On polynomial differential systems of degree 3 in $\mathbb{R}^2$ and $\mathbb{R}^3$

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Abstract. In this paper, we investigate on polynomial differential systems of degree 3 in $\mathbb{R}^2$ (planar systems) and $\mathbb{R}^3$. First, we present the normal forms of polynomial differential systems in $\mathbb{R}^2$ having an invariant non-singular irreducible cubic and also a cubic with a circle. It can be a new classification of polynomial differential systems in $\mathbb{R}^2$ of degree 3. By using Darboux theorem specific conditions which become integrable are obtained. It is shown that their first integrals are invariant under an affine transformation.
Second, we consider all the cubic polynomial differential systems in $\mathbb{R}^3$. Some conditions in which systems have nonchaotic behavior are provided. The normal forms of polynomial differential systems in $\mathbb{R}^3$ that the surface $f(x,y) = z$ is an invariant, where $f(x,y) = 0$ is a cubic polynomial curve are presented. It is proved that these surfaces are Darboux invariant and their associated systems behave nonchaotically. Also, we present a new way to build normal forms in $\mathbb{R}^3$ via a class of planar systems.

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1 Introduction

Let $\mathbb{R}[x,y]$ be the ring of polynomials in the variables x and y with the coefficients in $\mathbb{R}$. In this paper, first we consider the planar polynomial differential systems of the form

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y) \tag{1.1}$$

where P and Q are relatively prime polynomials in $\mathbb{R}[x,y]$. We find the normal forms of the systems of the form (1.1) which have an invariant non-singular irreducible cubic.
In [5] Llibre et al., characterized all the planar polynomial differential systems with a unique invariant algebraic curve given by a real conic and provided their Darboux

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invariants.
Here we present a normal form $\dot{x} = AC - D\bar{C_y}, \quad \dot{y} = BC + D\bar{C_x}$, where $A, B, D \in \mathbb{R}[x, y]$, C is an invariant algebraic curve and C_x and C_y may not be relatively prime. In Theorem 3.2 it is shown that the condition in which C_x and C_y are relatively prime is not necessary. The normal forms of planar systems which have a cubic and the circle $\theta = (x-\alpha)^2 + (y-\beta)^2 - r^2 = 0$ as invariant algebraic curves are provided. By Darboux theorem and some conditions on their coefficients these normal forms are integrable and the resulted integrable planar systems are invariant by an affine transformation (see Theorem 3.5).

Second, we consider $3D$ polynomial differential systems. In [8] the authors gave an algebraic criterion to determine the nonchaotic behavior via Darboux theory for quadratic $3D$ polynomial differential systems with symmetric Jacobian matrix. By using Messias's algorithm we present some conditions for recognizing the nonchaotic behavior of cubic $3D$ polynomial differential systems.
In [7] Llibre et al., found the normal forms for polynomial differential systems in $\mathbb{R}^3$ having an invariant quadric and a Darboux invariant. In Theorem 4.2 the normal forms in $\mathbb{R}^3$ which one of their invariant manifolds is the surface $f(x, y) = z$ such that $f(x, y) = 0$ is a cubic polynomial curve is obtained and then some conditions on coefficients of normal forms that the resulted systems have nonchaotic behavior are provided. In Theorem 4.3 we present a new way to make normal forms for 3-dimensional systems by using of a class of planar systems such that the deduced systems have an invariant algebraic cubic surface.

2 Preliminaries

We start with some concepts in Darboux theory of integrability from [1, 3, 5, 6, 7] and [8]. This theory was named for Jean–Gaston Darboux. In 1878 he provided a method to make first integrals of polynomial differential systems via a sufficient number of invariant algebraic curves, surfaces, or hyper surfaces.
The vector field associated to the system (1.1) is:

$$X(x, y) = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}. \tag{2.1}$$

The degree of the system (1.1) is defined by $m = max\{deg(P), deg(Q)\}$. A polynomial $f(x, y) \in \mathbb{C}[x, y]$ is called a Darboux polynomial for system (1.1) if

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf \textit{ for some polynomial } K(x, y) \in \mathbb{C}[x, y]. \tag{2.2}$$

$K(x, y)$ is called a cofactor of f . It is simple to prove that the degree of K is less than or equal to $m - 1$. If $f(x, y)$ is a Darboux polynomial of the system (1.1), then the algebraic surface $f = 0$ in $\mathbb{R}^2$ is called an invariant algebraic curve. Because if a solution of (1.1) has a point on the invariant algebraic curve, then the whole solution is contained in it.
A nonconstant real function $H(x, y, t) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is an invariant of the system (1.1), if it is constant on all the solution curves $(x(t), y(t))$ of the system (1.1), i.e.,

$H(x(t), y(t), t)$ is a constant for all values of t for which the solution $(x(t), y(t))$ is defined on $\mathbb{R}^2$. It is clear, if H is differentiable on $\mathbb{R}^2 \times \mathbb{R}$, then H is an invariant of the system (1.1) if and only if along every solution of the system (1.1) we have

$$P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} + \frac{\partial H}{\partial t} \equiv 0.$$

If the invariant H is independent of the time, then it is called a first integral. Whether the first integral H is a polynomial then it is called a polynomial first integral. If the form of invariant H is $f(x, y)e^{st}$ where f is a polynomial in x and y, and s is a nonzero real constant, then H is called a Darboux invariant.

Let $g, h \in \mathbb{C}[x, y] \setminus \{0\}$ and assume that g and h are relatively prime in the ring $\mathbb{C}[x, y]$ or that $h = 1$. Then, the function $exp(g/h)$ is called an exponential factor of system (1.1) if for some polynomial $L \in \mathbb{C}[x, y]$ of degree at most $m - 1$, we have $X(exp(g/h)) = L\ exp(g/h)$. All above concepts are satisfied for $3D$ polynomial differential systems.

3 Results on polynomial differential systems in $\mathbb{R}^2$

In [9] Nadjafikah classified cubic polynomials in seven classes up to the affine transformations, and gave a complete set of representations of the these classes.

In this section, we characterize normal forms for planar polynomial differential systems having an invariant non-singular irreducible cubic where the cubic is chosen from the seven classes which are mentioned in the next theorem. After that we deduce the most general normal form of planar polynomial vector fields having two invariant algebraic curves which are a cubic and a circle.

