Geodesic flow on Finsler manifolds of hyperbolic type

F. Houenou, C. Ogouyandjou and L. Todjihounde

Abstract. Let (X, F) be a compact Finsler (no reversibility is assumed) manifold and $\tilde{X}$ be its Finsler universal covering. In this work we study the geodesic flow restricted to the set of all geodesics that are minimal on $\tilde{X}$. In particular we give a comparison result between the topological entropy and the volume entropy of (X, F).

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1 Introduction

Let (X, g) be a closed (compact without boundary) and connected Riemannian manifold, $p : (\tilde{X}, \tilde{g}) \to (X, g)$ its Riemannian universal covering map and $\pi : TX \to X$ the canonical projection from its tangent bundle. Let us denote by $\tilde{x}$, the lift in $\tilde{X}$, for any picked $x \in X$. An interesting asymptotic invariant h_g (volume entropy also called volume growth) has been introduced in [8] as follows: if $\mathrm{vol}_{\tilde{g}} B(\tilde{x}, r)$ denotes the volume of the open ball $B(\tilde{x}, r)$ centered at $\tilde{x}$ with radius r in the universal covering $(\tilde{X}, \tilde{g})$ of (X, g), then the quantity

$$h_g := \lim_{r \to +\infty} \frac{\log \mathrm{vol}_{\tilde{g}} B(\tilde{x}, r)}{r},$$

where the limit on the right hand side exists for all $\tilde{x} \in \tilde{X}$ and, in fact, is independent of $\tilde{x}$. This asymptotic invariant describes the exponential growth rate of the volume on the universal covering and is related to the geometry, the topology and the dynamic of manifolds (see [10]).

Besides, iterating any homeomorphism φ on a metric space (X, d), leads to a dynamical system. The study and the understanding of this system strongly relies on the answers to the questions such as how many different orbits it has, how fast it mixes together various sets, etc. In fact, it is well known that to any homeomorphism of a compact Hausdorff topological space, there is an attached real which characterizes the induced action by the homeomorphism on the finite open covering of the space: the topological entropy. It represents the exponential growth rate of the number of distinguishable orbits of the iterate for the dynamical system given by an iterated homeomorphism. Denoting by $\{\phi^t\}_t$ the geodesic flow of g in the unit tangent bundle

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$SX = \{(x,v) \in TX \ ; \ g_x(v,v) = 1\} \subset TX$ and by $h_{\text{top}}(\phi^t) = h_{\text{top}}(\phi^t_{SX})$, the topological entropy of $\{\phi^t\}_t$, it was established in [8], that

$$h_{\text{top}}(\phi^t) \geqslant h_g$$

with equality when g is of non positive curvature. This estimation was generalized by A. Freire and R. Mañé to Riemannian metrics without conjugate points (see [4]). Later on, the authors in [6] proved that it is sufficient to consider minimal geodesics to generate exponential complexity (providing $h_g > 0$). Indeed, denoting by $\tilde{\mathcal{X}}$ the closed and ϕ^t-invariant subset of $S\tilde{X}$ which consists of all $(\tilde{x},\tilde{v}) \in S\tilde{X}$ such that the geodesic $c_{\tilde{v}}$ with speed $\dfrac{d}{dt}c_{\tilde{v}}(0) = \tilde{v}$ is globally length-minimizing, by $\mathcal{X} = p_*(\tilde{\mathcal{X}})$ the projection of $\tilde{\mathcal{X}}$ onto SX, by $\{\phi^t_{\mathcal{X}}\}_t$ and $\{\phi^t_{\tilde{\mathcal{X}}}\}_t$ the geodesic flow restricted to $\mathcal{X}$ and $\tilde{\mathcal{X}}$ respectively :

Theorem 1.1 (see [6]). *Let (X,g) be a compact Riemannian manifold and $\phi^t_{\mathcal{X}}$ be the geodesic flow ϕ^t restricted to the set of the minimal geodesics $\mathcal{X} \subset SX$. Then*

$$h_{top}(\phi^t_{\mathcal{X}}) \geqslant h_g. \tag{1.1}$$

Recall that a compact manifold of hyperbolic type is a compact manifold admitting a strictly negative curvature. A fruitful attempt to prove the reverse inequality of (1.1) has been made on compact manifold of hyperbolic type :

Theorem 1.2 (see [5]). *Let (X,g) be a compact Riemmannian manifold of hyperbolic type. There exists a constant β depending only on (X,g) such that for each compact set $K \subset \tilde{X}$:*

$$h_{top}(\phi^t, \pi^{-1}(K) \cap \tilde{\mathcal{X}}, \beta) \leqslant h_g. \tag{1.2}$$

The inequalities (1.1) and (1.2) highlighted the relationship between the volume entropy and the topological entropy on a Riemmannian manifold. Since Finsler manifold is a generalization of Riemmanian manifold, it is natural to study the equivalence, the properties and the implication of such relationship in Finsler geometry setting ; this is the aim of the present work. Let us mention that some researchers have treated question related to volume entropy of fundamental group of Finsler manifold (see [10] and references therein), but according to the authors knowledge, nothing is yet known about the topological entropy and its relationship with the volume one in Finsler geometry setting. This work brings them into light.

To do so, let us assume that (X,F) is a Finsler manifold with $p : (\tilde{X}, p^*F) \longrightarrow (X,F)$ its Finsler universal covering map and $\pi : TX \longrightarrow X$ the natural projection from the tangent bundle TX onto X. In the sequel, we will still denote p^*F by F. Pick any $x \in X$ and let $\tilde{x}$ be its lift in $\tilde{X}$ and $S_F\tilde{X} = \{(\tilde{x},v) \in T\tilde{X} \ ; F(\tilde{x},v) = 1\}$ the unit tangent bundle. The geodesic flow $\{\phi^t_F\}_t$ of F is defined by

$$\begin{array}{rccl} \phi^t_F : & S_F\tilde{X} & \longrightarrow & S_F\tilde{X} \\ & v & \longmapsto & \dfrac{d}{dt}c_v(0) = v. \end{array}$$

Let us denote by $\tilde{\mathcal{X}}_F$ the closed and ϕ_F^t-invariant subset of $S_F\tilde{X}$ consisting of all $(\tilde{x}, v) \in S_F\tilde{X}$ such that the geodesic $c_{\tilde{v}}$ with speed $\frac{d}{dt}c_v(0) = v$ is globally length-minimizing. Let $\mathcal{X}_F = p_*(\tilde{\mathcal{X}}_F)$ stands for the projection of $\tilde{\mathcal{X}}_F$ onto S_FX, while $\{\phi^t_{\mathcal{X}_F}\}_t$ and $\{\phi^t_{\tilde{\mathcal{X}}_F}\}_t$ will denote the geodesic flow restricted to $\mathcal{X}_F$ and $\tilde{\mathcal{X}}_F$ respectively.

We obtain the following result showing that the topological entropy of a geodesic flow on a compact Finsler manifold is bounded from bellow by its volume entropy.

Theorem A *Let (X, F) be a compact Finsler manifold and let $\{\phi^t_{\mathcal{X}_F}\}_t$ denotes the restriction to $\mathcal{X}_F \subset S_FX$ of the geodesic flow $\{\phi^t_F\}_t$ of (X, F). Then*

$$h_{top}(\phi^t_{\mathcal{X}_F}) \geqslant h_F. \tag{1.3}$$

Assuming that (X, F) is of hyperbolic type, we prove that the topological entropy is bounded above by the volume entropy as follows :

Theorem B *Let (X, F) be a compact Finsler manifold of hyperbolic type and $K \subset \tilde{X}$ a compact set in the universal cover $\tilde{X}$ of X. Let $\mathcal{F} = SK \cap \tilde{\mathcal{X}}_F$, where $SK = \pi^{-1}(K) \cap S_F\tilde{X}$. Then there is some constant β depending only on (X, F) such that*

$$h_{top}(\phi^t_F, \mathcal{F}, \beta) \leqslant h_F. \tag{1.4}$$

After gathering in Section 2 some needed materials for the sake of completeness and easy reading we study the ideal boundary of a Finsler manifold of hyperbolic type and the Morse Lemma. In Section 3, we recall the Bowen's definition for the topological entropy, afterward we present and prove our comparison results for the topological entropy.

2 Background materials

In this section we give an overview of some basics tools needed for easy reading and understanding of the present work for the sake of completeness. We refer to [2, 10] and references therein for more details.

