Differential geometric aspects of lightlike f-rectifying curves in Minkowski space-time

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Abstract. This paper is concerned with a class of null (lightlike) curves, called null f-rectifying curves, in Minkowski space-time $\mathbb{E}_1^4$. Here, f is a nowhere vanishing real-valued integrable function in pseudo-arc length parameter of concerned null curve in $\mathbb{E}_1^4$. A null f-rectifying curve in $\mathbb{E}_1^4$ is introduced as a curve γ parametrized by pseudo-arc length s in $\mathbb{E}_1^4$ such that its f-position vector field γ_f, defined by $\gamma_f(s) = \int f(s)d\gamma$, always lies in its rectifying space (i.e., the orthogonal complement ${N_\gamma}^\perp$ of its unit principal normal vector field N_γ) in $\mathbb{E}_1^4$. In particular, some characterizations and classification of such null curves in $\mathbb{E}_1^4$ are thoroughly investigated.

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1 Introduction

In a *proper pseudo-Riemannian manifold* M (i.e., a smooth manifold M endowed with an everywhere non-degenerate metric g of index ν satisfying $1 \leq \nu \leq \dim M - 1$), a tangent vector v to M can have exactly one among three *causal characters*: it is *spacelike*, *null* or *timelike* if and only if $g(v, v) > 0$, $g(v, v) = 0$ or $g(v, v) < 0$, respectively [1]. This terminology is adopted from *Theory of Relativity*. Accordingly, an arbitrary curve γ in M may locally or globally have exactly one among these three causal characters completely determined by causal characters of its velocity vectors $\gamma'(t)$. Thus, a curve γ in M is *spacelike*, *null* or *timelike* if and only if all of its velocity vectors $\gamma'(t)$ are respectively *spacelike*, *null* or *timelike* [1]. In particular, null vectors and null curves in a *Lorentz manifold* (i.e., a pseudo-Riemannian manifold endowed with a metric of index $\nu = 1$) are also known as *lightlike vectors* and *lightlike curves* respectively. In the study of null curves, the main obstacle is to normalizing their velocity vectors in the usual way because their arc length functions vanish identically. To overcome this, one needs to introduce another parameter, called *pseudo-arc length function*, which normalizes acceleration vectors of null curves. In [2], W.B. Bonnor defined the curvature functions of a null curve in Minkowski space-time $\mathbb{E}_1^4$ in terms

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of pseudo-arc length where the first curvature can have only two values: 0 if the curve is linear or 1 in the remaining cases.

In [3], B.Y. Chen introduced the notion of a *rectifying curve* in the *Euclidean 3-space* $\mathbb{E}^3$ as a tortuous space curve whose position vector field always lies in its *rectifying plane*, i.e., the plane generated by its unit tangent and unit binormal vector fields. This is equivalent to saying that a rectifying curve in $\mathbb{E}^3$ is a tortuous space curve whose position vector field always lies in the orthogonal complement of its unit principal normal vector field.

Let I denote a non-trivial interval in $\mathbb{R}$, i.e., a connected set in $\mathbb{R}$ containing at least two points. Let $\gamma : I \longrightarrow \mathbb{E}^3$ be a *unit-speed* curve (i.e., a curve parametrized by its arc length function s) possessing continuous derivatives at least up to fourth order. We consider the *Frenet apparatus* $\{T_\gamma, N_\gamma, B_\gamma, \kappa_\gamma, \tau_\gamma\}$ for the curve γ which is defined as follows: T_γ is the unit tangent vector field of γ, N_γ is the unit principal normal vector field of γ obtained by normalizing T'_γ, $B_\gamma = T_\gamma \times N_\gamma$ is the unit binormal vector field of γ such that the dynamic *Frenet frame* $\{T_\gamma, N_\gamma, B_\gamma\}$ is positive definite along γ having the same orientation as that of $\mathbb{E}^3$; $\kappa_\gamma : I \longrightarrow (0, \infty)$ is the curvature function and $\tau_\gamma : I \longrightarrow \mathbb{R}$ is the torsion of γ. The planes spanned by $\{T_\gamma, N_\gamma\}$, $\{T_\gamma, B_\gamma\}$ and $\{N_\gamma, B_\gamma\}$ are respectively called the *osculating plane*, *rectifying plane* and *normal plane* of γ in $\mathbb{E}^3$. Consequently, the rectifying plane of γ is the orthogonal complement of its unit principal normal vector field N_γ. More elaborations may be found in [3, 24, 25, 26, 27].

Thus, the position vector field of a rectifying curve $\gamma : I \longrightarrow \mathbb{E}^3$ parametrized by arc length s satisfies equation

$$\gamma(s) = \lambda(s)T_\gamma(s) + \mu(s)B_\gamma(s)$$

for all $s \in I$, where $\lambda, \mu : I \longrightarrow \mathbb{R}$ are two smooth functions in parameter s. In [3], B.Y. Chen investigated some characterizations of rectifying curves in $\mathbb{E}^3$ in terms of distance functions, tangential, normal and binormal components of their position vector fields and also in terms of ratios of their curvatures and torsions. Moreover, in the same paper, the author explored a classification of such curves in $\mathbb{E}^3$ based on a sort of dilation applied on unit-speed curves on the unit-sphere $\mathbb{S}^2(1)$. Thereafter, in [4], B.Y. Chen and F. Dillen studied rectifying curves as *extremal curves* and *centrodes* in $\mathbb{E}^3$. In particular, they established a relation between rectifying curves and centrodes in $\mathbb{E}^3$ which plays a significant role in defining the curves of constant precession in *Differential Geometry* as well as *Kinematics* or, in general, *Mechanics*.

Furthermore, several characterizations of rectifying curves in *Euclidean spaces* were inquired by B.Y. Chen in [5], by S. Deshmukh, B.Y. Chen and S. Alshamari in [6], by K. İlarslan and E. Nesovic in [7] and by S. Cambie, W. Goemans and I. Van den Bussche in [8]. Meanwhile, the notion of rectifying curves were extended to several sorts of *Riemannian* and *pseudo-Riemannian spaces*. As for example, rectifying curves in 3D sphere $\mathbb{S}^3(r)$ and 3D hyperbolic space $\mathbb{H}^3(-r)$ were studied by P. Lucas and J.A. Ortega-Yagües in [9] and [10] respectively. Again, many characterizations of rectifying curves in *Minkowski 3-space* $\mathbb{E}^3_1$ were investigated by K. İlarslan, E. Nešović and T. M. Petrović in [11] and by K. İlarslan and E. Nešović in [12], and the same for null, pseudo-null and partially null rectifying curves in *Minkowski space-time* $\mathbb{E}^4_1$ were explored by K. İlarslan and E. Nešović in [13]. Also, some characterizations of rectifying spacelike curves in $\mathbb{E}^4_1$ were investigated by T.A. Ali and M. Onder in

[14]. Moreover, some relations between normal curves and rectifying curves in $\mathbb{E}_1^4$ were presented by K. İlarslan and E. Nešović in [15]. In [16], F. Hathout introduced a new kind of curves which generalizes rectifying curves and helices in $\mathbb{E}^3$. Also, some characterizations and classification of non-null and null f-rectifying curves in *Minkowski* 3-*space* $\mathbb{E}_1^3$ were investigated by Z. Iqbal and J. Sengupta in [17] and [18] respectively. In this paper, we study null f-rectifying curves in *Minkowski space-time* $\mathbb{E}_1^4$.