Theorem 3.1. *[9] Any cubic can be transformed by an affine transformation to one and only one of the following cubic polynomials:*
1) $C^1_{a,b} : x^3 + x^2y = y^2 + ax + b$, *with* $a, b \in \mathbb{R}$ *and* $a \geq 2$;
2) $C^2_{c,d} : x^3 + x^2y = y^2 + cx - 3y + d$, *with* $c \in \{-2, 0\}$ *and* $d \in \mathbb{R}$;
3) $C^3_e : x^3 + x^2y = y^2 - x - 3y + e$, *with* $e \in \{1, 2, 3\}$;
4) $C^4_f : x^3 + x^2y = fx + 1$, *with* $f \in \{-1, 1\}$;
5) $C^5 : x^3 + x^2y = 1$;
6) $C^6_h : x^3 + x^2y = hx$, *with* $h \in \{-1, 1\}$; *and*
7) $C^7 : x^3 + x^2y = 0$.

A non-singular curve means the curve that

$$S = \{(x, y) \in \mathbb{R}^2 \mid C(x, y) = C_x(x, y) = C_y(x, y) = 0\} = \varnothing. \qquad (*)$$

Let's recall the Hilbert's Nullstellensatz (see [4]). Let $A, B_i \in \mathbb{C}[x, y]$ for $i = 1, ..., k$, and for A and all of the $B_1, B_2, ..., B_{k-1}$ and B_k vanish in the same points of $\mathbb{C}^2$. Then there exist polynomials $M_i \in \mathbb{C}[x, y]$ and a nonnegative integer n such that $A^n = \sum_{i=1}^{k} M_iB_i$. In particular, if all B_i have no any common zero, then there exist polynomials M_i such that $\sum_{i=1}^{k} M_iB_i = 1$.

Now we state a theorem which is a generalization of a lemma which have been proved by Christopher et al., in [1]. They assume the grate common divisor of C_x and C_y is 1 here it is assumed $(C_x, C_y) = M$.

Theorem 3.2. *Assume that polynomial system (1.1) has an irreducible non-singular invariant algebraic curve $C = 0$. If $(C_x, C_y) = M$ and $M \neq C_x$, C_y, then system (1.1) has the following normal form:*

$$\dot{x} = AC - D\bar{C}_y \qquad \dot{y} = BC + D\bar{C}_x \tag{3.1}$$

where $A, B, D, M \in \mathbb{R}[x, y]$, $C_x = M\bar{C}_x$ and $C_y = M\bar{C}_y$

Proof. Because $S = \varnothing$ and according to the Hilbert's Nullstellensatz there exist polynomials E, F and G such that $EC_x + FC_y + GC = 1$. From (2.2) and former equality we can find cofactor $K = (KE + GP)C_x + (KF + GQ)C_y$.
By substituting K into (2.2), we get

$$(P - (KE + GP)C)M\bar{C}_x = -(Q - (KF + GQ)C)M\bar{C}_y.$$

Since $(\bar{C}_x, \bar{C}_y) = 1$, then there exists a polynomial D such that

$$P - (KE + GP)C = -D\bar{C}_y, \qquad Q - (KF + GQ)C = D\bar{C}_x.$$

If we replace A with $KE + GP$ and B with $KF + GQ$, then we have (3.1). □

Now we state the main theorem of this section.

Theorem 3.3. *Assume that the polynomial differential system (1.1) of degree m has a non-singular irreducible cubic. Then system (1.1) with invariant non-singular irreducible cubic can be written to one of the five following forms:*

I) If the system (1.1) has $C^1_{a,b}$ as an invariant algebraic curve with the conditions $2 \leq a \neq x_0^3 + 3x_0^2$ and $b \neq -2x_0^3 - \frac{3}{4}x_0^4$ where $(x_0, y_0 = \frac{x_0}{2})$ is a singular point of $C^1_{a,b}$, then the system (1.1) has the following form

$$\begin{aligned} \dot{x} &= A(x^3 + x^2y - y^2 - ax - b) - D(x^2 - 2y), \\ \dot{y} &= B(x^3 + x^2y - y^2 - ax - b) + D(3x^2 + 2xy - a). \end{aligned} \tag{3.2}$$

II) If the system (1.1) has $C^2_{c,d}$ as an invariant algebraic curve then we can write the two below modes:

i) If $c = 0$ and $d \neq \frac{9}{4}$ then

$$\begin{aligned} \dot{x} &= A(x^3 + x^2y - y^2 + 3y - d) - D(x^2 - 2y + 3), \\ \dot{y} &= B(x^3 + x^2y - y^2 + 3y - d) + D(3x^2 + 2xy). \end{aligned} \tag{3.3}$$

ii) If $c = -2$ and $d \neq \frac{99}{4}$ then

$$\begin{aligned} \dot{x} &= A(x^3 + x^2y - y^2 + 2x + 3y - d) - D(x^2 - 2y + 3), \\ \dot{y} &= B(x^3 + x^2y - y^2 + 2x + 3y - d) + D(3x^2 + 2xy + 2). \end{aligned} \tag{3.4}$$

III) If the system (1.1) has C_e^3 where $e \neq 2$ as an invariant algebraic curve then the system (1.1) has the following form

$$\begin{aligned} \dot{x} &= A(x^3 + x^2y - y^2 + x + 3y + e) - D(x^2 - 2y + 3), \\ \dot{y} &= B(x^3 + x^2y - y^2 + x + 3y + e) + D(3x^2 - 2xy + 1). \end{aligned} \tag{3.5}$$

IV) If the system (1.1) has C_f^4 as an invariant algebraic curve then we can write two below modes
i) If $f = 1$ then

$$\begin{aligned} \dot{x} &= A(x^3 + x^2y - x - 1) - D(x^2), \\ \dot{y} &= B(x^3 + x^2y - x - 1) - D(3x^2 + 2xy - 1). \end{aligned} \tag{3.6}$$

ii) If $f = -1$ then

$$\begin{aligned} \dot{x} &= A(x^3 + x^2y + x - 1) - D(x^2), \\ \dot{y} &= B(x^3 + x^2y + x - 1) - D(3x^2 + 2xy + 1). \end{aligned} \tag{3.7}$$

V) If the system (1.1) has C^5 as an invariant algebraic curve then it has the following form

$$\begin{aligned} \dot{x} &= A(x^3 + x^2y - 1) - Dx, \\ \dot{y} &= B(x^3 + x^2y - 1) + D(3x + 2y) \end{aligned} \tag{3.8}$$

Proof. For the proof, it is enough to find the conditions so that each curve related to each part be a non-singular curve. By using of non-singularity of the curves and Theorem 3.2 we have the deduced normal forms which are mentioned in the theorem.