2.1 Finsler geometry structure

Roughly speaking, as Riemannian geometry extends Euclidean geometry, Finsler geometry gives a larger environnement than the Riemannian one to study geometrical objets.

Definition 2.1. Let X be a differentiable manifold. A Finsler structure on X is a function

$$F : TX \longrightarrow [0; +\infty)$$

on the tangent bundle TX of X with the following properties:

1. F is C^∞ on the slit tangent bundle $TX \setminus \mathbf{0}$ where $\mathbf{0}$ stands for the zero section (Regularity) ;

2. $F(x, \lambda v) = \lambda F(x, v)$ for all $x \in X$, $v \in T_xX$ and $\lambda \geqslant 0$ (positive 1-homogeneity) ;

3. The $n \times n$ Hessian matrix $(g_{ij})_{ij} := \left(\frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}F^2\right)_{ij}$ is positively defined at every point (x, v) of $TX \smallsetminus \mathbf{0}$ given then a metric $g_{x,v}$ (strict convexity).

Therefore, every Riemannian manifold (M, g) is a natural example of a Finsler manifold with $F = \sqrt{g}$.

Consider a and b reals satisfying $a \leqslant b$. Given $\gamma : [a, b] \to X$, a piecewise C^∞ curve with velocity $\frac{d\gamma}{dt} = \frac{d\gamma^i}{dt}\frac{\partial}{\partial x^i} \in T_{\gamma(t)}X$ on a Finsler manifold (X, F), one computes its length as $l_F(\gamma) = \int_a^b F\big(\gamma, \frac{d\gamma}{dt}\big)dt$.

For two elements x and y in a Finsler manifold (X, F), let us denote by $C^\infty(x, y)$ the collection of all piecewise C^∞ curves $\gamma : [a, b] \to X$ starting from $\gamma(a) = x$ and ending to $\gamma(b) = y$. The metric distance from x to y is defined by:

$$d_F(x, y) = \inf_{\gamma \in C^\infty(x,y)} l_F(\gamma).$$

In general, F is non reversible then the induced metric d_F is non symmetric.

If a Finsler function F satisfies

$$F(x, \lambda v) = |\lambda| F(x, v) \quad \text{ for all } \quad \lambda \in \mathbb{R},$$

one says that it is absolute homogeneous and for such Finsler function, the associated distance d_F is symmetric.

It is important for the whole theory of Finlser manifolds that one does not symmetrize or reduce the general Finsler metric to the special Riemannian or reversible Finsler case. The non-reversibility is essential, which should be preserved because it results from the main difference between Finsler geometry and Riemannian geometry.

The metric tensor g on a Riemannian manifold (X, g) depends only on the points x of that manifold X, namely $g = g_x$; whereas in Finsler geometry, although the metric tensor g on a Finsler manifold (X, F) depends on a point x of X it also depends on tangent vector v to X at x i.e. $g = g_{x,v}$. The non reversibility of the associated metric to a Finsler function strongly relies on the dependence in the tangent vector. Nevertheless there is a way to overcome the non reversibility on some Finsler manifold without symmetrization.

Definition 2.2. A Finsler manifold (X, F) is called uniform if there exists a constant α_F such that

$$\frac{1}{\alpha_F} \cdot g_{x,v_1} \leqslant g_{x,v_2} \leqslant \alpha_F \cdot g_{x,v_1} \qquad \forall\ x \in X \text{ and } \forall\ v_1,\ v_2 \in T_xX \smallsetminus \{0\}. \tag{2.1}$$

The constant α_F in (2.1) is called the uniformity constant of (X, F) and in this case α_F is finite it is given by

$$\alpha_F = \sup_{x \in X}\left\{\sup_{u,v_1,v_2 \in T_xX \smallsetminus \{0\}} \frac{\sqrt{g_{x,v_1}(u,u)}}{\sqrt{g_{x,v_2}(u,u)}}\right\}.$$

Let us remark that since d_F is non symmetric, the inequality (2.1) gives a helpful and powerful tool for estimation. In fact, as a consequence to (2.1), we have for all distinct pair of points x and y :

$$d_F(y,x) \leqslant \alpha_F^2 d_F(x,y).$$

Observe that compact Finsler manifolds and their Finsler universal coverings are uniform with the same uniformity constant α_F.

Let us denote by $B_F^+(x,r) = \{y \in X,\ d_F(x,y) < r\}$ the forward open ball of radius r centered at x and $B_F^-(x,r) = \{y \in X,\ d_F(y,x) < r\}$ the backward open ball of radius r centered at x (namely the set of center of balls of radius r containing x). Every Finsler manifold comes with a natural volume form, which is described as follows:

fix an arbitrary Riemannian metric g on X, let dv_g be its volume form ; pick $x \in X$ and denote by $\mathrm{vol}_g B_g(x,1)$ and $\mathrm{vol}_g B_F^+(x,1)$ the volume with respect to g of the unit open ball in (X,g) and (X,F) respectively ; the Finsler volume form on X is given by

$$dv_F(x) = \frac{\mathrm{vol}_g B_g(x,1)}{\mathrm{vol}_g B_F^+(x,1)} dv_g(x),$$

and is independent of the choice of the Riemannian metric g. Then the Finsler volume for any subset A of (X,F) is $\mathrm{vol}_F(A) = \displaystyle\int_A dv_F(x)$.

Proposition 2.1. *Let α be a strictly positive real number and let Ω be a fundamental domain with diameter α in $\tilde{X}$. Then for all $r > \alpha$ and for all $\tilde{x}$ and $\tilde{y}$ in Ω one has:*

$$B_F^+(\tilde{x}, r-\alpha) \subseteq B_F^+(\tilde{y}, r) \subseteq B_F^+(\tilde{x}, r+\alpha).$$

Consequently for all $\tilde{x}$ and $\tilde{y}$ in $\tilde{X}$ the following holds

$$vol_F(B_F^+(\tilde{x}, r-\alpha)) \leqslant vol_F(B_F^+(\tilde{y}, r)) \leqslant vol_F(B_F^+(\tilde{x}, r+\alpha)).$$

Proof. Take $\tilde{z} \in B_F^+(\tilde{x}, r-\alpha)$ meaning $d_F(\tilde{x},\tilde{z}) < r-\alpha$ which infers that $d_F(\tilde{x},\tilde{z})+\alpha < r$. Since Ω is of diameter α then for all $\tilde{x}$ and $\tilde{y}$ in Ω one has $\max(d_F(\tilde{x},\tilde{y}), d_F(\tilde{y},\tilde{x})) \leqslant \alpha$. Therefore $d_F(\tilde{y},\tilde{z}) \leqslant d_F(\tilde{y},\tilde{x}) + d_F(\tilde{x},\tilde{z}) < r$. In the other hand taking $\tilde{z} \in B_F^+(y,r)$, i.e $d_F(\tilde{y},\tilde{z}) < r$, we have by the same argument that $d_F(\tilde{x},\tilde{z}) \leqslant d_F(\tilde{x},\tilde{y}) + d_F(\tilde{y},\tilde{z}) < \alpha + r$. □

Proposition 2.2. *Let (X,F) be a closed Finsler manifold, $\tilde{x} \in \tilde{X}$ and $B_F^+(\tilde{x},r)$ the forward ball of radius r in the universal covering $\tilde{X}$ of X. Then*

$$h_F := \lim_{r\to+\infty} \frac{1}{r} \log vol_F(B_F^+(\tilde{x},r))$$

exists and is independent of $\tilde{x}$.