We organize this paper with four sections. In the first section, we revisit some requisite basic ideas and results. Then, in the next section, we introduce the notion of null f-rectifying curves in Minkowski space-time $\mathbb{E}_1^4$, where f is a nowhere vanishing integrable real-valued function in pseudo-arc length parameter of concerned null curve. Thereafter, the third section is devoted to investigate some characterizations of null f-rectifying curves in $\mathbb{E}_1^4$ having everywhere vanishing or nowhere vanishing tangential components. Finally, in the concluding section, we attempt for some classification of such null f-rectifying curves in terms of parametrizations based on their f-position vector fields in $\mathbb{E}_1^4$.

2 Preliminaries

Minkowski space-time $\mathbb{E}_1^4$ is an 4D pseudo-Euclidean space on which metric has index $\nu = 1$, i.e., an 4D real vector space $\mathbb{R}^4$ endowed with an everywhere non-degenerate metric g having signature $(-, +, +, +)$ or $(+, -, -, -)$ [1, 22]. It describes a flat space when no mass is present. A metric g on $\mathbb{E}_1^4$ is called *Minkowski inner product* or *Lorentz inner product* or sometimes *pseudo-inner product* on $\mathbb{E}_1^4$ [1, 22]. In general, *Geometers* and *Relativists* prefer Minkowski inner product having signature $(-, +, +, +)$ while *Particle Physicists* tend to choose the same having signature $(+, -, -, -)$. Throughout this paper, we consider Minkowski space-time according to signature $(-, +, +, +)$. The Minkowski inner product g having signature $(-, +, +, +)$ is defined by

$$g(v, w) = -v_1w_1 + v_2w_2 + v_3w_3 + v_4w_4$$

for all tangent vectors $v = (v_1, v_2, v_3, v_4)$, $w = (w_1, w_2, w_3, w_4)$ to $\mathbb{E}_1^4$. Since the metric g is non-degenerate everywhere on $\mathbb{E}_1^4$, it follows that a tangent vector v to $\mathbb{E}_1^4$ can have exactly one among the following *Lorentzian causal characters* ([1]):

1. v is a *spacelike vector* if and only if $g(v, v) > 0$ or $v = 0$.
2. v is a *lightlike vector* or a *null vector* if and only if $g(v, v) = 0$ and $v \neq 0$.
3. v is a *timelike vector* if and only if $g(v, v) < 0$.

For each point $p \in \mathbb{E}_1^4$, the set of null vectors in the tangent space $T_p\mathbb{E}_1^4$ is called the *null-cone* at the point p. The *norm* of a tangent vector v to $\mathbb{E}_1^4$ having any one of three causal characters is denoted and defined by $\|v\| = \sqrt{|g(v, v)|}$. It is trivial to mention that a tangent vector v to $\mathbb{E}_1^4$ is called a *unit vector* if and only if $\|v\| = 1$, i.e., if and only if $g(v, v) = \pm 1$. As usual, two tangent vectors v and w to $\mathbb{E}_1^4$ are called *orthogonal* if and only if $g(v, w) = 0$.

A basis for $\mathbb{E}_1^4$ is a set consisting of four mutually orthogonal tangent vectors to $\mathbb{E}_1^4$. In particular, an orthonormal basis for $\mathbb{E}_1^4$ is a basis consisting of four mutually orthogonal unit tangent vectors to $\mathbb{E}_1^4$. It is necessary that an orthonormal basis for $\mathbb{E}_1^4$ must include one timelike together with three spacelike unit tangent vectors. The standard orthonormal basis for $\mathbb{E}_1^4$ is the orthonormal basis $\{e_1, e_2, e_3, e_4\}$ given by

$$e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}, \delta_{i4}) \text{ for each } i \in \{1,2,3,4\}, \quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

such that

$$g(e_1, e_1) = -1, \ \ g(e_2, e_2) = 1, \ \ g(e_3, e_3) = 1, \ \ g(e_4, e_4) = 1.$$

For any three tangent vectors $u = (u_1, u_2, u_3, u_4)$, $v = (v_1, v_2, v_3, v_4)$ and $w = (w_1, w_2, w_3, w_4)$ to $\mathbb{E}_1^4$, the *vector product* of u, v and w is the unique tangent vector to $\mathbb{E}_1^4$, denoted by $u \wedge v \wedge w$, such that for any tangent vector x to $\mathbb{E}_1^4$, the following equation is satisfied

$$g(u \wedge v \wedge w, x) = \det(u, v, w, x).$$

Here, $\det(u, v, w, x)$ is the determinant of the matrix $[u, v, w, x]$ formed by positioning the coordinates of u, v, w and x by columns with respect to the standard orthonormal basis $\{e_1, e_2, e_3, e_4\}$ for $\mathbb{E}_1^4$. Then we easily find

$$\begin{cases} g(e_1 \wedge e_2 \wedge e_3, e_4) = 1, & g(e_2 \wedge e_3 \wedge e_4, e_1) = -1, \\ g(e_3 \wedge e_4 \wedge e_1, e_2) = 1, & g(e_4 \wedge e_1 \wedge e_2, e_3) = -1. \end{cases}$$

Thus, one finds the following expression for $u \wedge v \wedge w$ in terms of coordinates of u, v and w with respect to the basis $\{e_1, e_2, e_3, e_4\}$ ([20, 21]):

$$u \wedge v \wedge w = - \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix}.$$

Let $\gamma : I \longrightarrow \mathbb{E}_1^4$ be an arbitrary curve in $\mathbb{E}_1^4$ and γ' denote its velocity vector field. Then γ may locally or globally have exactly one of the following causal characters completely determined by the causal characters of its velocity vectors $\gamma'(t)$ ([1]):

1. γ is *spacelike* if and only if all of its velocity vectors $\gamma'(t)$ are spacelike (i.e., $g(\gamma'(t), \gamma'(t)) > 0$ or $\gamma'(t) = 0$) for all possible t.
2. γ is *null* (*lightlike*) if and only if all of its velocity vectors $\gamma'(t)$ are null (lightlike) (i.e., $g(\gamma'(t), \gamma'(t)) = 0$ and $\gamma'(t) \neq 0$) for all possible t.
3. γ is *timelike* if and only if all of its velocity vectors $\gamma'(t)$ are timelike (i.e., $g(\gamma'(t), \gamma'(t)) < 0$) for all possible t.

