits of infinite-dimensional Lie groups compared to direct limits in related categories*, Journal of Functional Analysis 245 (2007) 19–61.

[Gra] J. Grabowski, *Derivative of the Exponential Mapping for Infinite Dimensional Lie Groups*, Annals of Global Analysis and Geometry **11** issue 3 (1993) 213–220.

[Ham] R.S. Hamilton, *The Inverse Function Theorem of Nash and Moser*, Bulletin of the American Mathematical Society **7** 1 (1982) 65–222.

[Jar] H. Jarchow, *Locally convex spaces*, Teubner, Stutgart, 1981.

[KaSw] N. J. Kalton R. C. Swanson, *A symplectic Banach space with no Lagrangian subspaces*, Transactions of the American Mathematical Society, Vol. 273, No. 1 (Sep., 1982), 385–392.

[Klo] M. Klotz, *The Automorphism Group of a Banach Principal Bundle with* $\{1\}-$*structure*, Geometriae Dedicata, October 2011, Volume 154, Issue 1, 161–182.

[Kob] S. Kobayashi, *Transformation groups in differential geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 70, Berlin: Springer, 1972.

[KrMi] A. Kriegel, P.W. Michor, *The convenient Setting of Global Analysis* (AMS Mathematical Surveys and Monographs) **53**, 1997.

[Kum1] P. Kumar, *Almost complex structure on Path space*, International Journal of Geometric Methods in Modern Physics Vol. 10, No. 3 (2013).

[Kum2] P. Kumar, *Darboux chart on projective limit of weak symplectic Banach manifold*, Int. J. Geom. Methods Mod. Phys., 12, (2015).

[Lan] S. Lang, *Fundamentals of Differential Geometry*, Springer, New York, 1999.

[Les] J.A. Leslie, *On a differential structure for the group of diffeomorphisms*, Topology **46** (1967) 263–271.

[Lib] P. Libermann, *Sur les structures presque complexes et autres structures infinitésimales régulières*, Bulletin de la S. M. F., tome 83 (1955) 195–224.

[Mar] J. E. Marsden *Lectures on Geometric Methods in Mathematical Physics*, SIAM, (1981).

[MaRa] J.E. Marsden, T.S. Ratiu, *Introduction to Mechanics and Symmetry*, Texts in Applied Mathematics 17, Springer-Verlag New-York, 1999.

[Mol] P. Molino, *Sur quelques propriétés des* G*-structures*, Journal of Differential Geometry 7 (1972) 489–518.

[Nee] K-H Neeb,*Towards a Lie theory for locally convex groups*, arXiv:1501.06269 [math.RT].

[Omo] H. Omori, *Infinite-dimensional Lie groups*, Translations of Mathematical Monographs vol 158 (American Mathematical Society) 1997.

[OnVi] A.L. Onishchik, E.B. Vinberg, *Lie Groups and Lie Algebras III: Structure of Lie Groups and Lie Algebras*, Encyclopaedia of Mathematical Sciences, vol. 41. Springer, Berlin 1994.

[Pat] I. Patyi, *On the* $\overline{\partial}$*-equation on a Banach space*, Bull. Soc. math. France, 128 (2000) 391–406.

[Pay] S. Paycha, *Basic Prerequisites in Differential Geometry end Operator Theory in View of Applications to Quantum Field Theory*, Preprint Université Blaise Pascal, Clermont, France, 2001.

[Pel1] F. Pelletier, *Integrability of weak distributions on Banach manifolds*, Indagationes Mathematicae 23 (2012) 214–242.

[Pel2] F. Pelletier, *On a Darboux theorem for symplectic forms on direct limits of ascending sequences of Banach manifolds*,IJMP https://doi.org/10.1142/S0219887818502067 (2018)

[PiTa] P. Piccione, D. V. Tausk, *The theory of connections and G-structures. Applications to affine and isometric immersions*, XIV escola de geometria diferencial – XIV school of differential geometry, Salvador, Brazil, July 17–21, 2006.

[RoRo] A. P. Robertson, W. Robertson, *Topological Vector Spaces*, Cambridge Tracts in Mathematics. 53. Cambridge University Press 1964.

[Sau] D. J. Saunders, *The Geometry of Jet Bundles*, **142**, London Mathematical Society Lectures Notes Series, Cambridge University Press, 1989.

[ThSc] G. Thompsonandu, U. Schwardmann, *Almost tangent and cotangent structure in the large*, Trans AMS vol 327 N1 (1991) 313–328.

[Tum1] B. Tumpach, *Hyperkähler structures and infinite-dimensional Grassmannians*, Journal of Functional Analysis 243 (2007) 158–206.

[Tum2] B. Tumpach, *Structures kählériennes et hyperkählériennes en dimension infinie*, Thèse, Ecole Polytechnique, 2005.

[Vai] I. Vaisman, *Lagrange geometry on tangent manifolds*, Intern. J. of Math. and Math. Sci., 2003(51) (2003) 3241–3266.

[Wei] A. Weinstein, *Symplectic Manifolds and Their Lagrangian Submanifolds*, Advances in Math. 6 (1971) 329–346.

[Wen] J. Wenzel, *Real and complex operator Ideals*, Quaestiones Mathematicae, 18 (1995).

Authors' addresses:

Patrick Cabau and Fernand Pelletier
Unité Mixte de Recherche 5127 CNRS,
Université de Savoie Mont Blanc,
Laboratoire de Mathématiques (LaMa),
Campus Scientifique,
73370 Le Bourget-du-Lac, France.
E-mail: patrickcabau@gmail.com, fernand.pelletier@univ-smb.fr