Proof. Choose a fundamental domain Ω in $\tilde{X}$ with diameter α. Then

$$B_F^+(\tilde{x}, r-\alpha) \subseteq B_F^+(\tilde{y}, r) \subseteq B_F^+(\tilde{x}, r+\alpha), \qquad \forall\ \tilde{x},\ \tilde{y} \in \Omega.$$

Consequently for all $\tilde{x}$ and $\tilde{y}$ in $\tilde{X}$ the following holds

$$\mathrm{vol}_F(B_F^+(\tilde{x}, r-\alpha)) \leqslant \mathrm{vol}_F(B_F^+(\tilde{y}, r)) \leqslant \mathrm{vol}_F(B_F^+(\tilde{x}, r+\alpha))$$

since the covering transformations that bring $\tilde{x}$ and $\tilde{y}$ in Ω are isometries. Also for all positive reals r and s,

$$B_F^+(\tilde{x}, r+s) = \bigcup_{\tilde{y} \in B_F^+(\tilde{x}, r)} B_F^+(\tilde{y}, s).$$

Let N be a maximal subset of $B_F^+(\tilde{x}, r)$ whose points are pair-wise b apart. We have

$$\bigcup_{\tilde{y} \in N} B_F^+(\tilde{y}, \frac{b}{2}) \subseteq B_F^+(\tilde{x}, r + \frac{b}{2})$$

which infers

$$\begin{aligned} &\textstyle\sum_{\tilde{y} \in N} \mathrm{vol}_F(B_F^+(\tilde{y}, \frac{b}{2})) \leqslant \mathrm{vol}_F(B_F^+(\tilde{x}, r + \frac{b}{2})) \\ &\qquad \Rightarrow \mathrm{card} N \cdot \inf_{y \in X} \mathrm{vol}_F(B_F^+(\tilde{y}, \tfrac{b}{2})) \leqslant \mathrm{vol}_F(B_F^+(\tilde{x}, r + \tfrac{b}{2})) \\ &\qquad \Rightarrow \mathrm{card} N \cdot \inf_{\tilde{y} \in X} \mathrm{vol}_F(B_F^+(\tilde{y}, \tfrac{b}{2})) \leqslant \mathrm{vol}_F(B_F^+(\tilde{x}, r + \tfrac{b}{2})). \end{aligned} \tag{2.2}$$

Therefore, a maximal subset N of $B_F^+(\tilde{x}, r)$ whose point are pair-wise b apart has cardinality at most $\frac{1}{c_b}\mathrm{vol}_F\big(B_F^+(\tilde{x}, r + \frac{b}{2})\big)$ where the constant c_b is $\inf\limits_{\tilde{z} \in X} \mathrm{vol}_F(B_F^+(\tilde{z}, \frac{b}{2}))$ and depends on the curvature. Since every point of $B_F^+(\tilde{x}, r)$ is within b of some $\tilde{y}$ in N then

$$B_F^+(\tilde{x}, r+s) = \bigcup_{\tilde{y} \in N} B_F^+(\tilde{y}, s+b).$$

One may assume that $\mathrm{vol}_F(B_F^+(\tilde{y}, r))$ is unbounded. Hence one can choose b so that $c_b = 1$. Put $\Lambda = \alpha + \frac{3b}{2}$ then

$$\mathrm{vol}_F B_F^+(\tilde{x}, r+s) \leqslant \mathrm{vol}_F B_F^+(\tilde{x}, r + \frac{b}{2}) \mathrm{vol}_F B_F^+(\tilde{x}, s + \alpha + b); \tag{2.3}$$

hence

$$\mathrm{vol}_F B_F^+(\tilde{x}, r+s) \leqslant \mathrm{vol}_F B_F^+(\tilde{x}, r) \mathrm{vol}_F B_F^+(\tilde{x}, s + \Lambda)$$

since

$$\begin{aligned} \mathrm{vol}_F B_F^+(\tilde{x}, r+s) &= \mathrm{vol}_F B_F^+(\tilde{x}, r - \tfrac{b}{2} + \tfrac{b}{2} + s) \\ &\leqslant \mathrm{vol}_F B_F^+(\tilde{x}, r) \mathrm{vol}_F B_F^+(\tilde{x}, r - \tfrac{b}{2} + \tfrac{b}{2} + s + \alpha). \end{aligned}$$

Using again the relation (2.3), if $ks \leqslant r < (k+1)s$ then

$$\begin{aligned} \mathrm{vol}_F B_F^+(\tilde{x}, r) &\leqslant \mathrm{vol}_F B_F^+(\tilde{x}, (k+1)s) \\ &\leqslant \mathrm{vol}_F B_F^+(\tilde{x}, ks) \mathrm{vol}_F B_F^+(\tilde{x}, s + \Lambda) \\ &\leqslant \cdots \leqslant \mathrm{vol}_F B_F^+(\tilde{x}, s) \mathrm{vol}_F B_F^+(\tilde{x}, s + \Lambda)^k, \end{aligned}$$

$$\begin{aligned}\frac{1}{r}\log \mathrm{vol}_F B_F^+(\tilde{x},r) &\leqslant \frac{1}{r}\log \mathrm{vol}_F B_F^+(\tilde{x},s)+\frac{k}{r}\log \mathrm{vol}_F B_F^+(\tilde{x},s+\Lambda)\\ &\leqslant \frac{1}{r}\log \mathrm{vol}_F B_F^+(\tilde{x},s)+\frac{1}{s}\log \mathrm{vol}_F B_F^+(\tilde{x},s+\Lambda)\end{aligned}$$

$$\limsup_{r\to+\infty}\frac{1}{r}\log \mathrm{vol}_F B_F^+(\tilde{x},r) \leqslant \frac{1}{s}\log \mathrm{vol}_F B_F^+(\tilde{x},s+\Lambda) \quad \forall\, s>0 \tag{2.4}$$

and so

$$\limsup_{r\to+\infty}\frac{1}{r}\log \mathrm{vol}_F B_F^+(\tilde{x},r) \leqslant \liminf_{s\to+\infty}\frac{1}{s}\log \mathrm{vol}_F B_F^+(\tilde{x},s+\Lambda) = \liminf_{s\to+\infty}\frac{1}{s}\log \mathrm{vol}_F B_F^+(\tilde{x},s)$$

which shows that $\limsup\limits_{r\to+\infty}\log \mathrm{vol}_F B_F^+(\tilde{x},r)$ exists. □

Definition 2.3. Let (X,F) be a compact Finsler manifold. Analogously to the Riemannian metric case, one defines the volume entropy of F as follows:

$$h_F=\lim_{r\to\infty}\frac{\log \mathrm{vol}_F(B_{\tilde{F}}^+(\tilde{x},r))}{r}.$$

Let us recall some terminologies for the sake of reading:

Definition 2.4. Let (X,F) be a Finsler manifold and $\tilde{X}$ its Finsler universal covering

1. A curve $c:[a,b]\to\tilde{X}$ satisfying $F(\dot{c})=1$ is said to be minimal if $l_F(c)=d_F(c(a),c(b))$.
2. A curve $c:[0;+\infty)\to\tilde{X}$ is called a forward ray if $c_{|_{[a,b]}}$ is minimal for allowed $[a,b]\subset[0;+\infty)$.
3. A curve $c:(-\infty,0]\to\tilde{X}$ is called a backward ray if $c_{|_{[a,b]}}$ is minimal for allowed $[a,b]\subset(-\infty,0]$.
4. A curve $c:\mathbb{R}\to\tilde{X}$ is said minimal if $c_{|_{[a,b]}}$ is minimal for allowed $[a,b]\subset\mathbb{R}$.
5. A $\mathcal{C}^\infty$ curve $c:\mathbb{R}\to\tilde{X}$ is called a geodesic if it has constant speed and for all sufficiently closed positive reals $a<b$ one has $l_F(c_{|_{[a,b]}})=d_F(c(a),c(b))$.

The Finsler manifold (X,F) is said to be forward complete if any geodesic defined over $[a,b[$ can be extended to a geodesic over $[a,+\infty[$. Similarly (X,F) is said to be backward complete if any geodesic defined over $]a,b]$ can be extended to a geodesic over $]-\infty,b]$. We say that (X,F) is complete if it is both forward complete and backward complete.

Lemma 2.3. *If a uniform Finsler manifold (X,F) is forward complete, it is also backward complete, thus complete.*

Let us denote by $S_F\tilde{X}$ the unit tangent bundle of $(\tilde{X},F)$, $\quad S_F\tilde{X}=\{(\tilde{x},v)\in S\tilde{X}: F(\tilde{x},v)=1\}$. The geodesic flow $\Phi_F^t:S_F\tilde{X}\to S_F\tilde{X}$ of F is defined by $\Phi_F^t(\tilde{x},v)=\dot{c}_v(t)$, where $c_v:\mathbb{R}\to\tilde{X}$ is the F-geodesic defined by $c_v(0)=\tilde{x}$ and $\dot{c}_v(0)=v$. Let $\tilde{\mathcal{X}}_F$ be the closed and Φ_F^t-invariant subset of $S_F\tilde{X}$ consisting of all $(\tilde{x},v)\in S_F\tilde{X}$ such that the geodesic c_v with $c_v(0)=\tilde{x}$ and $\dot{c}_v(0)=v$ is a minimal geodesic. We denote by $\mathcal{X}_F=p_*(\tilde{\mathcal{X}}_F)$ the projection of $\tilde{\mathcal{X}}_F$ to S_FX and by $\phi^t_{\mathcal{X}_F},\phi^t_{\tilde{\mathcal{X}}_F}$ the geodesic flow restricted to $\mathcal{X}_F$ and $\tilde{\mathcal{X}}_F$, respectively.