Thus, γ is globally spacelike, null or timelike in $\mathbb{E}_1^4$ if and only if its velocity vector field γ' is respectively everywhere spacelike, null or timelike along γ. If γ is a non-null (spacelike or timelike) curve in $\mathbb{E}_1^4$ and we change an arbitrary parameter t by arc length function $s = s(t)$ starting at some $t_0 \in I$ given by $s(t) = \int_{t_0}^{t} \|\gamma'(u)\| \, du$ such that $g(\gamma'(s), \gamma'(s)) = \pm 1$ for all possible s, then γ is said to be parametrized by arc length s or unit-speed in $\mathbb{E}_1^4$. Again, if γ is a null curve in $\mathbb{E}_1^4$ with $g(\gamma''(t), \gamma''(t)) \neq 0$ for all $t \in I$ and we change an arbitrary parameter t by pseudo-arc length function $s = s(t)$ starting at some $t_0 \in I$ given by $s(t) = \int_{t_0}^{t} \sqrt{\|\gamma''(u)\|}\, du$ such that $g(\gamma''(s), \gamma''(s)) = 1$ for all possible s, then γ is said to be parametrized by pseudo-arc length s or unit-speed in $\mathbb{E}_1^4$.

Let $\gamma : I \longrightarrow \mathbb{E}_1^4$ be a unit-speed null curve (parametrized by pseudo-arc length s) in $\mathbb{E}_1^4$. Then its velocity vector field γ' is a null vector field and its acceleration vector field γ'' is normalized by the pseudo-arc length s, i.e., $g(\gamma''(s), \gamma''(s)) = 1$ for all $s \in I$ unless γ is linear. We consider the Frenet apparatus $\{T_\gamma, N_\gamma, B_{\gamma_1}, B_{\gamma_2}, \kappa_{\gamma_1}, \kappa_{\gamma_2}, \kappa_{\gamma_3}\}$ for the curve γ in $\mathbb{E}_1^4$ which is defined as follows:

- $T_\gamma = \gamma'$ is the tangent vector field of γ which is a null vector field along γ.
- N_γ is the principal normal vector field of γ which is a spacelike unit vector field along γ such that $N_\gamma = T_\gamma'$ unless the curve γ is linear.
- B_{γ_1} is the first binormal vector field of γ which is a null vector field along γ and is completely determined by $g\left(T_\gamma(s), B_{\gamma_1}(s)\right) = 1$ for all $s \in I$.
- B_{γ_2} is the second binormal vector field of γ which is the unique spacelike unit vector field along γ orthogonal to each three dimensional subspace of $\mathbb{E}_1^4$ spanned by $\left\{T_\gamma(s), N_\gamma(s), B_{\gamma_1}(s)\right\}$ such that $\{T_\gamma, N_\gamma, B_{\gamma_1}, B_{\gamma_2}\}$ forms the dynamic Frenet frame along γ having the same orientation as that of $\mathbb{E}_1^4$. That is, B_{γ_2} can have the expression $B_{\gamma_2} = \epsilon\, T_\gamma \wedge N_\gamma \wedge B_{\gamma_1}$, where ϵ can take value $+1$ or -1 to make the Frenet frame $\{T_\gamma, N_\gamma, B_{\gamma_1}, B_{\gamma_2}\}$ positively oriented along γ.
- κ_{γ_1}, κ_{γ_2} and κ_{γ_3} denote the first curvature, second curvature and the third curvature of the curve γ in $\mathbb{E}_1^4$, respectively.

Then the Frenet formulae for the curve γ in $\mathbb{E}_1^4$ are given by [2, 19]

$$\begin{pmatrix} T_\gamma' \\ N_\gamma' \\ B_{\gamma_1}' \\ B_{\gamma_2}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_{\gamma_1} & 0 & 0 \\ \kappa_{\gamma_2} & 0 & -\kappa_{\gamma_1} & 0 \\ 0 & -\kappa_{\gamma_2} & 0 & \kappa_{\gamma_3} \\ -\kappa_{\gamma_3} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T_\gamma \\ N_\gamma \\ B_{\gamma_1} \\ B_{\gamma_2} \end{pmatrix}, \tag{2.1}$$

where

$$\begin{cases} g(T_\gamma(s), T_\gamma(s)) = 0, & g(T_\gamma(s), N_\gamma(s)) = 0, \\ g\left(T_\gamma(s), B_{\gamma_1}(s)\right) = 1, & g\left(T_\gamma(s), B_{\gamma_2}(s)\right) = 0, \\ g(N_\gamma(s), N_\gamma(s)) = 1, & g\left(N_\gamma(s), B_{\gamma_1}(s)\right) = 0, \\ g\left(N_\gamma(s), B_{\gamma_2}(s)\right) = 0, & g\left(B_{\gamma_1}(s), B_{\gamma_1}(s)\right) = 0, \\ g\left(B_{\gamma_1}(s), B_{\gamma_2}(s)\right) = 0, & g\left(B_{\gamma_2}(s), B_{\gamma_2}(s)\right) = 1 \end{cases} \tag{2.2}$$

for all $s \in I$. In this case, the first curvature κ_{γ_1} can take only two values: $\kappa_{\gamma_1} \equiv 0$ if γ is linear or $\kappa_{\gamma_1} \equiv 1$ in the remaining cases [2]. Thus, γ has only two significant curvatures: the second curvature κ_{γ_2} and the third curvature κ_{γ_3}. The second curvature κ_{γ_2} has less obvious geometrical significance whereas the third curvature κ_{γ_3} performs a role analogous to the third curvature in the Euclidean 4-space $\mathbb{E}^4$ [23]. It follows that $\kappa_{\gamma_3} \equiv 0$ if and only if the curve γ lies wholly in a hypersurface in $\mathbb{E}_1^4$. This is equivalent to saying that $\kappa_{\gamma_3} \not\equiv 0$ if and only if the curve γ lies wholly in $\mathbb{E}_1^4$. There exist null helices having everywhere vanishing second curvature in $\mathbb{E}_1^4$. If γ has everywhere vanishing second and third curvatures (i.e., $\kappa_{\gamma_2} \equiv 0$ and $\kappa_{\gamma_3} \equiv 0$), then γ is a parametrization of the null cubic in $\mathbb{E}_1^4$. Furthermore, if γ has constant second curvature and nowhere vanishing constant third curvature (i.e., κ_{γ_2} is a constant and κ_{γ_3} is a non-zero constant), then γ is a null helix lying on a circular cylinder in $\mathbb{E}_1^4$ [2, 19]. Throughout this paper, we consider null curves in $\mathbb{E}_1^4$ having first curvature $\kappa_\gamma \equiv 1$, (vanishing or non-vanishing) second curvature κ_{γ_2} and nowhere vanishing third curvature κ_{γ_3}.