Proof (I).

By using of $(*)$ the curve $C^1_{a,b}$ is a non-singular curve. If $(x_0, y_0) \in S$ then $\dfrac{\partial C^1_{a,b}}{\partial y} = x_0^2 - 2y_0 = 0$ therefore $x_0^2 = 2y_0$. By substituting $x_0^2 = 2y_0$ in $\dfrac{\partial C^1_{a,b}}{\partial x}$ we find $x_0^3 + 3x_0^2 = a$. If we put $a = x_0^3 + 3x_0^2$ in $C^1_{a,b}$ then we obtain $b = -2x_0^3 - \dfrac{3x_0^4}{4}$. Thus $2 \leq a \neq x_0^3 + 3x_0^2$ and $b \neq -2x_0^3 - \dfrac{3}{4}x_0^4$. Hence $C^1_{a,b}$ is non-singular.

Proof (II).

Case (i). If $c = 0$, then $\dfrac{\partial C^2_{0,d}}{\partial x} = 3x^2 + 2xy$ and $\dfrac{\partial C^2_{0,d}}{\partial y} = x^2 - 2y + 3$. We put $\dfrac{\partial C^2_{0,d}}{\partial x} = 0$ so we have $x = 0$ or $x = \dfrac{-2}{3}y$. If $x = 0$, then $\dfrac{\partial C^2_{c,d}}{\partial y} = -2y + 3$. We take $\dfrac{\partial C^2_{0,d}}{\partial y} = 0$ then we have $y = \dfrac{3}{2}$. Now we put $x = 0, y = \dfrac{3}{2}$ in the curve $C^2_{0,d}$, and we obtain $d = \dfrac{9}{4}$. Since $C^2_{0,d}$ must be non-singular then $C^1_{a,b}$ is non-singular if we assume $d \neq \dfrac{9}{4}$. If $x = \dfrac{-2y}{3}$, then $\dfrac{\partial C^2_{0,d}}{\partial y}$ dose not have real root. So $C^2_{0,d}$ is a non-singular curve.

Case (ii). If $c=-2$, then $\dfrac{\partial C^2_{-2,d}}{\partial x}=3x^2+2xy+2$ and $\dfrac{\partial C^2_{-2,d}}{\partial y}=x^2-2y+3$. We take $\dfrac{\partial C^2_{-2,d}}{\partial y}=x^2-2y+3=0$, so $y=\dfrac{x^2+3}{2}$ by substituting it in $\dfrac{\partial C^2_{-2,d}}{\partial x}$ we obtain $x^3+3x^2+3x+2=0$. $x=-2$ is the root of this equation and we have $y=\dfrac{7}{2}$. By putting $x=-2$ and $y=\dfrac{7}{2}$ in $C^2_{-2,d}$ we get $d=\dfrac{99}{4}$. Thus the condition $d\neq\dfrac{99}{4}$ implies that $C^1_{-2,d}$ is non-singular.

Proof (III).

We have $\dfrac{\partial C^3_e}{\partial x}=3x^2+2xy+1$ and $\dfrac{\partial C^3_e}{\partial y}=x^2-2y+3$. If $\dfrac{\partial C^3_e}{\partial y}=x^2-2y+3$, then $y=\dfrac{x^2+3}{2}$. We substitute y it in $\dfrac{\partial C^3_e}{\partial x}=0$, and we deduce $x^3+3x^2+3x+1=0$ which $x=-1$ is a root of it. It implies $y=2$. By substituting $x=-1$, and $y=2$ in $C^3_e=0$ we obtain $e=2$. Thus if $e\neq 2$, then the curve C^3_e is a non-singular curve.

Proof (IV).

We have $\dfrac{\partial C^4_f}{\partial x}=3x^2+2xy-f$ and $\dfrac{\partial C^4_f}{\partial y}=x^2$. By substituting $x=0$ in the curve C^4_f we find a contradiction. So C^4_f is a non-singular curve.

Proof (V).

$\dfrac{\partial C^5}{\partial x}=3x^2+2xy$ and $\dfrac{\partial C^5}{\partial y}=x^2$. By substituting $x=0$ in the curve we reach to $0=1$ which is a contradiction. So it is non-singular.

The reader must pay attention to this point that in this case we have $M=x$. □

Remark 3.1. In Theorem 3.3 we don't consider curve C^6 and C^7 because they are not irreducible.

In [6] Llibre et al., have deduced the most general normal form of planar polynomial vector fields having two invariant algebraic curves $f_1=0$ and $f_2=0$ such that the Jacobian $\{f_1,f_2\}\not\equiv 0$. They deduced normal form is $X=\lambda_1 f_1\mathcal{H}_{f_2}+\lambda_2 f_2\mathcal{H}_{f_1}+f_1f_2\tilde{X}$, where $\mathcal{H}_{f_i}$ denotes the Hamiltonian vector field $(\dfrac{-\partial f_i}{\partial y},\dfrac{\partial f_i}{\partial x})$, λ_i are arbitrary rational functions, for $i=1,2$ and $\tilde{X}$ is an arbitrary polynomial vector field.

By using of this form we calculate the normal forms which have a cubic and the circle

$$\theta=(x-\alpha)^2+(y-\beta)^2-r^2=0$$

as invariant algebraic curves. For the convenience we put $\tilde{X}\equiv 0$.

I) If system (1.1) has $C^1_{a,b}$ and θ as invariant algebraic curves with $J\not\equiv 0$, then the system (1.1) has the following form

$\dot{x}=-\lambda_1(x^3+x^2y-y^2-ax-b)(2y-2\beta)-\lambda_2((x-\alpha)^2+(y-\beta)^2-r^2)(x^2-2y)$,
$\dot{y}=\lambda_1(x^3+x^2y-y^2-ax-b)(2x-2\alpha)+\lambda_2((x-\alpha)^2+(y-\beta)^2-r^2)(3x^2+2xy-a)$.