2.2 Morse lemma and ideal boundary of a Finsler manifold of hyperbolic type

For the use made of them in the work, the ideal boundary and the Morse lemma was presented and discussed in this section. For more details, we refer to [1] and references therein.

Definition 2.5. Let (X, F) be a Finsler manifold. The function F is uniformly equivalent to a Riemannian metric g on X, if there is a positive constant α_0 such that

$$\frac{1}{\alpha_0} \cdot F \leqslant \| \cdot \|_g \leqslant \alpha_0 \cdot F.$$

Consequently one has for the associated metrics d_F and d_g:

$$\frac{1}{\alpha_0} \cdot d_F(\cdot, \cdot) \leqslant d_g(\cdot, \cdot) \leqslant \alpha_0 \cdot d_F(\cdot, \cdot).$$

If in addition, the Riemmanian metric g is of strictly negative curvature then the Finsler manifold (X, F) is said to be of hyperbolic type and the metric g is called the associated metric of strictly negative curvature to the Finsler structure F on X.

For any subset A of X and any $x \in X$ the distance from x to A denoted by $d_F(x, A)$ and the distance from A to x denoted by $d_F(A, x)$ are respectively defined as follows:

$$d_F(x, A) = \inf_{a \in A} d_F(x, a) \quad \text{resp.} \quad d_F(A, x) = \inf_{a \in A} d_F(a, x).$$

Also for any two subsets A and B the F-Hausdorff distance between A and B is defined as follows:

$$d_F^H(A, B) = \sup_{a \in A} d_F(a, B).$$

Observe that $d_F(x, A) \leqslant \alpha_F^2 d_F(A, x)$ and $d_F^H(A, B) \leqslant \alpha_F^2 d_F^H(B, A)$.

The following result is fundamental for the study of the topological entropy of $\phi^t_{\mathcal{X}_F}$ when (X, F) is a compact Finsler manifold of hyperbolic type ; it was proved by Morse on a 2-dimensionnal Riemannian manifold [9] and was extended in Finsler geometry setting by Zaustinsky [11]. Thanks to Klingenberg [7] the Morse lemma holds in arbitrary dimension.

Let (X, F) be a compact Finsler manifold (so F is uniform) of hyperbolic type (so F is uniformly equivalent to a Riemannian metric of strictly negative curvature). From now on, we assume the existence of a fixed Riemannian metric g_0 on X with strictly negative curvature associated to F ; we denoted by α_F the uniformity constant of F and by α_0 the uniformly equivalence constant of F to g_0.

Proposition 2.4 (Morse Lemma)**.** *Let (X, F) be a Finsler manifold of hyperbolic type. Then there is a constant $r_0 = r_0(F, g_0) > 0$ with the following properties:*

(i) if $\gamma : [a, b] \longrightarrow \tilde{X}$ and $\chi : [a_0, b_0] \longrightarrow \tilde{X}$ are minimizing geodesic segments with respect to F and g_0, respectively, joining $\gamma(a) = \chi(a_0)$ to $\gamma(b) = \chi(b_0)$, then

$$d_F^H(\gamma[a, b], \chi[a_0, b_0]) \leqslant r_0.$$

(ii) *for any minimizing F-geodesic $\gamma : \mathbb{R} \to \tilde{X}$ there is a g_0-geodesic $\chi : \mathbb{R} \to \tilde{X}$ and conversely for any g_0-geodesic $\chi : \mathbb{R} \to \tilde{X}$ there is a minimizing F-geodesic $\gamma : \mathbb{R} \to \tilde{X}$ with*

$$d_F^H(\gamma(\mathbb{R}), \chi(\mathbb{R})) \leqslant r_0.$$

Now let (X, F) be a compact Finsler manifold of hyperbolic type and $(\tilde{X}, F)$ be its Finsler universal covering. Let g_0 denote the associated metric of strictly negative curvature on X. Note that the Riemannian universal covering $\tilde{X}_0$ of (X, g_0) is a Hadamard manifold. Let us denote by $\tilde{X}_0(\infty)$ its ideal boundary. Two F-geodesics c and c' are said to be asymptotic if there exists a constant $\eta \geqslant 0$ such that $d_F^H\big(c(\mathbb{R}_+), c'(\mathbb{R}_+)\big) \leqslant \eta$, where d_F^H is the Hausdorff distance with respect to the distance d_F. This defines an equivalence relation on the set of minimizing F-geodesics rays of $\tilde{X}$. Let $\tilde{X}(\infty)$ be the coset of asymptotic minimizing F-geodesics rays c of $\tilde{X}$. For each minimizing F-geodesic ray c of $\tilde{X}$, it follows from Morse lemma that there exists a g_0-geodesic ray γ such that $d_F^H(c(\mathbb{R}_+), \gamma(\mathbb{R}_+)) \leqslant \eta$, where η is the constant in Morse Lemma. Let $[c]$ be the equivalence class of minimizing F-geodesic ray c and let $[\gamma]$ be the equivalence class of the g_0-geodesic γ. The map f defined by

$$\begin{array}{cccc} f: & \tilde{X}(\infty) & \longrightarrow & \tilde{X}_0(\infty) \\ & [c] & \longmapsto & [\gamma] \end{array}$$

is bijective. Then f defines on $\tilde{X}(\infty)$ a natural topology with respect to which $\tilde{X}(\infty)$ and $\tilde{X}_0(\infty)$ are homeomorphic.

Let $\mathcal{G}_0$ be the set of the g_0-geodesics $\gamma : \mathbb{R} \longrightarrow \tilde{X}$ and denote by $\tilde{\mathcal{X}_F}(\gamma)$ the following set

$$\tilde{\mathcal{X}_F}(\gamma) := \{(x, v) \in \mathcal{X}_F : c_v(-\infty) = \gamma(-\infty) \text{ and } c_v(+\infty) = \gamma(+\infty)\}.$$

Using the Morse Lemma we have

$$\tilde{\mathcal{X}_F} = \bigcup_{\gamma \in \mathcal{G}_0} \tilde{\mathcal{X}_F}(\gamma).$$

3 Topological entropy estimation in Finsler geometry

In this section we present and prove our main results on estimation of the topological entropy in Finsler geometry setting by providing a lower and an upper bounds for the topological entropy.

3.1 Bowen's definition of topological entropy

Here we recall Bowen's definition of topological entropy. Let $\varphi : V \to V$ be a homeomorphism of a metric space (V, d), not necessarily compact. By iterating φ, one obtains a dynamical system $(X, \mathbb{N}, \psi)$ where $\psi : (x, n) \longmapsto \psi(x, n) = \varphi^n(x)$. For each $n \in \mathbb{N}$, a metric on V is defined by

$$d_n(x, y) := \max_{0 \leqslant j < n} d(\varphi^j(x), \varphi^j(y)).$$

Let H be a subset of V. For any $\varepsilon > 0$ and any integer n, we say that a set $Y \subset V$ is (n, ε)-spanning for H if the closed balls $\bar{B}_n(y, \varepsilon) = \{x \in V : d_n(y, x) \leqslant \varepsilon\}$ with $y \in Y$ cover H. If $Y \subset H$ and $\bar{B}_n(y, \varepsilon) \cap Y = \{y\}$ for all $y \in Y$, we say that Y is an (n, ε)-separated subset of H. Let $r_n(F, \varepsilon)$ denote the minimal cardinality of (n, ε)-spanning sets for H and let $s_n(H, \varepsilon)$ denote the maximal cardinality of (n, ε)-separated subsets of H. It is easy to see that for any $\varepsilon > 0$ we have

$$r_n(H, \varepsilon) \leqslant s_n(H, \varepsilon) \leqslant r_n(H, \varepsilon/2).$$

If H is compact then $r_n(H, \varepsilon) < \infty$,

We recall the following definitions of topological entropy:

- $h_{\text{top}}(f, H, \varepsilon) := \limsup\limits_{n \to \infty} \dfrac{1}{n} \log r_n(H, \varepsilon);$
- $h_{\text{top}}(f, H) := \lim\limits_{\varepsilon \to 0} h_{\text{top}}(f, H, \varepsilon)$;
- $h_{\text{top}}(f) := \sup\limits_{H \subset V \text{ compact}} h_{\text{top}}(f, H).$

Note that for any $\varepsilon > 0$ we have $h_{\text{top}}(f, H, \varepsilon) \leqslant h_{\text{top}}(f, H)$ and if V is itself compact, we have $h_{\text{top}}(f) = h_{\text{top}}(f, V)$. If we use $s_n(H, \varepsilon)$ instead of $r_n(H, \varepsilon)$, we obtain the same value for $h_{\text{top}}(f, H)$. For details on topological entropy we refer to [12].