We recall that the hypersurface in $\mathbb{E}_1^4$ defined by

$$\mathbb{S}_1^3(1) := \left\{v \in \mathbb{E}_1^4 : g(v,v) = 1\right\}$$

is called the *pseudo-sphere* of unit radius with centre at the origin in $\mathbb{E}_1^4$, and the hypersurface in $\mathbb{E}_1^4$ defined by

$$\mathbb{H}_0^3(1) := \left\{v \in \mathbb{E}_1^4 : g(v,v) = -1\right\}$$

is called the *pseudo-hyperbolic space* of unit radius with centre at the origin in $\mathbb{E}_1^4$. Both the hypersurfaces $\mathbb{S}_1^3(1)$ and $\mathbb{H}_0^3(1)$ are the *central hyperquadrics* in $\mathbb{E}_1^4$ [1]. We also recall that the rectifying space of γ in $\mathbb{E}_1^4$ is the orthogonal complement ${N_\gamma}^\perp$ of its unit principal normal vector field N_γ in $\mathbb{E}^4$ defined by

$${N_\gamma}^\perp := \left\{v \in \mathbb{E}_1^4 : g\left(v, N_\gamma\right) = 0\right\}.$$

Consequently, ${N_\gamma}^\perp$ at each point $\gamma(s)$ on γ is a three dimensional subspace of $\mathbb{E}_1^4$ spanned by $\left\{T_\gamma(s), B_{\gamma_1}(s), B_{\gamma_2}(s)\right\}$. Since γ is a null curve in $\mathbb{E}_1^4$, it follows that N_γ is a spacelike vector field along γ. Thus ${N_\gamma}^\perp$ at each point $\gamma(s)$ on γ is a three dimensional timelike subspace of $\mathbb{E}_1^4$.

3 Null f-rectifying curves in $\mathbb{E}_1^4$

Let $\gamma : I \longrightarrow \mathbb{E}_1^4$ be a unit-speed null curve (parametrized by pseudo-arc length s) with Frenet apparatus $\{T_\gamma, N_\gamma, B_{\gamma_1}, B_{\gamma_2}, \kappa_{\gamma_1}, \kappa_{\gamma_2}, \kappa_{\gamma_3}\}$. Then $\gamma : I \longrightarrow \mathbb{E}_1^4$ is a rectifying curve in $\mathbb{E}_1^4$ if and only if its position vector field always lies in its rectifying space, i.e., if and only if its position vector field satisfies equation

$$\gamma(s) = \lambda(s)T_\gamma(s) + \mu_1(s)B_{\gamma_1}(s) + \mu_2(s)B_{\gamma_2}(s)$$

for all $s \in I$, where $\lambda, \mu_1, \mu_2 : I \longrightarrow \mathbb{R}$ are smooth functions in parameter s ([13]).

Now, for some nowhere vanishing integrable function $f : I \longrightarrow \mathbb{R}$ in parameter s, the f-position vector field of the curve γ in $\mathbb{E}_1^4$ is denoted by γ_f and is defined by

$$\gamma_f(s) := \int f(s)\, d\gamma$$

for all $s \in I$. Here the integral sign is used in this sense that after differentiating previous equation, one finds

$$\gamma_f'(s) = f(s)T_\gamma(s)$$

for all $s \in I$, and so γ_f is an integral curve of fT_γ. Using this notion of f-position vector field of a curve in $\mathbb{E}_1^4$, we define an f-rectifying curve which is null in $\mathbb{E}_1^4$ as follows:

Definition 3.1. Let $\gamma : I \longrightarrow \mathbb{E}_1^4$ be a unit-speed null curve (parametrized by pseudo-arc length s) with Frenet apparatus $\{T_\gamma, N_\gamma, B_{\gamma_1}, B_{\gamma_2}, \kappa_{\gamma_1}(\equiv 1), \kappa_{\gamma_2}, \kappa_{\gamma_3}(\not\equiv 0)\}$ and let $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function in parameter s. The curve γ is called an f-rectifying curve in $\mathbb{E}_1^4$ if its f-position vector field $\gamma_f = \int f \, d\gamma$ always lies in its timelike rectifying space ${N_\gamma}^\perp$ in $\mathbb{E}_1^4$, i.e., if its f-position vector field $\gamma_f = \int f \, d\gamma$ satisfies equation

$$\gamma_f(s) = \lambda(s) T_\gamma(s) + \mu_1(s) B_{\gamma_1}(s) + \mu_2(s) B_{\gamma_2}(s) \tag{3.1}$$

for all $s \in I$, where $\lambda, \mu_1, \mu_2 : I \longrightarrow \mathbb{R}$ are three smooth functions in parameter s.

Remark 3.2. In particular, if f is a non-zero constant function on I, then, up to isometries of $\mathbb{E}_1^4$, a null f-rectifying curve $\gamma : I \longrightarrow \mathbb{E}_1^4$ is congruent to a null rectifying curve in $\mathbb{E}_1^4$ and the study coincides with the same incorporated in [13].

4 Some characterizations of null f-rectifying curves in $\mathbb{E}_1^4$

In many papers, several interesting results were found primarily attempting towards characterizations of rectifying curves. For example, in three dimensional Euclidean space [3, 4, 6] or Minkowski space [11, 12], some characterizations of rectifying curves are found in terms of components of their position vector fields and ratios of their curvatures and torsions. In higher dimensional Euclidean spaces [7, 8] or in Minkowski space-time [13, 14], characterizations of rectifying curves are mostly based on components of their position vector fields and their curvatures. In this section, we characterize unit-speed null f-rectifying curves in $\mathbb{E}_1^4$ in terms of their curvatures and components of their f-position vector fields.

First, in the following theorem, we establish a necessary as well as sufficient condition for a unit-speed null curve to be an f-rectifying curve in $\mathbb{E}_1^4$.

Theorem 4.1. *Let $\gamma : I \longrightarrow \mathbb{E}_1^4$ be a unit-speed null curve (parametrized by pseudo-arc length s) having first curvature $\kappa_{\gamma_1} \equiv 1$, second curvature κ_{γ_2} and nowhere vanishing third curvature κ_{γ_3}. Also, let $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function in parameter s. Then, up to isometries of $\mathbb{E}_1^4$, γ is congruent to an f-rectifying curve in $\mathbb{E}_1^4$ if and only if for each $s \in I$, the following equation is satisfied:*

$$c_1 \, {\kappa_{\gamma_2}}'(s) + \left(c_1 \int \kappa_{\gamma_3}(s) \, ds - c_2 \right) \kappa_{\gamma_3}(s) = f(s), \tag{4.1}$$

where c_1 is a constant and c_2 is a non-zero constant.