II) If system (1.1) has $C^2_{c,d}$ and θ as invariant algebraic curves with $J\not\equiv 0$, then the system (1.1) has the following form

$\dot{x}=-\lambda_1(x^3+x^2y-y^2-cx+3y-d)(2y-2\beta)-\lambda_2((x-\alpha)^2+(y-\beta)^2-r^2)(x^2-2y+3)$

$\dot{y} = \lambda_1(x^3+x^2y-y^2-cx+3y-d)(2x-2\alpha)+\lambda_2((x-\alpha)^2+(y-\beta)^2-r^2)(3x^2+2xy-c)$

III) If system (1.1) has C_e^3 and θ as invariant algebraic curves with $J \not\equiv 0$, then the system (1.1) has the following form
$\dot{x} = -\lambda_1(x^3+x^2y-y^2+x+3y+e)(2y-2\beta)-\lambda_2((x-\alpha)^2+(y-\beta)^2-r^2)(x^2-2y+3)$
$\dot{y} = \lambda_1(x^3+x^2y-y^2+x+3y+e)(2x-2\alpha)+\lambda_2((x-\alpha)^2+(y-\beta)^2-r^2)(3x^2+2xy+1)$

IV) If system (1.1) has C_f^4 and θ as invariant algebraic curves with $J \not\equiv 0$, then the system (1.1) has the following form
$\dot{x} = -\lambda_1(x^3+x^2y-fx-1)(2y-2\beta)-\lambda_2((x-\alpha)^2+(y-\beta)^2-r^2)x^2$
$\dot{y} = \lambda_1(x^3+x^2y-fx-1)(2x-2\alpha)+\lambda_2((x-\alpha)^2+(y-\beta)^2-r^2)(3x^2+2xy-f)$

V) If system (1.1) has C^5 and θ as invariant algebraic curves with $J \not\equiv 0$, then the system (1.1) has the following form
$\dot{x} = -\lambda_1(x^3+x^2y-1)(2y-2\beta)-\lambda_2((x-\alpha)^2+(y-\beta)^2-r^2)x^2$
$\dot{y} = \lambda_1(x^3+x^2y-1)(2x-2\alpha)+\lambda_2((x-\alpha)^2+(y-\beta)^2-r^2)(3x^2+2xy)$

Now we find the conditions which the former systems to be integrable. For this purpose we use of the following theorem.

Theorem 3.4. *[3] I) If the polynomial differential system* (1.1) *of degree m admits p irreducible invariant algebraic curves $f_i = 0$ with cofactors K_i, for $i = 1, ..., p$, and q exponential factors $F_j = exp(g_j/h_j)$ with cofactors L_j, for $j = 1, ..., q$, then there exist $\alpha_i, \beta_j \in \mathbb{C}$ not all zero such that*

$$\Sigma_{i=1}^{p}\alpha_i K_i + \Sigma_{j=1}^{q}\beta_j L_j = div(X) \tag{3.9}$$

where $div(X)$ is the divergence of the vector field X, the real (multivalued) function $V = f_1^{\alpha_1}...f_p^{\alpha_p}F_1^{\beta_1}...F_q^{\beta_q}$ is an inverse factor of the system (1.1).
II) Suppose that the polynomial differential system (1.1) *of degree m admits p irreducible invariant algebraic curves $f_i = 0$ with cofactors K_i, for $i = 1, ..., p$, and q exponential factors $F_j = exp(g_j/h_j)$ with cofactors L_j, for $j = 1, ..., q$.*
There exist $\alpha_i, \beta_j \in \mathbb{C}$ not zero such that $\Sigma_{i=1}^{p}\alpha_i K_i + \Sigma_{j=1}^{q}\beta_j L_j = -s$, for some $s \in \mathbb{R} \setminus \{0\}$, if and only if the real function $V = f_1^{\alpha_1}...f_p^{\alpha_p}F_1^{\beta_1}...F_q^{\beta_q}e^{st}$ is a Darboux invariant of system (1.1).

The above theorem is also correct for polynomial systems in $\mathbb{R}^3$.

Theorem 3.5. *I) Let the coefficients A, B, D and M be real numbers. Then the normal forms $I - IV$ of Theorem 3.3 are integrable.*
II) Let $X = \lambda_1 f_1 \mathcal{H}_{f_2} + \lambda_2 f_2 \mathcal{H}_{f_1} + f_1 f_2 \tilde{X}$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $J = \{f_1, f_2\} \not\equiv 0$, $\tilde{X} \equiv 0$. Then the vector field X is integrable on an open set $U \subseteq \mathbb{R}^2$ if $f_1 f_2 \neq 0$ on U.
III) If the system (1.1) varies by an affine transformation then any first integral associated to the system (1.1) is transformed to a first integral.

Proof. Proof I) The cofactor K for the invariant curve C equals $AC_x + BC_y$, which is $div(X)$. So the curves $C = C_{a,b}^1, C_{c,d}^2, C_e^3$ and C_f^4 are inverse factors for their normal forms. Therefore the system $\dot{x} = A - (D/C)\bar{C}_y$, $\dot{y} = B + (D/C)\bar{C}_x$ is a Hamiltonian system.