We need the following less known concept of local entropy introduced by Bowen [3]. For $x \in V$ and $\beta > 0$ set

$$Z_\beta(x) := \big\{y \in V \ ; \ d(f^n(x), f^n(y)) \leqslant \beta, \ \forall\, n \in \mathbb{Z}\big\}.$$

The β-local entropy $h_{\text{top,loc}}(f, \beta)$ of f is given by: $h_{\text{top,loc}}(f, \beta) := \sup\limits_{x \in V} h_{\text{top}}(f, Z_\beta(x))$.

One says that f is β-entropy-expansive for $\beta > 0$ if $h_{\text{top,loc}}(f, \beta) = 0$.

We consider the following setting: let $(\tilde{V}, \tilde{d})$ be a metric space and $\Gamma \subset \text{Iso}(\tilde{V})$ acting on $\tilde{V}$. Assume that the quotient $V := \tilde{V}/\Gamma$ is compact and equipped with a metric d such that the projection $p : \tilde{V} \longrightarrow V$ is a local isometry. Let $\tilde{f} : \tilde{V} \longrightarrow \tilde{V}$ be a homeomorphism which commutes with the group Γ and let $f : V \longrightarrow V$ be the projection defined by $f(x) = p\tilde{f}p^{-1}(x)$ (this is well-defined since $\tilde{f}$, Γ-commutes). Note that f is a homeomorphism as well.

The following theorem is a slight extension of a result of Bowen (see [3]). It allows us to estimate the topological entropy using coverings and will be crucial for our applications.

Theorem 3.1. *(see [5] for details) Let $K \subset \tilde{V}$ be a compact set such that $p(K) = V$. Then for any $\beta > 0$ we have*

$$h_{top}(f) \leqslant h_{top}(\tilde{f}, K, \beta) + h_{top,loc}(\tilde{f}, \beta). \tag{3.1}$$

3.2 Lower bound for the topological entropy

Recall the notation $p : \tilde{X} \longrightarrow X$ for the universal cover of X and

$$\tilde{\mathcal{X}}_F = \{(\tilde{x}, v) \in S_F\tilde{X} \ ; \ c_v(0) = \tilde{x}, \ c_v \text{ is a minimal } F\text{-geodesic } \} \subset S_F\tilde{X},$$
$$\mathcal{X}_F = p_*(\tilde{\mathcal{X}}_F) \subset S_F X.$$

The following theorem is an extension to Finsler manifolds of the result of Katok and Hasselblatt in Riemannian geometry (see [6]).

Theorem 3.2. *Let (X, F) be a compact Finsler manifold and $\phi^t_{\mathcal{X}_F}$ be the geodesic flow ϕ^t_F restricted to $\mathcal{X}_F \subset S_F X$. Then*

$$h_{top}(\phi^t_{\mathcal{X}_F}) \geqslant h_F. \tag{3.2}$$

The following lemma is useful for the proof of Theorem 3.2. For any positive real numbers δ and T, recall that $s_T(Y, \delta)$ denotes the maximal cardinality of a (T, δ)-separated subsets of Y.

Lemma 3.3. *Assume that (X, F) is a Finsler manifold with associated Finsler distance d_F. Let $(\phi^t)_t$ be a continuous flow on X and $Y \subset X$. For times $0 = t_0 < t_1 < \cdots < t_m = T$ and $\delta > 0$, we have*

$$\prod_{i=1}^{m} s_{t_i - t_{i-1}}(\phi^{t_{i-1}} Y, \delta) \geqslant s_T(Y, 2\delta), \qquad \forall\ i = 1, \cdots, m.$$

Proof. Let L be a maximal $(T, 2\delta)$-separated subset of Y and let L_i be a maximal $(t_i - t_{i-1}, \delta)$-separated subsets of $\phi^{t_{i-1}}(Y)$ for $i = 1, 2, \cdots, m$. For $(x_1, x_2, \cdots, x_m) \in L_1 \times L_2 \times \cdots \times L_m$ set

$$B(x_1, x_2, \cdots, x_m) := \Big\{ z \in L \ ; \ d_F\big(\phi^{t+t_{i-1}}(z), \phi^t(z)\big) < \delta, \ \forall\, 1 \leqslant i \leqslant m, \ t \in [0, t_i - t_{i-1}] \Big\}.$$

Since L is $(T, 2\delta)$-separated, the triangle inequality implies that $\mathrm{card}B(x_1, \cdots, x_m) \leqslant 1$. Therefore, since the cardinalities of the L_i are maximal implying that they are also $(t_i - t_{i-1}, \delta)$-spanning,

$$\mathrm{card}L = \mathrm{card}\left(\bigcup_{(x_1, x_2, \cdots, x_m)} B(x_1, x_2, \cdots, x_m) \right) \leqslant \prod_{i=1}^{m} \mathrm{card}L_i.$$

□

Lemma 3.4. *Let (X, F) be a closed Finsler manifold of injectivity radius $\delta = inj(X, F)$. Denote by $(\tilde{X}, F)$ the Finsler universal covering of (X, F) with uniformity constant α_F. Let ω be a positive real number satisfying $0 < \omega < \frac{2}{1+\alpha_F^2}$. Let $(T_k)_k \in \mathbb{N}$ be a sequence tending to $+\infty$ starting from T_0 (sufficiently large) and defined by $T_{k+1} = T_k + \omega\frac{\delta}{2}$. Fix $\tilde{x} \in \tilde{X}$, for any $\tilde{y}$ in N, the maximal 2δ-separated set contains in the annulus $\overline{B}^+_F(\tilde{x}, T_k + \omega\frac{\delta}{2}) \setminus B^+_F(\tilde{x}, T_k)$, let $c_{\tilde{y}} : [0, d_F(\tilde{x}, \tilde{y})] \longrightarrow \tilde{X}$ denotes the minimal geodesic segment from $\tilde{x} = c_{\tilde{y}}(0)$ to $\tilde{y} = c_{\tilde{y}}(d_F(\tilde{x}, \tilde{y}))$. Then for any pair of different points $\tilde{y}_1$ and $\tilde{y}_2$ in N_k we have*

$$d_F\big(c_{\tilde{y}_1}(T_k), c_{\tilde{y}_2}(T_k)\big) > \delta.$$

Proof. The proof is a straightforward computation using the triangular inequality and the fact that $c_{\tilde{y}_i}$ for $i = 1, 2$ are geodesic segments from $\tilde{x}$ to $\tilde{y}_i$. In one hand we have:

$$d_F(\tilde{y}_1, \tilde{y}_2) \leqslant d_F\big(\tilde{y}_1, c_{\tilde{y}_1}(T_k)\big) + d_F\big(c_{\tilde{y}_1}(T_k), c_{\tilde{y}_2}(T_k)\big) + d_F\big(c_{\tilde{y}_2}(T_k), \tilde{y}_2\big),$$

which infers

$$d_F\big(c_{\tilde{y}_1}(T_k), c_{\tilde{y}_2}(T_k)\big) \geqslant d_F(\tilde{y}_1, \tilde{y}_2) - d_F\big(\tilde{y}_1, c_{\tilde{y}_1}(T_k)\big) - d_F\big(c_{\tilde{y}_2}(T_k), \tilde{y}_2\big).$$

On the other hand we have

$$d_F(\tilde{x}, \tilde{y}_i) = d_F(\tilde{x}, c_{\tilde{y}_i}(T_k)) + d_F(c_{\tilde{y}_i}(T_k), \tilde{y}_i) < T_k + \frac{\omega\delta}{2},$$

whence $d_F(c_{\tilde{y}_i}(T_k), \tilde{y}_i) < \frac{\omega\delta}{2}$, $\forall i = 1, 2$. Using the uniform property of F, one gets

$$d_F(\tilde{y}_1, c_{\tilde{y}_1}(T_k)) < \alpha_F^2 d_F(c_{\tilde{y}_1}(T_k), \tilde{y}_1) < \alpha_F^2 \frac{\omega\delta}{2}. \tag{3.3}$$

Therefore

$$d_F\big(c_{\tilde{y}_1}(T_k), c_{\tilde{y}_2}(T_k)\big) \geqslant 2\delta - \frac{\omega\delta}{2}(1 + \alpha_F^2).$$

Since ω is such that $0 < \omega < \dfrac{2}{1 + \alpha_F^2}$, one obtains $d_F\big(c_{\tilde{y}_1}(T_k), c_{\tilde{y}_2}(T_k)\big) > \delta$. □

We can now give the proof of Theorem 3.2.