Proof. Let us first assume that $\gamma : I \longrightarrow \mathbb{E}_1^4$ be a unit speed null f-rectifying curve having first curvature $\kappa_{\gamma_1} \equiv 1$, second curvature κ_{γ_2} and nowhere vanishing third curvature κ_{γ_3}, where $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in parameter s. Then its f-position vector field γ_f satisfies equation (3.1) for some smooth functions $\lambda, \mu_1, \mu_2 : I \longrightarrow \mathbb{R}$ in parameter s. Differentiating (3.1) and then applying (2.1), we obtain

$$\begin{aligned} f(s)T_\gamma(s) \quad = \quad & \big(\lambda'(s) - \mu_2(s)\kappa_{\gamma_3}(s)\big)\, T_\gamma(s) + \big(\lambda(s) - \mu_1(s)\kappa_{\gamma_2}(s)\big)\, N_\gamma(s) \\ & + \mu_1'(s)B_{\gamma_1}(s) + \big(\mu_2'(s) + \mu_1(s)\kappa_{\gamma_3}(s)\big)\, B_{\gamma_2}(s) \end{aligned} \tag{4.2}$$

for all $s \in I$. Equating the coefficients of like-terms from both sides of (4.2), we get

$$\begin{cases} \lambda'(s) - \mu_2(s)\kappa_{\gamma_3}(s) = f(s), \\ \lambda(s) - \mu_1(s)\kappa_{\gamma_2}(s) = 0, \\ \mu_1'(s) = 0, \\ \mu_2'(s) + \mu_1(s)\kappa_{\gamma_3}(s) = 0 \end{cases} \tag{4.3}$$

for all $s \in I$. From the last three equations of (4.3), we find

$$\begin{cases} \lambda(s) = c_1\, \kappa_{\gamma_2}(s), \\ \mu_1(s) = c_1, \\ \mu_2(s) = -c_1 \displaystyle\int \kappa_{\gamma_3}(s)\, ds + c_2 \end{cases} \tag{4.4}$$

for all $s \in I$, where c_1 is a constant and c_2 is a non-zero constant. Substituting the first and third of relations (4.4) in the first one of equations (4.3), we obtain our desired equation (4.1).

Conversely, we assume that $\gamma : I \longrightarrow \mathbb{E}^4_1$ is a unit-speed null curve having first curvature $\kappa_{\gamma_1} \equiv 1$, second curvature κ_{γ_2} and nowhere vanishing third curvature κ_{γ_3}, and $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in parameter s such that for each $s \in I$, equation (4.1) is satisfied. We define a vector field V along γ by

$$V(s) = \gamma_f(s) - c_1\, \kappa_{\gamma_2}(s)\, T_\gamma(s) - c_1\, B_{\gamma_1}(s) - \left(-c_1 \int \kappa_{\gamma_3}(s)\, ds + c_2\right) B_{\gamma_2}(s)$$

for all $s \in I$. Differentiating the previous equation and then applying (2.1) and (4.1), we find that $V'(s) = 0$ for all $s \in I$. Consequently, V is a constant vector field along γ. Hence, up to isometries of $\mathbb{E}^4_1$, γ is congruent to an f-rectifying curve in $\mathbb{E}^4_1$. $\square$

Remark 4.1. Since $g\big(T_\gamma(s), B_{\gamma_1}(s)\big) = 1$ for all $s \in I$, by using (3.1) and (4.4), the tangential component $g(\gamma_f, T_\gamma)$ of the f-position vector field γ_f is obtained by

$$g\left(\gamma_f(s), T_\gamma(s)\right) = \mu_1(s) = c_1$$

for all $s \in I$, where c_1 is a constant. Consequently, the following two cases come up for consideration:

Case I. If $c_1 = 0$, then the tangential component $g(\gamma_f, T_\gamma)$ of γ_f vanishes everywhere.

Case II. If $c_1 \neq 0$, then the tangential component $g(\gamma_f, T_\gamma)$ of γ_f vanishes nowhere.

Accordingly, in the following theorem, we exhibit some characterizations of unit speed null f-rectifying curves in $\mathbb{E}^4_1$ having everywhere vanishing tangential component.

Theorem 4.2. *Let $\gamma : I \longrightarrow \mathbb{E}_1^4$ be a unit-speed null curve (parametrized by pseudo-arc length s) having first curvature $\kappa_{\gamma_1} \equiv 1$, second curvature κ_{γ_2} and nowhere vanishing third curvature κ_{γ_3}. Also, let $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function in parameter s. If γ is an f-rectifying curve in $\mathbb{E}_1^4$ having everywhere vanishing tangential component (i.e. $g(\gamma_f, T_\gamma) \equiv 0$), then the following statements are true:*

1. *γ has everywhere spacelike f-position vector field γ_f and the norm function $\rho = \|\gamma_f\|$ is a positive constant.*
2. *The third curvature κ_{γ_3} of γ is a non-zero constant multiple of the function f.*
3. *The first binormal component $g(\gamma_f, B_{\gamma_1})$ of γ_f vanishes everywhere and the second binormal component $g(\gamma_f, B_{\gamma_2})$ of γ_f is a non-zero constant given by*

$$g\left(\gamma_f(s), B_{\gamma_2}(s)\right) = -\frac{f(s)}{\kappa_{\gamma_3}(s)} \tag{4.5}$$

for all $s \in I$.

Conversely, if $\gamma : I \longrightarrow \mathbb{E}_1^4$ is a unit-speed null curve having first curvature $\kappa_{\gamma_1} \equiv 1$, second curvature κ_{γ_2} and nowhere vanishing third curvature κ_{γ_3}, and if $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in parameter s and any one of the statements (1), (2) or (3) is true, then γ is an f-rectifying curve in $\mathbb{E}_1^4$ having everywhere vanishing tangential component.

Proof. We first assume that $\gamma : I \longrightarrow \mathbb{E}_1^4$ is a unit-speed null f-rectifying curve having first curvature $\kappa_{\gamma_1} \equiv 1$, second curvature κ_{γ_2}, nowhere vanishing third curvature κ_{γ_3} and having everywhere vanishing tangential component (i.e. $g(\gamma_f, T_\gamma) \equiv 0$), where $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in parameter s. Then its f-position vector field γ_f satisfies equation (3.1) for some smooth functions $\lambda, \mu_1, \mu_2 : I \longrightarrow \mathbb{R}$. Also, from the proof of Theorem 4.1, we have sets of expressions (4.3) and (4.4). Since $g(\gamma_f, T_\gamma) \equiv 0$, it follows that $c_1 = 0$ and therefore relations (4.4) reduce to

$$\lambda(s) = 0,\ \mu_1(s) = 0 \text{ and } \mu_2(s) = c_2 \tag{4.6}$$

for all $s \in I$, where $c_2 \neq 0$ is a constant. Then equation (3.1) becomes

$$\gamma_f(s) = c_2 B_{\gamma_2}(s)$$

for all $s \in I$. We obtain the following:

1. Using the last equation, we find

$$g\left(\gamma_f(s), \gamma_f(s)\right) = c_2^2 > 0$$

for all $s \in I$. This implies that γ_f is everywhere spacelike and the norm function $\rho = \|\gamma_f\|$ is a positive constant.

2. Substituting (4.6) in the first equation of (4.3), we obtain

$$\kappa_{\gamma_3}(s) = -\frac{1}{c_2} f(s) \tag{4.7}$$

for all $s \in I$, where $c_2 \neq 0$ is a constant. This proves that $\kappa_{\gamma 3}$ is a non-zero constant multiple of f.

