On 3-dimensional Lorentzian para-Kenmotsu manifolds

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Abstract. the object of the present paper is to study the special results on 3-dimensional Lorentzian para-Kenmotsu manifolds. In Section 1, we introduce the historical background of Kenmotsu manifolds. Next in Section 2, some rudimentary facts and related properties of Lorentzian para-Kenmotsu manifolds are discussed. In Section 3 we study of Ricci tensor of Lorentzian para-Kenmotsu manifolds. Further, in Section 4, it is shown that a 3-dimensional Lorentzian para-Kenmotsu manifold satisfying the condition $R(X,\Upsilon).S = 0$ is a special manifold. Sections 5 and 6 deal with φ-symmetry, Ricci symmetry, and in Section 7, it is shown that a 3-dimensional Lorentzian para-Kenmotsu manifold with η-parallel Ricci tensor is of the positive constant scalar curvature.

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1 Introduction

K. Kenmotsu [8] studied a contact Riemannian manifold which satisfies a special type of condition, characterized different geometric properties of the manifolds of class (3), the obtained structure was called a Kenmotsu structure. In 1972, K. Kenmotsu gave the notion of Kenmotsu manifolds [8]. Further, G. Pitis [9], De U.C. and Pathak [2], Jun, De U.C., and G. Pathak [7] and many other authors [10] provided results on Kenmotsu manifolds. Recently Dileo G. [5] and A. M. Pastore [4] studied almost Kenmotsu manifolds satisfying n-parallelism and locally symmetry, respectively. In [6] was produced a complete classification of 3-dimensional almost Kenmotsu manifolds, assuming thatξ belongs to the (K,μ)-nullity distribution. In 1977 Takahashi [11] introduced the local φ-symmetry of manifolds [1]. A Sasakian manifold is locally φ-symmetric if

$$\varphi^2((\nabla_W R)(X,\Upsilon)) = 0, \tag{1.1}$$

for each horizontal vector fields X, Υ, Z and W on M.

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U.C. De and Sarkar [3] called a Sasakian manifold as being φ-Ricci symmetric if

$$\varphi^2(\nabla_X Q)(\Upsilon) = 0 \quad \text{and} \quad S(X,\Upsilon) = g(QX,\Upsilon), \tag{1.2}$$

for each X, Υ on M.

The present paper studies 3-dimensional Lorentzian para-Kenmotsu manifolds.

2 Preliminaries

Let M^n be Lorentzian metric manifold, with an (1,1) tensor field φ. We consider a vector field ξ, a Lorentzian metric g and a 1 form η on M and assume that the structure given tensor (φ, ξ, η, g) satisfies [7]:

$$\varphi^2(X) = X + \eta(X)\xi, \tag{2.1}$$

$$g(\varphi X, \varphi\Upsilon) = g(X,\Upsilon) + \eta(X)\eta(\Upsilon) \tag{2.2}$$

$$\eta(\xi) = -1, \quad \eta(\varphi X) = 0. \tag{2.3}$$

This provides a Lorentzian almost para-contact manifold for all X, Υ on M. In Lorentzian almost para-contact 3-dimensional manifolds, we have

$$\varphi\xi = 0 \quad \eta(\varphi X) = 0, \tag{2.4}$$

$$\varphi(X,\Upsilon) = \varphi(\Upsilon, X), \quad \text{where} \varphi(X,\Upsilon) = g(X, \varphi\Upsilon) \tag{2.5}$$

The para-contact structure is called K-para-contact if ξ is a Killing vector field. In such case, we have

$$\nabla_X \xi = 0. \tag{2.6}$$

Definition 2.1. An almost Lorentzian para-contact manifold M is called Lorentzian para-Sasakian 3-dimensional manifold if

$$(\nabla_X \varphi)\Upsilon = g(X,\Upsilon)\xi + \eta(\Upsilon)X + 2\eta(X)\eta(\Upsilon). \tag{2.7}$$

Definition 2.2. A Lorentzian almost para-contact 3-dimensional manifold M is a Lorentzian para-Kenmotsu manifold if for any vector fields X, Υ on M, we have

$$(\nabla_X \varphi)\Upsilon = -g(X,\Upsilon) - \eta(\Upsilon)\varphi X, \tag{2.8}$$

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{2.9}$$

$$(\nabla_X \eta)\Upsilon = -g(X,\Upsilon) - \eta(X)\eta(\Upsilon), \tag{2.10}$$

for all vector fields X, Υ on M, where ∇ denotes covariant differentiation.

