A note on SCR-lightlike warped product submanifolds of indefinite Kaehler manifolds

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Abstract. It is shown that there does not exist any non-trivial warped product SCR-lightlike submanifold of the type $M_\perp \times_\lambda M_T$ in an indefinite Kaehler manifold. Then, the existence of SCR-lightlike warped product submanifolds of the type $M_T \times_\lambda M_\perp$ in an indefinite Kaehler manifold is proved. Further, it is derived that for a proper SCR-lightlike warped product submanifold of an indefinite Kaehler manifold, the induced connection ∇ can never be a metric connection. We also find some characterization theorems in terms of the canonical structures f and ω on a SCR-lightlike submanifold of an indefinite Kaehler manifold forcing it to be a SCR-lightlike warped product submanifold. Finally, we classify SCR-lightlike warped product submanifolds of indefinite Kaehler manifolds by developing a sharp inequality for the squared norm of the second fundamental form h in terms of the warping function λ.

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1 Introduction

From last few decades, the geometry of warped product manifolds is one of the most popular research topic for mathematicians and physicists. The concept of warped product manifolds have several significant contributions in differential geometry and mathematical physics, especially, in the general theory of relativity. The available literature on warped product manifolds demonstrates that various geometrical objects take canonical forms in warped products. For example, the general formulae for Levi-Civita connection and the Riemann curvature tensor can be expressed in terms of the warped products (see [14]). In [16], Rajaratnam et. al., discussed the application of warped product decompositions of a given space, for construction of Killing tensors and coordinates to separate the Hamilton-Jacobi equation. Apart from differential geometric studies, the warped product manifolds provide an excellent setting to model spacetime near black holes or bodies with large gravitational

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field (see [15]). For instance, Robertson-Walker spacetimes, asymptotically flat spacetimes, Schwarzschild spacetimes and Reissner- Nordstr$\ddot{o}m$ spacetimes are examples of warped product manifolds (for details, see [10]). Also, the warped product manifolds give solutions to Einstein's field equations (see [2]), thus the study of this class of manifolds assumes significance in general.

However, the notion of warped product manifolds was proposed by Bishop and O'Neill (see [4]). But, the study of warped products attained momentum, when Chen, introduced the concept of CR-warped products in Kaehler manifolds, by proving the non-existence of non-trivial warped product CR-submanifolds of the type $M_\perp \times_\lambda M_T$ in a Kaehler manifold (see [5]). Later on, many research articles appeared exploring various geometrical aspects of warped product submanifolds in Kaehler manifolds. Far less common are studies, where the warped products are considered in the semi-Riemannian settings. In the present scenario, the need of such studies is growing because the relativity theory leads to the geometry of semi-Riemannian manifolds, which turns out to be the most general framework for the study of warped products and may result in some remarkable applications. In [17], Sahin brought our attention to geometry of warped product lightlike submanifolds and obtained some basic results for this class of warped products. Recently, Kumar [12]-[13], studied warped product lightlike submanifolds of indefinite nearly Kaehler manifolds and obtained some significant characterizations on warped product lightlike submanifolds. Moreover, the geometry of indefinite Kaehler manifolds is very important from mathematical point of view and the lightlike submanifolds of indefinite Kaehler manifolds have extensive applications in mathematical physics.

Therefore, in present paper, we investigate warped product SCR-lightlike submanifolds of indefinite Kaehler manifolds. We prove that there does not exist any non-trivial warped product SCR-lightlike submanifold of the type $M_\perp \times_\lambda M_T$ in an indefinite Kaehler manifold. Then, we prove the existence of SCR-lightlike warped product submanifolds of the type $M_T \times_\lambda M_\perp$ in an indefinite Kaehler manifold by a characterization in terms of the shape operator. Further, we prove that for a proper SCR-lightlike warped product submanifold of an indefinite Kaehler manifold, the induced connection ∇ can never be a metric connection. We also find some characterization theorems in terms of the canonical structures f and ω on a SCR-lightlike submanifold of an indefinite Kaehler manifold forcing it to be a SCR-lightlike warped product submanifold. Finally, we classify SCR-lightlike warped product submanifolds of indefinite Kaehler manifolds by developing a sharp inequality for the squared norm of the second fundamental form h in terms of the warping function λ.

2 Preliminaries

2.1 Geometry of lightlike submanifolds

In this section, we recall some basic formulae and notations for lightlike submanifolds following [7].