3. Using (2.2), (3.1) and (4.6), the first binormal component $g(\gamma_f, B_{\gamma 1})$ of γ_f is given by

$$g\left(\gamma_f(s), B_{\gamma 1}(s)\right) = \lambda(s) = 0$$

for all $s \in I$, and the second binormal component $g(\gamma_f, B_{\gamma 2})$ of γ_f is given by

$$g\left(\gamma_f(s), B_{\gamma 2}(s)\right) = \mu_2(s) = c_2$$

for all $s \in I$. Substituting (4.7) in the previous equation, we obtain (4.5).

Conversely, we assume that $\gamma : I \longrightarrow \mathbb{E}_1^4$ is a unit-speed null curve having first curvature $\kappa_{\gamma 1} \equiv 1$, second curvature $\kappa_{\gamma 2}$ and nowhere vanishing third curvature $\kappa_{\gamma 3}$, $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in parameter s and the statement (1) is true. Then there exists a constant $c > 0$ such that

$$g\left(\gamma_f(s), \gamma_f(s)\right) = c$$

for all $s \in I$. Differentiating and using the nowhere vanishing nature of f, we obtain

$$g\left(\gamma_f(s), T_\gamma(s)\right) = 0$$

for all $s \in I$. Differentiating again and applying the Frenet formulae (2.1), we find

$$g\left(\gamma_f(s), N_\gamma(s)\right) = 0$$

for all $s \in I$. This implies that γ_f lies in the rectifying space ${N_\gamma}^\perp$ of γ in $\mathbb{E}_1^4$. Hence γ is an f-rectifying curve in $\mathbb{E}_1^4$ having everywhere vanishing tangential component.

Next, we assume that the statement (2) is true. Then there exists a non-zero constant b such that $\kappa_{\gamma 3}$ can be expressed as

$$\kappa_{\gamma 3}(s) = b f(s)$$

for all $s \in I$. Now, applying Frenet formulae (2.1) in the previous equation, after some computations, one finds

$$\frac{d}{ds}\left(\gamma_f(s) + \frac{1}{b}\, B_{\gamma 2}(s)\right) = 0$$

for all $s \in I$. Then, up to isometries of $\mathbb{E}_1^4$, γ_f has the expression

$$\gamma_f(s) = -\frac{1}{b}\, B_{\gamma 2}(s)$$

for all $s \in I$. Therefore, using last equation and (2.2), we obtain for all $s \in I$,

$$g\left(\gamma_f(s), T_\gamma(s)\right) = 0, \quad g\left(\gamma_f(s), N_\gamma(s)\right) = 0.$$

Consequently, up to isometries of $\mathbb{E}_1^4$, γ is congruent to an f-rectifying curve in $\mathbb{E}_1^4$ having everywhere vanishing tangential component.

Finally, we assume that the statement (3) is true. Then we have

$$g\left(\gamma_f(s), B_{\gamma_1}(s)\right) = 0, \tag{4.8}$$

$$g\left(\gamma_f(s), B_{\gamma_2}(s)\right) = -\frac{f(s)}{\kappa_{\gamma_3}(s)} \tag{4.9}$$

for all $s \in I$. Differentiating (4.8) and using (2.1), (2.2), (4.9), we obtain for all $s \in I$,

$$g\left(\gamma_f(s), N_\gamma(s)\right) = 0.$$

This implies that γ_f lies in the rectifying space ${N_\gamma}^\perp$ of γ in $\mathbb{E}_1^4$. Again, differentiating (4.9) and applying Frenet formulae (2.1), we get

$$g\left(\gamma_f(s), T_\gamma(s)\right) = 0$$

for all $s \in I$. Hence γ is an f-rectifying curve in $\mathbb{E}_1^4$ having everywhere vanishing tangential component. □

In the next theorem, we explore some characterizations of unit speed null f-rectifying curves in $\mathbb{E}_1^4$ having nowhere vanishing tangential component.

Theorem 4.3. *Let $\gamma : I \longrightarrow \mathbb{E}_1^4$ be a unit-speed null curve (parametrized by pseudo-arc length s) having first curvature $\kappa_{\gamma_1} \equiv 1$, second curvature κ_{γ_2} and nowhere vanishing third curvature κ_{γ_3}. Also, let $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function in parameter s having a primitive function F. If γ is an f-rectifying curve in $\mathbb{E}_1^4$ having nowhere vanishing tangential component (i.e. $g(\gamma_f, T_\gamma) \not\equiv 0$), then the following statements are true:*

1. *The norm function $\rho = \|\gamma_f\|$ is given by*

$$\rho(s) = \sqrt{|2\, c_1 F(s)|} \tag{4.10}$$

for all $s \in I$, where $c_1 \neq 0$ is a constant.

2. *The first binormal component $g(\gamma_f, B_{\gamma_1})$ and the second binormal component $g(\gamma_f, B_{\gamma_2})$ of γ_f are respectively given by*

$$\begin{cases} g\left(\gamma_f(s), B_{\gamma_1}(s)\right) = c_1 \kappa_{\gamma_2}(s), \\ g\left(\gamma_f(s), B_{\gamma_2}(s)\right) = \dfrac{c_1 {\kappa_{\gamma}}_2'(s) - f(s)}{\kappa_{\gamma_3}(s)} \end{cases} \tag{4.11}$$

for all $s \in I$, where $c_1 \neq 0$ is a constant.

3. *The third curvature κ_{γ_3} of γ is given by*

$$\kappa_{\gamma_3}(s) = \frac{\epsilon\, c_1 {\kappa_{\gamma}}_2'(s) - \epsilon\, f(s)}{\sqrt{2\, c_1 \left(F(s) - c_1 \kappa_{\gamma_2}(s)\right)}} \tag{4.12}$$

for all $s \in I$, where $c_1 \neq 0$ is a constant and ϵ can take value $+1$ or -1 depending strictly on sign of the second binormal component $g(\gamma_f, B_{\gamma_2})$ of γ_f.

Conversely, if $\gamma : I \longrightarrow \mathbb{E}_1^4$ is a unit-speed null curve having first curvature $\kappa_{\gamma_1} \equiv 1$, second curvature κ_{γ_2} and nowhere vanishing third curvature κ_{γ_3}, and if $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in parameter s having a primitive function F and any one of the statements (1), (2) or (3) is true, then γ is an f-rectifying curve in $\mathbb{E}_1^4$ having nowhere vanishing tangential component.