Remark. In any Lorentzian para-Kenmotsu 3-dimensional manifold M, the following relations hold:

$$g(R(X,\Upsilon)Z,\xi)=\eta(R(X,\Upsilon)Z)=g(\Upsilon,Z)\eta(X)-g(X,Z)\eta(\Upsilon) \tag{2.11}$$

$$R(\xi,X)\Upsilon=g(X,\Upsilon)\xi-\eta(\Upsilon)X \tag{2.12}$$

$$R(X,\Upsilon)\xi=\eta(\Upsilon)X-\eta(X)\Upsilon \tag{2.13}$$

$$R(\xi,X)\xi=X+\eta(X)\xi \tag{2.14}$$

$$S(X,\xi)=(n-1)\eta(X) \tag{2.15}$$

$$Q\xi=(n-1)\xi \tag{2.16}$$

$$S(\varphi X,\varphi\Upsilon)=S(X,\Upsilon)+(n-1)\eta(X)\eta(\Upsilon). \tag{2.17}$$

We denote by R and S the Riemannian curvature tensor and the Ricci tensor, respectively. In an M^3 Riemannian manifold, we have

$$R(X,\Upsilon)Z=g(\Upsilon,Z)QX-g(X,\Upsilon)Q\Upsilon+S(\Upsilon,Z)\Upsilon-\frac{r}{2}[g(\Upsilon,Z)X-g(X,Z)\Upsilon], \tag{2.18}$$

where $Q(g(QX,\Upsilon)=S(X,\Upsilon))$ and r are the Ricci Operator and the scalar curvature, respectively.

Definition 2.3. A Lorentzian para-Kenmotsu manifold is an η-Einstein manifold if its Ricci tensor S takes the form

$$S(X,\Upsilon)=ag(X,\Upsilon)+b\eta(X)\eta(\Upsilon), \tag{2.19}$$

where a and b are scalar functions on M.

3 The Ricci tensor on a 3-dimensional Lorentzian para-Kenmotsu manifold

Theorem 3.1. *Any 3-dimensional Lorentzian para-Kenmotsu manifold M which is an η-Einstein manifold, satisfies $a-b=(n-1)$.*

Proof. By replacing $Z=\xi$ in (2.19), we get

$$\begin{aligned}R(X,\Upsilon)Z&=g(\xi,Z)QX-g(X,Z)Q\xi-g(X,Z)Q\xi+S(\xi,Z)X\\&-S(X,Z)\xi-\frac{r}{2}[g(\xi,Z)X-g(X,Z)\xi].\end{aligned} \tag{3.1}$$

Now using (2.13) and (2.15) we get

$$\eta(\Upsilon)QX-\eta(X)Q\Upsilon=\left(\frac{r}{2}-1\right)[\eta(\Upsilon)X-\eta(X)\Upsilon] \tag{3.2}$$

Further replacement in (2.19), by means of (2.13) and (2.15) infer

$$QX=\frac{1}{2}[(r-2)X+(r-6)\eta(X)\eta(\Upsilon)], \tag{3.3}$$

$$S(X,\Upsilon)=\frac{1}{2}[(r-2)g(X,\Upsilon)+(r-6)\eta(X)\eta(\Upsilon)], \tag{3.4}$$

then (3.4) concludes the proof. □

Lemma 3.1. *If the scalar curvature is constant $r=6$, then the Riemannian manifold M^3 is of constant positive curvature. Proof.* Using (3.4) in (2.19) we get

$$\begin{aligned} R(X,\Upsilon)Z=&\left(\frac{r-4}{2}\right)[g(\Upsilon,Z)X-g(X,Z)\Upsilon]+\left(\frac{r-6}{2}\right)[g(\Upsilon,Z)\eta(X)\xi,\\ &-g(X,Z)\eta(\Upsilon)\xi-\eta(X)\eta(Z)\Upsilon+\eta(\Upsilon)\eta(Z)X]. \end{aligned} \tag{3.5}$$

If $r=6$, we get

$$R(X,\Upsilon)Z=g(\Upsilon,Z)X-g(X,Z)\Upsilon, \tag{3.6}$$

and Lemma 3.1 follows. □.

4 Special 3-dimensional Lorentzian para-Kenmotsu manifolds

We consider a 3-dimensional Riemannian manifold which satisfies the condition

$$R(X,\Upsilon).S=0. \tag{4.1}$$

From (4.1), we obtain

$$S(R(X,\Upsilon)U,V)+S(U,R(X,\Upsilon)V)=0. \tag{4.2}$$

Putting $X=\xi$ and using (3.1), we get

$$S(V,\xi)g(\Upsilon,U)-S(V,\Upsilon)\eta(U)-S(U,\Upsilon)\eta(V)+S(\Upsilon,\xi)g(\Upsilon,V)=0. \tag{4.3}$$