We put $\dot{x} = \dfrac{\partial H}{\partial y}$ and $\dot{y} = -\dfrac{\partial H}{\partial x}$ where H is a nonconstant function. Hence $H = \int (A - (D/C)\bar{C}_y)dy = Ay - (D/M)\int (C_y/C)dy = Ay - (D/M)Ln \mid C \mid + g(x)$ and $\dfrac{\partial H}{\partial x} = -(D/M)(C_x/C) + g'(x) = -(D/C)(C_x/M) - B$.
Therefore $H = Ay - Bx - (D/M)Ln \mid C \mid + \alpha$, where α is a constant.
Proof II) Since f_1 and f_2 are invariant algebraic curves then
$k_1 = -\lambda_1 J + f_2(\tilde{P}\dfrac{\partial f_1}{\partial x} + \tilde{Q}\dfrac{\partial f_1}{\partial y})$ and $k_2 = \lambda_2 J + f_1(\tilde{P}\dfrac{\partial f_2}{\partial x} + \tilde{Q}\dfrac{\partial f_2}{\partial y})$ are the cofactors of f_1 and f_2 respectively.
On the other hand $div(X) = (\lambda_2 - \lambda_1)J + \dfrac{\partial}{\partial x}(f_1 f_2 \tilde{P}) + \dfrac{\partial}{\partial y}(f_1 f_2 \tilde{Q})$. Using (3.9), $(-\alpha_1\lambda_1 + \alpha_2\lambda_2)J = (-\lambda_1 + \lambda_2)J$ if $\alpha_1 = \alpha_2 = 1$. Hence $f_1 f_2$ is an inverse factor.
So $\dot{x} = \dfrac{-\lambda_1}{f_2}\dfrac{\partial f_2}{\partial y} - \dfrac{\lambda_2}{f_1}\dfrac{\partial f_1}{\partial y}$ and $\dot{y} = \dfrac{\lambda_1}{f_2}\dfrac{\partial f_2}{\partial x} + \dfrac{\lambda_2}{f_1}\dfrac{\partial f_1}{\partial x}$. Thus there is a C^1 function $H : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ such that $\dot{x} = \dfrac{\partial H}{\partial y}$ and $\dot{y} = -\dfrac{\partial H}{\partial x}$. Therefore $H = \int (\dfrac{-\lambda_1}{f_2}\dfrac{\partial f_2}{\partial y} - \dfrac{\lambda_2}{f_1}\dfrac{\partial f_1}{\partial y})dy = -(\lambda_1 Ln \mid f_2 \mid + \lambda_2 Ln \mid f_1 \mid) + G(x)$. Hence $\dfrac{\partial H}{\partial x} = \dfrac{-\lambda_1}{f_2}\dfrac{\partial f_2}{\partial x} - \dfrac{\lambda_2}{f_1}\dfrac{\partial f_1}{\partial x} + G'(x)$. So $G(x)$ is a constant.
Proof III) Consider the polynomial system (1.1).
Let $X = \alpha x + \beta y + \gamma$, $Y = \alpha' x + \beta' y + \gamma'$ and $D = \alpha\beta' - \beta\alpha' \neq 0$. Then $x = [\beta'(X - \gamma) - \beta(Y - \gamma')]/D$, $y = [\alpha(Y - \gamma') - \alpha'(X - \gamma)]/D$ and

$$\dot{X} = P(X,Y) = \alpha p(x,y) + \beta q(x,y), \dot{Y} = P(X,Y) = \alpha' p(x,y) + \beta' q(x,y). \tag{3.10}$$

Let $H(X,Y)$ be a first integral of (3.10) and $H(X,Y) = h(x,y)$. Then $h(x,y)$ is a first integral of the system (1.1). Thus
$P\dfrac{\partial H}{\partial X} + Q\dfrac{\partial H}{\partial y} = \dfrac{1}{D}[(\alpha p + \beta q)(\beta'\dfrac{\partial h}{\partial x} - \alpha'\dfrac{\partial h}{\partial y}) + (\alpha' p + \beta' q)(-\beta\dfrac{\partial h}{\partial x} + \alpha\dfrac{\partial h}{\partial y})] = p\dfrac{\partial h}{\partial x} + q\dfrac{\partial h}{\partial y} = 0$. □

4 Results on polynomial differential systems in $\mathbb{R}^3$

In this section we consider $3D$ polynomial differential system 4.1.

$$\dot{x} = P(x,y,z), \quad \dot{y} = Q(x,y,z), \quad \dot{z} = R(x,y,z) \tag{4.1}$$

where $P, Q, R \in \mathbb{R}[x,y,z]$. In [8] Massias et al., proved the next theorem by finding an algebraic criterion to recognize when a $3D$ polynomial system with a symmetric Jacobian matrix has a nonchaotic behavior.

Theorem 4.1. *[8] Let X be the vector field associated to the* 3*-dimensional differential system (4.1). If X has an invariant algebraic surface $f = 0$ with a non-zero constant cofactor k, then the α-limit and ω-limit sets of the orbit $\phi_p(t) = (x(t), y(t), z(t))$ with $\phi_p(0) = p \in \mathbb{R}^3$, are both contained in $\{f = 0\} \cup S^2$, where S^2 represent the points at infinity of $\mathbb{R}^3$. In particular X dose not present chaotic behavior.*

They have mentioned their method is expendable for the case of non symmetric Jacobian matrix. Firstly, we apply their algorithm for non symmetric case to find a class of nonchaotic systems of degree 3. For a 3-dimensional cubic polynomial system

$$
(4.2) \quad \begin{cases}
\dot{x} = a_0 + a_1x + a_2y + a_3z + a_4x^2 + a_5y^2 + a_6z^2 + a_7xy + a_8xz + a_9yz \\
+a_{10}x^3 + a_{11}y^3 + a_{12}z^3 + a_{13}x^2y + a_{14}x^2z + a_{15}y^2x + a_{16}y^2z + a_{17}z^2x \\
+a_{18}z^2y + a_{19}xyz, \\
\dot{y} = b_0 + b_1x + b_2y + b_3z + b_4x^2 + b_5y^2 + b_6z^2 + b_7xy + b_8xz + b_9yz \\
+b_{10}x^3 + b_{11}y^3 + b_{12}z^3 + b_{13}x^2y + b_{14}x^2z + b_{15}y^2x + b_{16}y^2z + b_{17}z^2x \\
+b_{18}z^2y + b_{19}xyz, \\
\dot{z} = c_0 + c_1x + c_2y + c_3z + c_4x^2 + c_5y^2 + c_6z^2 + c_7xy + c_8xz + c_9yz \\
+c_{10}x^3 + c_{11}y^3 + c_{12}z^3 + c_{13}x^2y + c_{14}x^2z + c_{15}y^2x + c_{16}y^2z + c_{17}z^2x \\
+c_{18}z^2y + c_{19}xyz.
\end{cases}
$$

We take the following conditions on it's coffecients

$I)$ $a_i = \dfrac{b_ih_2(k) - c_ih_3(k)}{h_1(k)}$ for $i = 4, ..., 19$.