Proof. First, we assume that for some nowhere vanishing integrable function $f : I \longrightarrow \mathbb{R}$ having a primitive function F, $\gamma : I \longrightarrow \mathbb{E}_1^4$ is a unit-speed null f-rectifying curve (parametrized by pseudo-arc length s) having first curvature $\kappa_{\gamma_1} \equiv 1$, second curvature κ_{γ_2}, nowhere vanishing third curvature κ_{γ_3} and having nowhere vanishing tangential component (i.e. $g(\gamma_f, T_\gamma) \not\equiv 0$). Then its f-position vector field γ_f satisfies equation (3.1) for some smooth functions $\lambda, \mu_1, \mu_2 : I \longrightarrow \mathbb{R}$ in parameter s. Also, from the proof of Theorem 4.1, we have (4.3) and (4.4). Since $g(\gamma_f, T_\gamma) \neq 0$, it follows that $c_1 \neq 0$. Now, from (4.3) and (4.4), we can write

$$\lambda'(s)\mu_1(s) + \lambda(s)\mu_1'(s) + \mu_2(s)\mu_2'(s) = c_1 f(s)$$

for all $s \in I$. Integrating the last equation with respect to s, we find

$$2\,\lambda(s)\mu_1(s) + \mu_2^2(s) = 2\,c_1 \int f(s)\,ds = 2\,c_1 F(s) \tag{4.13}$$

for all $s \in I$. Again, substituting first two relations of (4.4) in the first equation of (4.3), we easily compute

$$\mu_2(s) = \frac{c_1 {\kappa_{\gamma_2}}'(s) - f(s)}{\kappa_{\gamma_3}(s)} \tag{4.14}$$

for all $s \in I$. We obtain the following:

1. Using (2.2), (3.1) and (4.13), the norm function $\rho = \|\gamma_f\|$ is given by

$$\rho^2(s) = \|\gamma_f(s)\|^2 = |g(\gamma_f(s), \gamma_f(s))| = \left|2\,\lambda(s)\mu_1(s) + \mu_2^2(s)\right| = |2\,c_1 F(s)|$$

for all $s \in I$, that is,

$$\rho(s) = \sqrt{|2\,c_1 F(s)|}$$

for all $s \in I$, where $c_1 \neq 0$ is a constant.

2. Using (2.2), (3.1) and (4.4), the first binormal component $g(\gamma_f, B_{\gamma_1})$ of γ_f is given by

$$g\left(\gamma_f(s), B_{\gamma_1}(s)\right) = \lambda(s) = c_1 \kappa_{\gamma_2}(s)$$

for all $s \in I$. Again, using (2.2), (3.1) and (4.14), the second binormal component $g(\gamma_f, B_{\gamma_2})$ of γ_f is given by

$$g\left(\gamma_f(s), B_{\gamma_2}(s)\right) = \mu_2(s) = \frac{c_1 {\kappa_{\gamma_2}}'(s) - f(s)}{\kappa_{\gamma_3}(s)}$$

for all $s \in I$.

3. Applying the first two relations of (4.4) in (4.13), we obtain

$$\mu_2^2(s) = 2\,c_1 \left(F(s) - c_1 \kappa_{\gamma_2}(s)\right)$$

for all $s \in I$. Since $\mu_2^2(s) \geq 0$ for all $s \in I$, it follows that

$$\mu_2(s) = \epsilon \sqrt{2\, c_1 \left(F(s) - c_1 \kappa_{\gamma_2}(s)\right)} \tag{4.15}$$

for all $s \in I$, where ϵ can take value $+1$ or -1 depending strictly on sign of the second binormal component $g(\gamma_f, B_{\gamma_2})$ of γ_f. Comparing the previous equation with the last one of (4.4), we have

$$-c_1 \int \kappa_{\gamma_3}(s)\, ds + c_2 = \epsilon \sqrt{2\, c_1 \left(F(s) - c_1 \kappa_{\gamma_2}(s)\right)}$$

for all $s \in I$. Differentiating the last equation, we find (4.12).

Conversely, we assume that $\gamma : I \longrightarrow \mathbb{E}_1^4$ is a unit-speed null curve having first curvature $\kappa_{\gamma_1} \equiv 1$, second curvature κ_{γ_2} and nowhere vanishing third curvature κ_{γ_3} such that for some nowhere vanishing integrable function $f : I \longrightarrow \mathbb{R}$ in parameter s having a primitive function F, the statement (1) is true. Then we must have

$$g\left(\gamma_f(s), \gamma_f(s)\right) = 2\, c_1 F(s)$$

for all $s \in I$, where $c_1 \neq 0$ is a constant. Differentiating and using the nowhere vanishing nature of f, we obtain

$$g\left(\gamma_f(s), T_\gamma(s)\right) = c_1$$

for all $s \in I$. Again, differentiating and applying Frenet formulae (2.1), we obtain for all $s \in I$,

$$g\left(\gamma_f(s), N_\gamma(s)\right) = 0.$$

This implies that γ_f lies in the rectifying space ${N_\gamma}^\perp$ of γ in $\mathbb{E}_1^4$. Hence γ is an f-rectifying curve in $\mathbb{E}_1^4$ having nowhere vanishing tangential component.

Next, we assume that the statement (2) is true. Then differentiating the first one of (4.11) and applying (2.1), (2.2) and the second one of (4.11), after simplification, we find for all $s \in I$,

$$g\left(\gamma_f(s), N_\gamma(s)\right) = 0.$$

This implies that γ_f lies in the rectifying space ${N_\gamma}^\perp$ of γ in $\mathbb{E}_1^4$. Furthermore, the tangential component $g\left(\gamma_f, T_\gamma\right)$ of γ_f is given by

$$g\left(\gamma_f(s), T_\gamma(s)\right) = c_1 \neq 0$$

for all $s \in I$. Hence γ is an f-rectifying curve in $\mathbb{E}_1^4$ having nowhere vanishing tangential component.

Finally, we assume that the statement (3) is true so that the third curvature κ_{γ_3} of γ is given by (4.12). Then the second binormal component $g(\gamma_f, B_{\gamma_2}) \equiv \mu_2$ of γ_f satisfies (4.15). Now, we define a vector field W along γ by

$$W(s) = \gamma_f(s) - c_1\, \kappa_{\gamma_2}(s)\, T_\gamma(s) - c_1\, B_{\gamma_1}(s) - \left(\epsilon \sqrt{2\, c_1 \left(F(s) - c_1 \kappa_{\gamma_2}(s)\right)}\right) B_{\gamma_2}(s)$$

for all $s \in I$, where $c_1 \neq 0$ is a constant. Differentiating the previous equation and then applying (2.1) and (4.12), we find that $W'(s) = 0$ for all $s \in I$. This implies that W is a constant vector field along γ. Hence, up to isometries of $\mathbb{E}_1^4$, γ is congruent to an f-rectifying curve in $\mathbb{E}_1^4$ having nowhere vanishing tangential component. This completes the proof. □

5 Classification of null f-rectifying curves in $\mathbb{E}_1^4$

In this section, we attempt for some classification of null f-rectifying curves in $\mathbb{E}_1^4$ based on the parametrizations of their f-position vector fields.

First, in the following theorem which is an immediate consequence of Theorem 4.2, we give a classification of null f-rectifying curves in $\mathbb{E}_1^4$ having everywhere vanishing tangential component.

Theorem 5.1. *Let $\gamma : I \longrightarrow \mathbb{E}_1^4$ be a unit-speed null curve (parametrized by pseudo-arc length s) having first curvature $\kappa_{\gamma_1} \equiv 1$, second curvature κ_{γ_2} and nowhere vanishing third curvature κ_{γ_3}. Also, let $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function in parameter s. Then γ is an f-rectifying curve in $\mathbb{E}_1^4$ having everywhere vanishing tangential component if and only if, up to a parametrization, its f-position vector field γ_f is given by*

$$\gamma_f(t) = c\,\alpha(t) \tag{5.1}$$

for all $t \in J$, where $c > 0$ is a constant and $\alpha : J \longrightarrow \mathbb{S}_1^3(1)$ is a unit-speed null curve having $t : I \longrightarrow J$ as pseudo-arc length function.