Given (2.15) and (4.3), we get

$$2g(\Upsilon,U)\eta(V)-S(\Upsilon,V)\eta(U)+2g(\Upsilon,V)\eta(U)-S(U,\Upsilon)\eta(V)=0. \tag{4.4}$$

Let $\{e_1,e_2,e_3\}$ be an orthonormal basis; then putting $\Upsilon=U=e_i$ in the above equation and taking the sum for $1\le i\le 3$, then we get

$$S(V,\xi)-8\eta(V)+r\eta(V)=0. \tag{4.5}$$

Using (2.15), we have

$$(r-6)\eta(V) = 0. \tag{4.6}$$

Since we have $\eta(V) \neq 0$, it follows $(r-6) = 0$, which gives $r = 6$, which states by Lemma 3.1 that the manifold is of constant positive curvature.

Then we can state the following result:

Theorem 4.1. *The Riemannian manifold* (M^3) *satisfying the condition* $R(X,\Upsilon).S = 0$ *is a 3-dimensional manifold of constant positive curvature 1.*

which infers

Lemma 4.1. The manifold $(M^3, \varphi, \xi, \eta, g)$ is a Lorentzian para-Kenmotsu 3-dimensional manifold of constant curvature 1.

5 Locally $\varphi-$ symmetric Lorentzian para-Kenmotsu 3-dimensional manifolds

Definition 5.1. A Lorentzian para-Kenmotsu 3-dimensional manifold is locally φ-symmetric if

$$\varphi^2(\nabla_W R)(X,\Upsilon)Z = 0, \tag{5.1}$$

for all vector fields W, X, Υ, and Z orthogonal to ξ

Takahashi introduced the notion of φ-symmetry on Sasakian manifold.

By covariant differentiation concerning W of (3.5), we get

(5.2)

$$\begin{aligned}(\nabla_W R)(X,\Upsilon)Z = {}& \frac{dr(W)}{2}[g(\Upsilon,Z)X - g(X,Z)\Upsilon] + \frac{dr(W)}{2}[g(\Upsilon,Z)\eta(X)\xi \\ & - g(X,Z)\eta(\Upsilon)\xi + \eta(\Upsilon)\eta(Z)X - \eta(X)\eta(Z)\Upsilon] + \left(\frac{r-6}{2}\right)[g(\Upsilon,Z)(\nabla_W\eta)(X)\xi \\ & + g(\Upsilon,Z)\eta(X)\nabla_W\xi - g(X,Z)(\nabla_W\eta)(\Upsilon)\xi - g(X,Z)\eta(\Upsilon)\nabla_W\xi + (\nabla_W\eta)(\Upsilon)\eta(Z)X \\ & - (\nabla_W\eta)(X)\eta(Z)\Upsilon + \eta(\Upsilon)(\nabla_W\eta)(Z)X - \eta(X)(\nabla_W\eta)(Z)\Upsilon]\end{aligned}$$

For X, Υ, Z and W orthogonal to ξ, from (2.9) and (2.10), we get from equation (5.2)

$$\begin{aligned}(\nabla_W R)(X,\Upsilon)Z = {}& \frac{dr(W)}{2}[g(\Upsilon,Z)X - g(X,Z)\Upsilon] \\ & + \left(\frac{r-6}{2}\right)[g(X,Z)g(W,\Upsilon)\xi - g(\Upsilon,Z)g(W,X)\xi]\end{aligned} \tag{5.3}$$

Then it follows that

$$\varphi^2(\nabla_W R)(X,\Upsilon)Z = \frac{dr(W)}{2}[g(\Upsilon,Z)\varphi^2 X - g(X,Z)\varphi^2\Upsilon] \tag{5.4}$$

Now taking X, Υ, Z and W orthogonal to ξ, from equation (2.1) we have

$$\varphi^2(\nabla_W R)(X,\Upsilon)Z = \frac{1}{2}dr(W)[g(\Upsilon,Z)X - g(X,Z)\Upsilon]. \tag{5.5}$$

These lead to the following

Theorem 5.1. *A 3-dimensional Lorentzian para-Kenmotsu manifold is locally φ-symmetric iff its scalar curvature is constant.*

From comparing Section 3 and Section 4, we can also state the following

Theorem 5.2. *If a 3-dimensional Lorentzian para-Kenmotsu manifold satisfies the condition $R(X,\Upsilon).S = 0$, then the manifold is locally φ-symmetric.*

6 φ-symmetric Lorentzian para-Kenmotsu 3-dimensional manifolds

Definition 6.1. A Lorentzian para-Kenmotsu 3-dimensional manifold M is said to be φ-Ricci symmetric if the Ricci operators satisfy the condition

$$\varphi^2(\nabla_X Q)(\Upsilon) = 0, \tag{6.1}$$

for all X, Υ on M, let $S(X,\Upsilon) = g(QX,\Upsilon)$.