$II)$ $c_3 = k - \dfrac{h_1(k)a_3 + b_3h_2(k)}{h_3(k)}$

where $h_1(k) = a_2b_3 - a_3b_2 + a_3k$, $h_2(k) = a_1b_3 - a_2a_3 - b_3k$ and $h_3(k) = a_1b_2 - a_2^2 - a_1k - b_2k + k^2$ and k is a nonzero real number with $h_1(k), h_3(k) \neq 0$.

According to the above theorem it has nonchaotic behavior when $a_0, a_1, a_2, a_3 \in \mathbb{R}$, $b_j \in \mathbb{R}$ for $j = 0, ..., 19$ and $c_l \in \mathbb{R}$ for $l = 0, 1, 2, 4, ..., 19$. In order to find above conditions by using of theorem 4.1 we put some conditions on coefficients of the system (4.2) which has a plan $\Pi : Ax + By + Cz + D = 0$ where $A = 1$ as a Darboux invariant with the cofactor $k \in \mathbb{R} \setminus \{0\}$.

The conditions are obtained by solving the equation

$$P\frac{\partial \Pi}{\partial x} + Q\frac{\partial \Pi}{\partial y} + R\frac{\partial \Pi}{\partial z} = k\Pi,$$

where P, Q and R are the right hand side of the system (4.2). All polynomial of the same degree from two sides are equaled. Hence it is obtained

$$a_0 + b_0B + c_0C = kD, \quad a_1 + b_1B + c_1C = k$$

$$a_2 + b_2B + c_2C = kB \quad \textit{and} \quad a_3 + b_3B + c_3C = kC$$

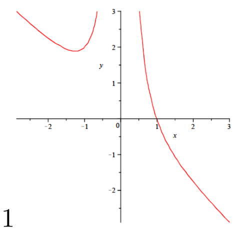
and $a_i + b_iB + c_iC = 0$ for $i = 4, ..., 19$.

So we obtain $B = \dfrac{-h_2(k)}{h_1(k)}, C = \dfrac{h_3(k)}{h_1(k)}, D = \dfrac{a_0h_1(k) + c_0h_3(k) - b_0h_2(k)}{k}$ and $c_3 = k - \dfrac{a_3h_1(k) + b_3h_2(k)}{h_3(k)}$ where $h_1(k) = a_2b_3 - a_3b_2 + a_3k$, $h_2(k) = a_1b_3 - a_2a_3 - b_3k$ and $h_3(k) = a_1b_2 - a_2^2 - a_1k - b_2k + k^2$ with $h_1(k), h_3(k) \neq 0$.

Remark 4.1. The system (4.2) with the conditions (I) and (II) has the invariant plan $x - \dfrac{h_2(0)}{h_1(0)}y + \dfrac{h_3(0)}{h_1(0)}z = 0$ as a first integral if we consider $k = D = 0$ and $b_0 = \dfrac{a_0h_1(0) + c_0h_3(0)}{h_2(0)}$.

Secondly, the normal forms for $3D$ polynomial differential system 4.1 such that the surface $f(x,y)=z$ is an invariant surface, where $f(x,y)=0$ is a cubic polynomial is provided. By substituting the seven forms of cubics in $f(x,y)-z=0$, the resulted surfaces is called $S_1, S_2, ..., S_6$ and S_7 respectively (see Figure 1). These surfaces are nondegenerate because they are irreducible in $\mathbb{C}[x,y,z]$. We present the normal form of $3D$ polynomial differential systems which have S_i, $i=1,...,5$ an invariant surface. By using of Darboux theorem 3.4, it is shown surfaces $S_1, S_2, ..., S_5$ can be as a Darboux invariant and so by theorem 4.1 they do not present chaotic behavior. Finally, we prove if the system (1.1) has an invariant curve $C=0$ with the cofactor $K=C_x+C_y$ then we can make a $3D$ polynomial differential system.

$x^3+x^2y=1$



$x^3+x^2y-1=z$

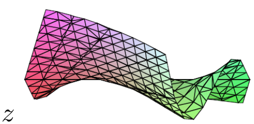


Figure 1:

In [7] Llibre et al., characterized all 3-dimensional polynomial differential system having a given invariant quadric algebraic surface. Base on their method we prove the next theorem. Let f_1, f_2 and f_3 be the real functions defined on an open subset U of $\mathbb{R}^3$. The Jocobian matrix of the functions f_1, f_2 and f_3 is defined by

$$J=\begin{bmatrix} f_{1x} & f_{1y} & f_{1z} \\ f_{2x} & f_{2y} & f_{2z} \\ f_{3x} & f_{3y} & f_{3z} \end{bmatrix}.$$

We denote determinant of J by $\{f_1, f_2, f_3\}$, where f_{ix}, f_{iy} and f_{iz}, $i=1,2,3$ are partial derivatives.

Suppose f_1 is a polynomial on $\mathbb{R}[x,y,z]$, then any polynomial differential system in $\mathbb{R}^3$ which admits $f_1=0$ as an invariant algebraic surface has the following form

$$\dot{v}=\lambda_1\{v,f_2,f_3\}+\lambda_2\{f_1,v,f_3\}+\lambda_3\{f_1,f_2,v\} \tag{4.3}$$

where v vary over the variables x,y and z, $\lambda_1=\varphi f_1$ and $\varphi, \lambda_2, \lambda_3$ are rational functions, f_2 and f_3 are arbitrary polynomial on $\mathbb{R}[x,y,z]$ which must be chosen in such a way that the *Jacobian* $\{f_1,f_2,f_3\}\not\equiv 0$. According to their proof in [7], f_1 ia an invariant algebraic surface of the polynomial vector field defined by the polynomial system $\dot{v}$ with the cofactor $K=\varphi\{f_1,f_2,f_3\}$. It is enough compute ϕ, λ_2 and λ_3. Authors showed the vector field $X(*)=\lambda_1\{*,f_2,f_3\}+\lambda_2\{f_1,*,f_3\}+\lambda_3\{f_1,f_2,*\}$, associated to the polynomial differential system (2) is the most general polynomial vector field which admits $f_1=0$ as an invariant algebraic surface. They supposed $Y=(Y_1(x,y,z),Y_2(x,y,z),Y_3(x,y,z))$ is a polynomial vector field having $f_1=0$ as an invariant algebraic and took

$$\lambda_j=\frac{Y(f_j)}{\{f_1,f_2,f_3\}} \tag{4.4}$$

for $j = 1, 2, 3$, where $Y(f_j) = Y_1 f_{jx} + Y_2 f_{jy} + Y_3 f_{j3}$ and by substituting these in $X(*)$ they obtained $X(F) = Y(F)$ for each $F \in \mathbb{R}[x, y, z]$.