Finally, the following theorem is concerned with a classification of null f-rectifying curves in $\mathbb{E}_1^4$ having nowhere vanishing tangential component.

Theorem 5.2. *Let $\gamma : I \longrightarrow \mathbb{E}_1^4$ be a unit-speed null curve (parametrized by pseudo-arc length s) having first curvature $\kappa_{\gamma_1} \equiv 1$, second curvature κ_{γ_2} and nowhere vanishing third curvature κ_{γ_3}. Also, let $f : I \longrightarrow \mathbb{R}$ be a nowhere vanishing integrable function in parameter s having a primitive function F. Then γ is an f-rectifying curve in $\mathbb{E}_1^4$ having spacelike (or timelike) f-position vector field and nowhere vanishing tangential component if and only if, up to a parametrization, its f-position vector field γ_f is given by*

$$\gamma_f(t) = \sqrt{2\,c_1 F(s_0)}\,\exp(t)\,\alpha(t) \tag{5.2}$$

for all $t \in J$, where $c_1 \neq 0$ is a constant and $s_0 \in I$ such that $c_1 F(s_0) > 0$ and $\alpha : J \longrightarrow \mathbb{S}_1^3(1)$ is a unit-speed timelike curve (or $\alpha : J \longrightarrow \mathbb{H}_0^3(1)$ is a unit-speed spacelike curve) having $t : I \longrightarrow J$ as arc length function starting at s_0.

Proof. We first assume that $\gamma : I \longrightarrow \mathbb{E}_1^4$ is a unit-speed null f-rectifying curve having spacelike f-position vector field and nowhere vanishing tangential component, where $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in parameter s with a primitive function F. Then we have

$$g\left(T_\gamma(s), T_\gamma(s)\right) = 0, \quad g\left(\gamma_f(s), \gamma_f(s)\right) > 0, \quad g\left(\gamma_f(s), T_\gamma(s)\right) \neq 0$$

for all $s \in I$. Therefore, by applying Theorem 4.3, we obtain

$$\rho^2(s) = g\left(\gamma_f(s), \gamma_f(s)\right) = 2\,c_1 F(s) > 0$$

for all $s \in I$, where $c_1 \neq 0$ is a constant. Now, we define a curve $\alpha = \alpha(s)$ by

$$\alpha(s) := \frac{1}{\rho(s)}\,\gamma_f(s)$$

for all $s \in I$. Then we have

$$\gamma_f(s) = \sqrt{2\,c_1 F(s)}\,\alpha(s) \tag{5.3}$$

for all $s \in I$. Also, we have

$$g\left(\alpha(s), \alpha(s)\right) = \frac{g\left(\gamma_f(s), \gamma_f(s)\right)}{\rho^2(s)} = 1$$

for all $s \in I$. This implies that α is a curve in $\mathbb{S}^3_1(1)$. Differentiating the previous equation, we obtain

$$g\left(\alpha(s), \alpha'(s)\right) = 0$$

for all $s \in I$. Again, differentiating (5.3) and applying last two equations, we easily find for all $s \in I$,

$$g\left(\alpha'(s), \alpha'(s)\right) = -\frac{f^2(s)}{4F^2(s)}$$

This indicates that α is a timelike curve. Using the previous equation, we compute

$$\|\alpha'(s)\| = \sqrt{\left|g\left(\alpha'(s), \alpha'(s)\right)\right|} = \frac{f(s)}{2\,F(s)}$$

for all $s \in I$. For some $s_0 \in I$, let $t : I \longrightarrow J$ be arc length function of α starting at s_0 in $\mathbb{S}^3_1(1)$ given by

$$t = \int_{s_0}^{s} \|\alpha'(u)\|\, du.$$

Then we easily find

$$t = \frac{1}{2}\,\ln F(s) - \frac{1}{2}\,\ln F(s_0)$$

Thus, $\alpha : J \longrightarrow \mathbb{S}^3_1(1)$ is a unit-speed timelike curve having arc length function t and it is obvious that $c_1 F(s_0) > 0$. Putting the last equation in (5.3), we obtain (5.2).

Conversely, we assume that $\gamma : I \longrightarrow \mathbb{E}^4_1$ be a unit-speed null curve and $f : I \longrightarrow \mathbb{R}$ is a nowhere vanishing integrable function in parameter s such that the f-position vector field γ_f of γ is given by equation (5.2), where $c_1 \neq 0$ is a constant and $s_0 \in I$ with $c_1 F(s_0) > 0$ and $\alpha : J \longrightarrow \mathbb{S}^3_1(1)$ is a unit-speed timelike curve having $t : I \longrightarrow J$ as arc length function starting at s_0. Then we have $g(\alpha'(t), \alpha'(t)) = -1$, $g(\alpha(t), \alpha(t)) = 1$ and consequently $g(\alpha(t), \alpha'(t)) = 0$ for all $t \in J$. Therefore, using (5.2), we find

$$g\left(\gamma_f(t), \gamma_f(t)\right) = 2\,c_1 F(s_0)\exp(2t) \tag{5.4}$$

for all $t \in J$. Then we observe that $g\left(\gamma_f(t), \gamma_f(t)\right) > 0$ for all $s \in I$. This implies that γ_f is spacelike. Now, γ may be reparametrized by

$$t = \frac{1}{2}\left(\ln F(s) - \ln F(0)\right).$$

Then s becomes pseudo-arc length function of γ. Therefore, (5.4) reduces to

$$g\left(\gamma_f(s), \gamma_f(s)\right) = 2\,c_1 F(s)$$

for all $s \in I$, and hence the norm function $\rho = \|\gamma_f\|$ is given by

$$\rho(s) = \sqrt{|g\left(\gamma_f(s), \gamma_f(s)\right)|} = \sqrt{|2\, c_1 F(s)|}$$

for all $s \in I$, where $c_1 \neq 0$ is a constant. Therefore, by applying Theorem 4.3, we conclude that γ is an f-rectifying curve in $\mathbb{E}_1^4$ having nowhere vanishing tangential component.

The proof is analogous when γ is considered as a unit-speed null f-rectifying curve in $\mathbb{E}_1^3$ having a timelike f-position vector field γ_f and nowhere vanishing tangential component. $\square$

6 Conclusions

In this paper, we presented some differential geometric aspects of null f-rectifying curves in Minkowski space-time. First, we introduced the notion of f-rectifying curves which are null in $\mathbb{E}_1^4$. Thereafter, we exhibited some characterizations of such curves in Theorem 4.1, Theorem 4.2 and Theorem 4.3. Finally, we attempted for some classifications in Theorem 5.1 and Theorem 5.2. Analogous characterizations and classifications may be obtained for f-rectifying curves which are non-null (spacelike or timelike) in Minkowski space-time.

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