If the manifold is φ-Ricci symmetric, then from (6.1) and (2.1),

$$(\nabla_X Q)(\Upsilon) + \eta(\nabla_X Q)(\Upsilon)\xi = 0. \tag{6.2}$$

It follows that

$$g((\nabla_X Q)(\Upsilon), Z) + \eta(\nabla_X Q)(\Upsilon)\eta(\xi) = 0. \tag{6.3}$$

Solving (6.3), we get

$$g((\nabla_X Q)(\Upsilon), Z) + S(\nabla_X \Upsilon, Z) + \eta(\nabla_X Q)(\Upsilon)\eta(Z) = 0. \tag{6.4}$$

Replacing $\Upsilon = \xi$ in (6.4), we get

$$g((\nabla_X Q)(\xi), Z) + S(\nabla_X \xi, Z) + \eta(\nabla_X Q)(\xi)\eta(Z) = 0. \tag{6.5}$$

From (2.9) and (2.13), we obtain

$$(n-1)[g(X,Z)+\eta(X)\eta(Z)] - S(X,Z) - S(\xi,Z)\eta(X) + \eta((\nabla_X Q)(\xi))\eta(Z) = 0. \tag{6.6}$$

Replacing X by φX and Z by φZ, we get

$$S(X,Z) = (n-1)g(X,Z), \tag{6.7}$$

which proves that we have an Einstein 3-dimensional manifold. We know that a symmetric Riemannian manifold is φ-Ricci symmetric.

Lemma 6.1. *Any φ-symmetric Lorentzian para-Kenmotsu manifold is an Einstein manifold.*

From result (6.7), we infer the following

Theorem 6.1. *If a 3-dimensional Lorentzian para-Kenmotsu manifold is an Einstein manifold, then it is φ-Ricci symmetric.*

Proof. From (6.2), for $n = 3$, we get

$$S(X, Z) = 2g(X, Z). \tag{6.8}$$

If $S(X, Z) = g(QX, Z)$, then $QX = 2X$ and we have

$$\varphi^2(\nabla_\Upsilon Q)(X) = 0, \tag{6.9}$$

which is the condition of φ-Ricci symmetry.□ From Theorems 4.1 and 5.1, we derive the following result

Theorem 6.2. *A 3-dimensional Lorentzian para-Kenmotsu manifold is φ-Ricci symmetric iff it is an Einstein Manifold.*

Corollary 6.1. A 3-dimensional Lorentzian para-Kenmotsu manifold is φ-Ricci symmetric if its scalar curvature r is constant.

7 Lorentzian para-Kenmotsu 3-dimensional manifold with η- parallel Ricci tensor

Definition 7.1. In a Lorentzian para-Kenmotsu 3-dimensional manifold M, the Ricci tensor S is called η-parallel if it is satisfies

$$(\nabla_X S)(\varphi X, \varphi \Upsilon) = 0, \tag{7.1}$$

for all vector fields X, Υ, and Z.

Let us consider a 3-dimensional Lorentzian para-Kenmotsu manifold with η-parallel Ricci tensor. Then from (2.3) and using (1.1) and (1.2) we get

$$S(\varphi X, \varphi \Upsilon) = \left(\frac{r-2}{2}\right)[g(X, \Upsilon) - \eta(X)\eta(\Upsilon)]. \tag{7.2}$$

By covariantly differentiating (7.2) Z, we yield

$$\begin{aligned}(\nabla_Z S)(\varphi X, \varphi \Upsilon) = \frac{dr(Z)}{2}[g(X, \Upsilon) - \eta(X)\eta(\Upsilon)]\\ - \left(\frac{r-2}{2}\right)[\eta(\Upsilon)(\nabla_Z \eta)(X) + \eta(X)(\nabla_Z \eta)(\Upsilon)] = 0\end{aligned} \tag{7.3}$$

By using (7.1) and (7.3) we get

$$dr(Z)[g(X, \Upsilon) - \eta(X)\eta(\Upsilon)] - (r-2)[\eta(\Upsilon)(\nabla_Z \eta)(X) - \eta(X)(\nabla_Z \eta)(X)] = 0. \tag{7.4}$$

Putting $X = \Upsilon = e_i$ in (7.4) and taking summation over $1 \leq i \leq 3$, we get $dr(Z) = 0$ for all Z.

Lemma 7.1. If a 3-dimensional Lorentzian para-Kenmotsu manifold is η-parallel Ricci tensor, then the scalar curvature is constant positive.

Theorem 7.1. *A 3-dimensional Lorentzian para-Kenmotsu Manifold with η-parallel Ricci tensor is locally φ-symmetric.*

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