Theorem 4.2. *Assume that a nondegenerate surface $S_1, ..., S_5$ is an invariant algebraic surface of polynomial differential system 4.1, then the normal form for it is the following form, where A, B, C, D, E, F are arbitrary polynomials of $\mathbb{R}[x, y, z]$.*
1) *System 4.1 with S_1 as an invariant algebraic surface has the following form.*
$\dot{x} = S_1 A + (2y - x^2)D - E,$
$\dot{y} = S_1 B + (3x^2 + 2xy - a)D + F,$
$\dot{z} = S_1 C + (-3x^2 - 2xy + a)E + (x^2 - 2y)F.$
If $C = A(3x^2 + 2xy - a) + B(x^2 - 2y) + \alpha$ then it has $I = S_1^{\alpha^{-1}} e^t$ as Darboux invariant where α is a non zero real number.

2) *System 4.1 with S_2 as an invariant algebraic surface has the following form.*
$\dot{x} = S_2 A + (2y - x^2 - 3)D - E,$
$\dot{y} = S_2 B + (3x^2 + 2xy - c)D + F,$
$\dot{z} = S_2 C + (-3x^2 - 2xy + c)E + (x^2 - 2y + 3)F.$
If $C = A(3x^2 + 2xy - c) + B(x^2 - 2y + 3) + \alpha$ then it has $I = S_2^{\alpha^{-1}} e^t$ as Darboux invariant where α is a non zero real number.

3) *System 4.1 with S_3 as an invariant algebraic surface has the following form.*
$\dot{x} = S_3 A + (2y - x^2 - 3)D - E,$
$\dot{y} = S_3 B + (3x^2 + 2xy + 1)D + F,$
$\dot{z} = S_3 C - (3x^2 + 2xy + 1)E + (x^2 - 2y + 3)F.$
If $C = A(3x^2 + 2xy + 1) + B(x^2 - 2y + 3) + \alpha$ then it has $I = S_3^{\alpha^{-1}} e^t$ as Darboux invariant where α is a non zero real number.

4) *System 4.1 with S_4 as an invariant algebraic surface has the following form.*
$\dot{x} = S_4 A - x^2 D - E,$
$\dot{y} = S_4 B + (3x^2 + 2xy - f)D + F,$
$\dot{z} = S_4 C + (-3x^2 - 2xy + f)E + x^2 F.$
If $C = A(3x^2 + 2xy - f) + Bx^2 + \alpha$ then it has $I = S_4^{\alpha^{-1}} e^t$ as Darboux invariant where α is a non zero real number.

5) *System 4.1 with S_5 as an invariant algebraic surface has the following form.*
$\dot{x} = S_5 A - x^2 D - E,$
$\dot{y} = S_5 B + (3x^2 + 2xy)D + F,$
$\dot{z} = S_5 C - (3x^2 + 2xy)E + x^2 F.$
If $C = A(3x^2 + 2xy) + Bx^2 + \alpha$ then it has $I = S_5^{\alpha^{-1}} e^t$ as Darboux invariant where α is a non zero real number.

Proof. We prove statement (1). For other cases the the proof is similar. Suppose the polynomial differential system 4.1 has S_1 as invariant algebraic surface. Due to system (4.3) we should compute $\dot{x}, \dot{y}$ and $\dot{z}$ so we compute
$\{x, f_2, f_3\} = f_{2y} f_{3z} - f_{2z} f_{3y} = J_1,\ \{y, f_2, f_3\} = f_{3x} f_{2z} - f_{2x} f_{3z} = J_2$ and $\{z, f_2, f_3\} = f_{2x} f_{3y} - f_{3x} f_{2y} = J_3$.
$\{S_1, x, f_3\} = -f_{3y} + f_{3z}(2y - x^2),\ \{S_1, y, f_3\} = f_{3z}(3x^2 + 2xy - a) + f_{3x},$
$\{S_1, z, f_3\} = (-3x^2 - 2xy + a)f_{3y} + (x^2 - 2y)f_{3x}.$

$\{S_1, f_2, x\} = f_{2z}(x^2 - 2y) + f_{2y},\ \{S_1, f_2, y\} = (-3x^2 - 2xy + a)f_{2z} - f_{2x}$,
$\{S_1, f_2, z\} = (3x^2 + 2xy - a)f_{2y} + (2y - x^2)f_{2x}$.
Then we put the above computed formulas in system (4.3) and it is deduced,
$\dot{x} = \varphi S_1 J_1 + \lambda_2(-f_{3y} + f_{3z}(2y - x^2)) + \lambda_3(f_{2z}(x^2 - 2y) + f_2 y) = S_1 A + (2y - x^2)D - E$
$\dot{y} = \varphi S_1 J_2 + \lambda_2(f_{3z}(3x^2 + 2xy - a) + f_{3x}) + \lambda_3((-3x^2 - 2xy + a)f_{2z} - f_{2x}) = S_1 B + (3x^2 + 2xy - a)D + F$
$\dot{z} = \varphi S_1 J_3 + \lambda_2((-3x^2 - 2xy + a)f_{3y} + (x^2 - 2y)f_{3x}.) + \lambda_3((3x^2 + 2xy - a)f_{2y} + (2y - x^2)f_{2x}) = S_1 C + (-3x^2 - 2xy + a)E + (x^2 - 2y)F$,
where $A = \varphi J_1,\ B = \varphi J_2,\ C = \varphi J_3,\ D = \lambda_3 f_{2z} - \lambda_2 f_{3z},\ E = \lambda_3 f_{2y} - \lambda_2 f_{3y}$ and $F = \lambda_2 f_{3x} - \lambda_3 f_{2x}$.
By using of (4.4) we have.
$\varphi = \dfrac{A(3x^2 + 2xy - a) + B(x^2 - 2y) - C}{(3x^2 + 2xy - a)J_1 + (x^2 - y)J_2 - J_3}$ and
$K = A(3x^2 + 2xy - a) + B(x^2 - 2y) - C$. By the same calculation we obtain
$\lambda_i = \dfrac{(Af_{ix} + Bf_{iy} + Cf_{iz})S_1 + E(-f_{ix} + (-3x^2 - 2xy + a)f_{iz})}{(3x^2 + 2xy - a)J_1 + (x^2 - 2y)J_2 - J_3} +$
$\dfrac{F(f_{iy} + (x^2 - 2y)f_{iz}) + D((2y - x^2)f_{ix} - (3x^2 + 2xy - a)f_{iy})}{(3x^2 + 2xy - a)J_1 + (x^2 - 2y)J_2 - J_3}$, for $i = 2, 3$.
To prove the second part of theorem we suppose this normal form has a Darboux invariant and it obtain uniquely with an invariant algebraic surface S_1. By Theorem 3.4 there exists non zero $\mu \in \mathbb{C}$ such that $\mu K = -s$ for some $s \in \mathbb{R} \setminus \{0\}$. So $\mu[A(3x^2+2xy-a)+B(x^2-2y)-C] = -s$ then $\mu[A(3x^2+2xy-a)+B(x^2-2y)]+\alpha = C$ where $\alpha = s/\mu$ is a non zero real number. Therefore $I = S_1^{s\alpha^{-1}} e^{st}$ is a Darboux invariant. Since there is no restriction on s then we can take $s = 1$. □