Proof. (Proof of Theorem 3.2)
Fix $\tilde{x} \in \tilde{X}$, $\varepsilon > 0$ and write

$$\delta := \text{inj } (X, F) > 0, \quad a := \sup_{y \in \tilde{X}_F} \text{vol} B_F^+(y, 2\delta), \quad b := h_{\text{top}}(\phi_{S_F X}^t).$$

There exists a constant ω satisfying $0 < \omega < \dfrac{2}{1 + \alpha_F^2}$ and a sequence $(T_k)_k$ tending to $+\infty$ such that

$$\text{vol} B_F^+(\tilde{x}, T_k + \omega\frac{\delta}{2}) \smallsetminus \text{vol} B_F^+(\tilde{x}, T_k) \geqslant e^{h_F(1-\varepsilon)T_k}, \tag{3.4}$$

for otherwise adding up the volume of the annuli $B_F^+\big(\tilde{x}, T_k + \omega\frac{\delta}{2}\big) \smallsetminus B_F^+(\tilde{x}, T_k)$ with $T_{k+1} = T_k + \omega\frac{\delta}{2}$ starting at T_0 sufficiently large would yield that the exponential growth rate is less than $h_F(1 - \varepsilon)$.

Let N_k be a maximal 2δ-separated set in the annulus $\bar{B}_F^+(\tilde{x}, T_k + \omega\frac{\delta}{2}) \smallsetminus B_F^+(\tilde{x}, T_k)$, then we have for all $k \in \mathbb{N}$

$$a \cdot \text{card } N_k \geqslant \text{vol}\left(\bigcup_{\tilde{y} \in N_k} B_F^+(\tilde{y}, 2\delta)\right) \geqslant \text{vol} B_F^+(\tilde{x}, T_k + \frac{\delta}{2}) \smallsetminus \text{vol} B_F^+(\tilde{x}, T_k) \geqslant e^{h_F(1-\varepsilon)T_k}.$$

For $\tilde{y} \in N_k$ let $c_{\tilde{y}} : [0, d_F(\tilde{x}, \tilde{y})] \to \tilde{X}$ be a minimal F-geodesic segment with $c_{\tilde{y}}(0) = \tilde{x}$ and $c_{\tilde{y}}(d_F(\tilde{x}, \tilde{y})) = \tilde{y}$. Now, if $\tilde{y}_1, \tilde{y}_2 \in N_k$ with $\tilde{y}_1 \neq \tilde{y}_2$ we have, by Lemma 3.4,

$$d_F(c_{\tilde{y}_1}(T_k), c_{\tilde{y}_2}(T_k)) \geqslant d_F(\tilde{y}_1, \tilde{y}_2) - d_F(\tilde{y}_1, c_{\tilde{y}_1}(T_k)) - d_F(c_{\tilde{y}_2}(T_k), \tilde{y}_2) > \delta, \tag{3.5}$$

and therefore, the following sets

$$\tilde{S}_k := \left\{ \frac{d}{dt} c_{\tilde{y}}(0) : \tilde{y} \in N_k \right\}$$

are (T_k, δ)-separated with respect to the following metric d_F^1 on $S_F\tilde{X}$,

$$d_F^1(u,v) = \max_{t\in[0,1]} d_F(c_u(t), c_w(t)), \qquad \forall\ (\tilde{x},u),\ (\tilde{x},v) \in S_F\tilde{X}.$$

In S_FX, the sets $S_k := p_*(\tilde{S}_k)$ are $(T_k, \frac{\delta}{2})$-separated. We define the decreasing sequence of compact sets

$$\mathcal{X}_k := p_* \left\{ (\tilde{x},v) \in S_F\tilde{X}\ ;\ c_v : \left[-\sqrt{T_k}, \sqrt{T_k}\right] \longrightarrow \tilde{X} \text{ is minimal} \right\} \quad \text{and take} \quad \bigcap_{k\in\mathbb{N}} \mathcal{X}_k = \mathcal{X}_F.$$

In order to find large separated sets in $\mathcal{X}_F$ we shall find them in the sets $\mathcal{X}_k$, observing that for $t \in \left[\sqrt{T_k}, T_k - \sqrt{T_k}\right]$ we have

$$\phi_F^t S_k \subset \mathcal{X}_k.$$

Assume that k is large enough, such that

$$s_{\sqrt{T_k}}\left(S_k, \frac{\delta}{4}\right) \leqslant e^{2b\sqrt{T_k}} \qquad \text{and} \qquad \sqrt{T_k} \geqslant \frac{2b}{\varepsilon h_F}.$$

We apply Lemma 3.3 and obtain

$$s_{T_k-\sqrt{T_k}}(\phi_F^{\sqrt{T_k}} S_k, \frac{\delta}{4}) \cdot s_{\sqrt{T_k}}(S_k, \frac{\delta}{4}) \geqslant s_{T_k}(S_k, \frac{\delta}{4}) \geqslant \text{card } N_k \geqslant \frac{1}{a} e^{h_F(1-\varepsilon)T_k},$$

showing that

$$s_{T_k-\sqrt{T_k}}(\phi_F^{\sqrt{T_k}} S_k, \frac{\delta}{4}) \geqslant \frac{1}{a} e^{h_F(1-\varepsilon)T_k - 2b\sqrt{T_k}} \geqslant \frac{1}{a} e^{h_F(1-2\varepsilon)T_k}.$$

Let now

$$T \in \left(0, T_k - \sqrt{T_k}\right] \quad \text{and set} \;\; m_k = \left\lfloor \frac{T_k - \sqrt{T_k}}{T} \right\rfloor \in \mathbb{N}.$$

Applying Lemma 3.3 again we have:

$$\begin{aligned} \left(\prod_{i=0}^{m_k-1} s_T(\phi_F^{iT+\sqrt{T_k}} S_k, \frac{\delta}{8}) \right) s_{T_k-\sqrt{T_k}-m_kT}(\phi_F^{m_kT+\sqrt{T_k}} S_k, \frac{\delta}{8}) \quad &\geqslant \quad s_{T_k-\sqrt{T_k}}(\phi_F^{\sqrt{T_k}} S_k, \frac{\delta}{4}) \\ &\geqslant \quad \frac{1}{a} e^{h_F(1-2\varepsilon)T_k} \end{aligned} \tag{3.6}$$

and hence

$$\begin{aligned} \prod_{i=0}^{m_k-1} s_T(\phi_F^{iT+\sqrt{T_k}} S_k, \frac{\delta}{8}) \quad &\geqslant \quad \frac{\frac{1}{a} e^{h_F(1-2\varepsilon)T_k}}{s_{T_k-\sqrt{T_k}-m_kT}(\phi_T^{m_kT+\sqrt{T_k}} S_k, \frac{\delta}{8})} \\ &\geqslant \quad \frac{\frac{1}{a} e^{h_F(1-2\varepsilon)T_k}}{s_T(S_FX, \frac{\delta}{8})} \\ &\geqslant \quad \frac{1}{a} e^{h_F(1-2\varepsilon)T_k - 2bT}, \end{aligned} \tag{3.7}$$

where in the last step we assumed that T is large, so that $s_T(SM, \frac{\delta}{8}) \leqslant e^{2bT}$. Hence one of the factors in the last product has to be " large", i.e., for some $i \in \{0, \cdots, m_k - 1\}$ we have

$$s_T(\phi_F^{iT+\sqrt{T_k}} S_k, \frac{\delta}{8}) \geqslant \frac{1}{a} e^{\frac{h_F(1-2\varepsilon)T_k - 2bT}{m_k}} \geqslant \frac{1}{a} e^{h_F(1-2\varepsilon)T} e^{-\frac{2bT}{m_k}}.$$