Remark 4.2. Due to Theorem 4.1 and the conditions on coefficients of normal forms in Theorem 4.2 when they have Darboux invariant, their behavior is nonchaotic.

Remark 4.3. In Theorem 4.2 we do not consider surfaces S_6 and S_7 because they are homeomorphic with surfaces S_4 and S_5 respectively.

In the next theorem we make a new 3-dimensional system via a planar system.

Theorem 4.3. *If system (1.1) has an invariant curve $C = 0$ with the cofactor $K = C_x + C_y$, then the 3-dimensional system*
$\tilde{P}(x, y, z) = P(x, y) - z,\quad \tilde{Q}(x, y, z) = Q(x, y) - z,\quad \tilde{R}(x, y, z) = h(C - z)$
with $h \in \mathbb{R}[x, y, z]$ has an invariant algebraic surface
$\tilde{C}(x, y, z) = C(x, y) - z$.

Proof. We have

$$\begin{aligned}
\tilde{P}\tilde{C}_x + \tilde{Q}\tilde{C}_y + \tilde{R}\tilde{C}_z &= (P - z)C_x + (Q - z)C_y - h(C - z) \\
&= PC_x + QC_y - z(C_x + C_y) - h\tilde{C} \\
&= KC - zK - h\tilde{C} \\
&= K(C - z) - h\tilde{C} \\
&= (K - h)\tilde{C}.
\end{aligned}$$

Thus $(K - h)$ is the cofactor of $\tilde{C}$. □

In system (3.1) if $A = B = 1$ then we can always build a 3-dimensional system with an invariant algebraic surface $C(x, y) = z$.

Example 4.4. By applying the former method to the system mentioned in (V) we deduce the following 3-dimensional system

$$\begin{aligned} \tilde{P}(x,y,z) &= x^3 + x^2y - Dx - z - 1 \\ \tilde{Q}(x,y,z) &= x^3 + x^2y + 3Dx + 2Dy - z - 1 \\ \tilde{R}(x,y,z) &= h(x^3 + x^2y - z - 1). \end{aligned} \tag{4.5}$$

$x^3 + x^2y - z - 1 = 0$ is an invariant surface for it.

5 Conclusions

The results in this paper can be a new classification for both polynomial differential systems in $\mathbb{R}^2$ and $\mathbb{R}^3$ having a cubic invariant algebraic curve and a cubic invariant algebraic surface respectively.
By using Darboux theorem we find specific conditions for integrability of planar systems with a cubic invariant algebraic curve. It is proved their first integrals are invariant under an affine transformation.
Some conditions for cubic polynomial differential systems in $\mathbb{R}^3$ which have non-chaotic behavior are provided. We compute Darboux invariant for normal forms of polynomial differential systems in $\mathbb{R}^3$ which the surface $f(x, y) = z$ is invariant where, $f(x, y) = 0$ is a cubic polynomial curve. Finally a new way to build normal forms in $\mathbb{R}^3$ via a class of planar systems is presented.

References

[1] C. Christopher, J. Llibre, C. Pantazi, X. Zhang, *Darboux integrability and invariant algebraic curves for planar polynomial systems*, J. Phys. A: Math Gen. 35 (2002), 2457–2476.

[2] A. Cima, J. Llibre, *Bounded polynomial vector fields*, Trans. Amer. Math. Soc. 318 (1990), 557–579.

[3] F. Dumortier, J. Llibre, J. C. Artés, *Qualitative theory of planar differential systems*, Springer-Verlag, NY 2006.

[4] W. Fulton, *Algebraic Curves. An Introduction to Algebraic Geometry*, W.A. Benjamin Inc. New York 1969.

[5] J. Llibre, M. Messias, A. C. Reinol, *Darboux invariants for planar polynomial differential systems having an invariant conic*, J. Z. Angew. Math. Phys. 65 (2013), 1127–1136.

[6] J. Llibre, M. Messias, A. C. Reinol, *Normal forms and global phase portraits of quadratic and cubic integrable vector fields having two non-concentric circles as invariant algebraic curves*, J. Dynamical Systems, 32 (2016), 374–390.

[7] J. Llibre, M. Messias, A. C. Reinol, *Normal forms for polynomial differential systems in $\mathbb{R}^3$ having an invariant quadric and a Darboux invariant*, Int. J. Bifurcation and Chaos. 25 (2015), 1-16.

[8] M. Messias, R. P. Silva, *Non-chaotic behavior in quadratic three-dimensional differential systems with a symmetric Jacobian matrix*, Int. J. Bifurcation and Chaos. 28 (2018), 1830006-1–1830006-12.

[9] M. Nadjafikah, *Classification of cubics up to affine transformations*, Diff. Geom. Dyn. Sys., Geometry Balkan Press, 8 (2006), 184–195.

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