Note also that $\phi_F^{iT+\sqrt{T_k}} S_k \subset \mathcal{X}_k$, so when letting $k \to \infty$ while fixing T and using $m_k \longrightarrow \infty$, we find a $(T, \frac{\delta}{8})$-separated set in $\mathcal{X}_F = \cap_k \mathcal{X}_k$ of cardinality at least

$$\frac{1}{a} e^{h_F(1-2\varepsilon)T} \lim_{k\to\infty} e^{-\frac{2bT}{m_k}} = \frac{1}{a} e^{h_F(1-2\varepsilon)T}.$$

Hence

$$h_F - 2\varepsilon \leqslant h_{\text{top}}(\phi^t_{\mathcal{X}_F}, \frac{\delta}{8}) \leqslant h_{\text{top}}(\phi^t_{\mathcal{X}_F}),$$

which completes the proof of Theorem 3.2. $\square$

3.3 Upper bound for the topological entropy of a Finsler manifolds of hyperbolic type

We equipped $S_F\tilde{X}$ with the metric d_F^1 defined as follows:

$$d_F^1(\tilde{u}, \tilde{v}) := \max_{t\in[0,1]} d_F(c_u(t), c_v(t)) \qquad \forall\ (\tilde{x}, u),\ (\tilde{x}, v) \in S_F\tilde{X}.$$

Let (X, F) be a compact Finsler manifold of hyperbolic type, g_0 its associated Riemannian metric with strictly negative curvature, α_F the uniformity constant of F, α_0 the uniformly equivalence constant of F to g_0 and r_0 the constant in Morse Lemma. In this subsection we prove the following theorem:

Theorem 3.5. *Let (X, F) be a compact Finsler manifold of hyperbolic type, $\tilde{X}$ its Finsler universal covering and K a compact subset of $\tilde{X}$. Let $\mathcal{F} = SK \cap \tilde{\mathcal{X}}_F$, where $SK = \pi^{-1}(K) \cap S_F\tilde{X}$. Then there is some constant $\beta = \beta(F, g_0, \alpha_F, \alpha_0, r_0)$ such that*

$$h_{top}(\phi_F^t, \mathcal{F}, \beta) \leqslant h_F.$$

In order to prove this theorem, we construct spanning sets for $\mathcal{F}$. Let $K \subset \tilde{X}$ be a compact set with diam $K = a$. For $r > a$ consider

$$K_r := \{\tilde{z} \in \tilde{X},\ \ r - a \leqslant d_F(\tilde{z}, K) \leqslant r\}$$

Let K^ε and K_r^ε be the minimal ε-spanning sets for K and K_r, respectively. For $\tilde{y} \in K^\varepsilon, \tilde{z} \in K_r^\varepsilon$, let $\chi_{\tilde{y}\tilde{z}} : \mathbb{R} \to \tilde{X}$ be the g_0-geodesic connecting $\tilde{y}$ and $\tilde{z}$ such that $\chi_{\tilde{y}\tilde{z}}(0) = \tilde{y}$ and $\chi_{\tilde{y}\tilde{z}}(d_g(\tilde{y}, \tilde{z})) = \tilde{z}$. By the Morse Lemma, there exists a (unit speed) minimizing F-geodesic $\gamma_{\tilde{y}\tilde{z}} : \mathbb{R} \to \tilde{X}$ that is r_0-close to $\chi_{\tilde{y}\tilde{z}}(\mathbb{R})$. Set

$$P_r := \{\dot{\gamma}_{\tilde{y}\tilde{z}}(0)\ ;\ \tilde{y} \in K^\varepsilon, \tilde{z} \in K_r^\varepsilon\} \subset \tilde{\mathcal{X}}_F. \tag{3.8}$$

Using similar arguments like in [5], we obtain the following result:

Lemma 3.6. *Assume that the subset P_r in relation* (3.8) *is defined, then P_r is a $(r-1,\beta)$-spanning set for $\mathcal{F} = SK \cap \tilde{\mathcal{X}}_F$ with respect to the metric d_F^1 where β is given by $\beta := (2+3\alpha_F^4)r_0 + (\alpha_0^2 + \alpha_F^4(\alpha_0^2+1))\varepsilon$.*

Proof. Let $\gamma : \mathbb{R} \longrightarrow \tilde{X}$ be a minimizing F-geodesic with $\gamma(0) \in K$. Then $\gamma(r) \in K_r$ and one can choose $\tilde{y} \in K^\varepsilon$ and $\tilde{z} \in K_r^\varepsilon$ such that

$$\max\big(d_F(\tilde{y},\gamma(0)), d_F(\gamma(0),\tilde{y})\big) \leqslant \varepsilon \quad \text{and} \quad \max\big(d_F(\tilde{z},\gamma(r)), d_F(\gamma(r),\tilde{z})\big) \leqslant \varepsilon.$$

Let χ be the g_0-geodesic connecting $\gamma(0)$ and $\gamma(r)$ parametrized such that $\chi(0) = \gamma(0)$ and $\chi(d_{g_0}(\tilde{y},\tilde{z})) = \gamma(r)$. Due to the negative curvature, the function

$$\psi : t \in [0, d_{g_0}(\tilde{y},\tilde{z}] \longrightarrow \psi(t) = d_{g_0}(\chi(t), \chi_{\tilde{y}\tilde{z}}(t))$$

is convex thus for all $t \in [0, d_{g_0}(y,z)]$, one has:

(3.9)
$$d_{g_0}(\chi(t), \chi_{yz}(t)) \leqslant \max\{d_{g_0}(\gamma(0),\tilde{y}), d_{g_0}(\gamma(r),\tilde{z})\} \leqslant \varepsilon\alpha_0\ , \qquad \forall\, t \in [0, d_{g_0}(\tilde{y},\tilde{z})].$$

In fact, for any $t \in [0, d_g(\tilde{y},\tilde{z})]$, we write $t = c \times 0 + (1-c) \times d_{g_0}(\tilde{y},\tilde{z})$ for some $c \in [0,1]$. Then

$$\begin{aligned}
\psi(t) = \psi\big(c \times 0 + (1-c) \times d_{g_0}(\tilde{y},\tilde{z})\big) &= c\,\psi(0) + (1-c)\psi(d_{g_0}(\tilde{y},\tilde{z})) \\
&= c\, d_{g_0}(\chi(0), \chi_{\tilde{y}\tilde{z}}(0)) + (1-c) d_{g_0}(\chi(d_{g_0}(\tilde{y},\tilde{z})), \chi_{\tilde{y}\tilde{z}}(d_{g_0}(\tilde{y},\tilde{z}))) \\
&= c\, d_{g_0}(\gamma(0),\tilde{y}) + (1-c) d_{g_0}(\gamma(r),\tilde{z}) \\
&\leqslant \max\{d_{g_0}(\gamma(0),\tilde{y})\ \ d_{g_0}(\gamma(r),\tilde{z})\}.
\end{aligned}$$

Also,

$$\begin{aligned}
d_{g_0}(\gamma(0),\tilde{y}) &= d_{g_0}(\tilde{y},\gamma(0)) \leqslant \alpha_0 d_F(\tilde{y},\gamma(0)) \leqslant \varepsilon\alpha_0, \\
d_{g_0}(\gamma(r),\tilde{z}) &= d_{g_0}(\tilde{z},\gamma(r)) \leqslant \alpha_0 d_F(\tilde{z},\gamma(r)) \leqslant \varepsilon\alpha_0,
\end{aligned}$$

and γ and χ are minimizing geodesics.

Let $A = \gamma[0,r]$ and $B = \gamma_{\tilde{y}\tilde{z}}[0,r']$ be the segment of $\gamma_{\tilde{y}\tilde{z}}$ lying r_0-close to $\chi_{\tilde{y}\tilde{z}}[0, d_g(\tilde{y},\tilde{z})]$ with respect to the F-Hausdorff metric d_F^H ($\gamma_{\tilde{y}\tilde{z}}$ is a minimal geodesic in the r_0-tube around $\chi_{\tilde{y}\tilde{z}}$, showing that B exists and $0 < r' \leqslant d_{g_0}(\tilde{y},\tilde{z})$), namely

$$d_F^H\big(\chi_{\tilde{y}\tilde{z}}([0, d_{g_0}(\tilde{y},\tilde{z})]), B\big) \leqslant r_0.$$

Using the Morse Lemma (Proposition 2.4) and equation (3.9), we find (omitting for the moment the intervals $[0, d_g(\tilde{y},\tilde{z})]$ for $\chi, \chi_{\tilde{y}\tilde{z}}$)

$$d_F^H(A,B) \leqslant d_F^H(A,\chi) + d_F^H(\chi, \chi_{\tilde{y}\tilde{z}}) + d_F^H(\chi_{\tilde{y}\tilde{z}}, B) \leqslant 2r_0 + \alpha_0^2\varepsilon.$$

By the definition of the F-Hausdorff distance d_F^H, for any $t \in [0,r]$ there is some $t' \in [0,r']$ satisfying

$$d_F(\gamma(t), \gamma_{\tilde{y}\tilde{z}}(t')) \leqslant 2r_0 + \varepsilon\alpha_0^2.$$

Recall that γ and $\gamma_{\tilde{y}\tilde{z}}$ are minimal F-geodesics and

$$d_F(\gamma(0), \gamma_{\tilde{y}\tilde{z}}(0)) \leqslant \max\big(d_F(\tilde{y},\gamma(0)), d_F(\gamma(0),\tilde{y})\big) \leqslant r_0 + \varepsilon.$$

- If $t \leqslant t'$ then

$$\begin{aligned} t+t' &= d_F\big(\gamma_{\tilde{y}\tilde{z}}(0),\gamma_{\tilde{y}\tilde{z}}(t+t')\big)\\ &\leqslant d_F\big(\gamma_{\tilde{y}\tilde{z}}(0),\gamma(0)\big)+d_F\big(\gamma(0),\gamma(t)\big)+d_F\big(\gamma(t),\gamma_{\tilde{y}\tilde{z}}(t')\big)+d_F\big(\gamma_{\tilde{y}\tilde{z}}(t'),\gamma_{\tilde{y}\tilde{z}}(t+t')\big)\\ &\leqslant r_0+\varepsilon+t+2r_0+\varepsilon\alpha_0^2+t=2t+3r_0+\varepsilon(\alpha_0^2+1)\end{aligned}$$

implying that $t'-t \leqslant 3r_0+\varepsilon(\alpha_0^2+1) \leqslant \alpha_F^2\big(3r_0+\varepsilon(\alpha_0^2+1)\big)$.

- If $t' < t$ then

$$\begin{aligned} t &= d_F(\gamma(0),\gamma(t))\\ &\leqslant d_F\big(\gamma(0),\gamma_{\tilde{y}\tilde{z}}(0)\big)+d_F\big(\gamma_{\tilde{y}\tilde{z}}(0),\gamma_{\tilde{y}\tilde{z}}(t')\big)+d_F\big(\gamma_{\tilde{y}\tilde{z}}(t'),\gamma(t)\big)\\ &\leqslant d_F\big(\gamma(0),\gamma_{\tilde{y}\tilde{z}}(0)\big)+d_F\big(\gamma_{\tilde{y}\tilde{z}}(0),\gamma_{\tilde{y}\tilde{z}}(t')\big)+\alpha_F^2 d_F\big(\gamma(t),\gamma_{\tilde{y}\tilde{z}}(t')\big)\\ &\leqslant r_0+\varepsilon+t'+\alpha_F^2(2r_0+\varepsilon\alpha_0^2)\\ &\leqslant \alpha_F^2(r_0+\varepsilon)+t'+\alpha_F^2(2r_0+\varepsilon\alpha_0^2)\end{aligned}$$

implying that $t-t' \leqslant \alpha_F^2\big(3r_0+\varepsilon(\alpha_0^2+1)\big)$.

Thus for any $t\in[0,r]$ there is some $t'\in[0,r']$ satisfying

$$d_F(\gamma(t),\gamma_{\tilde{y}\tilde{z}}(t')) \leqslant 2r_0+\varepsilon\alpha_0^2$$

such that one has in one hand

$$|t-t'| \leqslant \alpha_F^2\big(3r_0+\varepsilon(\alpha_0^2+1)\big)$$

and in another hand

$$\begin{aligned} d_F(\gamma_{\tilde{y}\tilde{z}}(t'),\gamma_{\tilde{y}\tilde{z}}(t)) &\leqslant \max\Big\{d_F\big(\gamma_{\tilde{y}\tilde{z}}(\min(t,t')),\gamma_{\tilde{y}\tilde{z}}(\max(t,t'))\big),\alpha_F^2 d_F\big(\gamma_{\tilde{y}\tilde{z}}(\min(t,t')),\gamma_{\tilde{y}\tilde{z}}(\max(t,t'))\big)\Big\}\\ &\leqslant \max(|t-t'|,\alpha_F^2|t-t'|)\\ &= \alpha_F^2|t-t'|\\ &\leqslant \alpha_F^4\big(3r_0+\varepsilon(\alpha_0^2+1)\big).\end{aligned}$$

Therefore

$$\begin{aligned} d_F(\gamma(t),\gamma_{\tilde{y}\tilde{z}}(t)) &\leqslant d_F(\gamma(t),\gamma_{\tilde{y}\tilde{z}}(t'))+d_F(\gamma_{\tilde{y}\tilde{z}}(t'),\gamma_{\tilde{y}\tilde{z}}(t))\\ &\leqslant 2r_0+\alpha_0^2\varepsilon+\alpha_F^4\big(3r_0+\varepsilon(\alpha_0^2+1)\big)\\ &= (2+3\alpha_F^4)r_0+(\alpha_0^2+\alpha_F^4(\alpha_0^2+1))\varepsilon. \end{aligned} \tag{3.10}$$

Taking $\beta := (2+3\alpha_F^4)r_0+(\alpha_0^2+\alpha_F^4(\alpha_0^2+1))\varepsilon$, we obtain

$$d_F^1(\dot\gamma(t),\dot\gamma_{\tilde{y}\tilde{z}}(t)) = \max_{s\in[0,1]} d_F(\gamma(t+s),\gamma_{\tilde{y}\tilde{z}}(t+s)) \leqslant \beta \quad \forall t\in[0,r-1].$$

□

We can now prove the theorem 3.5.

Proof of Theorem 3.5. Using inequality (2.2), we have for any $x \in K$,

$$\text{card } K_r^\varepsilon \leqslant C_\varepsilon \cdot \mathrm{vol} B_F^+\left(\tilde{x}, r+a+\frac{\varepsilon}{2}\right), \quad \text{where } C_\varepsilon := \left(\inf_{y \in X} \mathrm{vol} B_F^+(\tilde{y}, \frac{\varepsilon}{2})\right)^{-1},$$

which implies that

$$\text{card } P_r \leqslant \text{card } K^\varepsilon \cdot \text{card } K_r^\varepsilon \leqslant \text{card } K^\varepsilon \cdot C_\varepsilon \cdot \mathrm{vol} B_F^+\left(\tilde{x}, r+a+\frac{\varepsilon}{2}\right).$$

Hence

$$h_{\text{top}}(\phi_F^t, \mathcal{F}, \beta) \leqslant \overline{\lim_{r \to +\infty}} \frac{1}{r-1} \log \text{card } P_r \leqslant \overline{\lim_{r \to +\infty}} \frac{1}{r-1} \log \mathrm{vol} B_F^+\left(\tilde{x}, r+a+\frac{\varepsilon}{2}\right).$$

Since

$$\begin{aligned} \overline{\lim_{r \to +\infty}} \frac{1}{r-1} \log \mathrm{vol} B_F^+\left(\tilde{x}, r+a+\frac{\varepsilon}{2}\right) &= \lim_{r \to +\infty} \left[\frac{r+a+\frac{\varepsilon}{2}}{r-1} \cdot \frac{\log \mathrm{vol} B_F^+\left(\tilde{x}, r+a+\frac{\varepsilon}{2}\right)}{r+a+\frac{\varepsilon}{2}}\right] \\ &= h_F. \end{aligned}$$

Then $h_{\text{top}}(\phi_F^t, \mathcal{F}, \beta) \leqslant h_F$. □

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Authors' addresses:

Franck Djideme Houenou Faculty of Science and Techniques,
University of Abomey Calavi, Benin.
E-mail: rdjeam@gmail.com

Carlos Ogouyandjou and Leonard Todjihounde
Institut of Mathematics and Physical Sciences,
University of Abomey Calavi, Benin.
E-mail: ogouyandjou@imsp-uac.org, leonardt@imsp-uac.org