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g be the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M, then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g

on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces, but no longer complementary. In this case, there exists a subspace $Rad(T_xM) = T_xM \cap T_xM^{\perp}$, which is known as radical (null) subspace. If the mapping $Rad(TM) : x \in M \longrightarrow Rad(T_xM)$, defines a smooth distribution on M of rank $r > 0$, then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $Rad(TM)$ is called the radical distribution on M.
The screen distribution $S(TM)$ is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is

$$TM = Rad(TM) \perp S(TM) \tag{2.1}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M}\mid_M$ and to $Rad(TM)$ in $S(TM^{\perp})^{\perp}$, respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{2.2}$$

$$T\bar{M}\mid_M = TM \oplus tr(TM) = (Rad(TM) \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{2.3}$$

For a quasi-orthonormal fields of frames on TM, we have

Theorem 2.1. *([7]). Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)\mid_u)$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}\mid_u$, where u is a coordinate neighborhood of M such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \text{for any} \quad i, j \in \{1, 2, .., r\}, \tag{2.4}$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$, then according to the decomposition (2.3), the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^t_X U, \tag{2.5}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^t_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, A_U is a linear operator on M and is known as shape operator.
According to (2.2), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$ respectively, then Gauss and Weingarten formulae become

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \quad \bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.6}$$

where we put $h^l(X, Y) = L(h(X, Y)), h^s(X, Y) = S(h(X, Y)), D^l_X U = L(\nabla^t_X U)$, $D^s_X U = S(\nabla^t_X U)$. As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called the lightlike second fundamental form and the screen second fundamental form on M. In particular,

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \quad \bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.7}$$

where $X \in \Gamma(TM), N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$. Using (2.6) and (2.7), we obtain

$$\bar{g}(h^s(X,Y),W) + \bar{g}(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.8}$$

$$\bar{g}(D^s(X,N),W) = \bar{g}(A_W X, N), \tag{2.9}$$

for any $X, Y \in \Gamma(TM)$, $W \in \Gamma(S(TM^{\perp}))$ and $N \in \Gamma(ltr(TM))$.
Let P be the projection morphism of TM on $S(TM)$, then using (2.1), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$\nabla_X PY = \nabla^*_X PY + h^*(X,Y), \quad \nabla_X \xi = -A^*_{\xi} X + \nabla^{*t}_X \xi, \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad(TM))$, where $\{\nabla^*_X PY, A^*_{\xi} X\}$ and $\{h^*(X,Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$, respectively. Using (2.6) and (2.10), we obtain

$$\bar{g}(h^l(X,PY),\xi) = g(A^*_{\xi} X, PY), \quad \bar{g}(h^*(X,PY),N) = g(A_N X, PY), \tag{2.11}$$

for any $X, Y \in \Gamma(TM), \xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$.
In general, the induced connection ∇ on M is not a metric connection. Since $\bar{\nabla}$ is a metric connection, by using (2.6), we get

$$(\nabla_X g)(Y,Z) = \bar{g}(h^l(X,Y),Z) + \bar{g}(h^l(X,Z),Y). \tag{2.12}$$

However, it is important to note that ∇^* is a metric connection on $S(TM)$.

2.2 Indefinite Kaehler manifolds

Let $\bar{M}$ be an indefinite almost Hermitian manifold with an almost complex structure $\bar{J}$ of type $(1,1)$ and Hermitian metric $\bar{g}$, then $(\bar{M}, \bar{g}, \bar{J})$ is called an indefinite Kaehler manifold (see ([1])), if

$$\bar{J}^2 = -I, \quad \bar{g}(\bar{J}U, \bar{J}V) = \bar{g}(U,V), \quad (\bar{\nabla}_U \bar{J})V = 0, \ \forall\, U, V \in \Gamma(TM), \tag{2.13}$$

where $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$.

2.3 Screen Cauchy-Riemann (SCR)-lightlike submanifolds

Definition 2.1. ([9]). Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite Kaehler manifold $(\bar{M}, \bar{g}, \bar{J})$, then M is called a Screen Cauchy-Riemann (SCR)-lightlike submanifold, if the following conditions are satisfied

(A) There exists a real non-null distribution $D \subset S(TM)$ such that

$$S(TM) = D \oplus D^{\perp}, \quad \bar{J}D^{\perp} \subset S(TM^{\perp}), \quad D \cap D^{\perp} = \{0\},$$

where $D^{\perp}$ is orthogonal complementary to D in $S(TM)$.

(B) $Rad(TM)$ is invariant with respect to $\bar{J}$.

Further, it follows that D and $ltr(TM)$ are invariant with respect to $\bar{J}$, that is, $\bar{J}D = D$, $\bar{J}ltr(TM) = ltr(TM)$, $TM = D' \oplus D^{\perp}$ and $D' = D \perp Rad(TM)$. Denote the orthogonal complement to $\bar{J}D^{\perp}$ in $S(TM^{\perp})$ by ν. Then $tr(TM) = ltr(TM) \perp \bar{J}D^{\perp} \perp \nu$.

Let Q and P be the projections on D' and $D^{\perp}$, respectively. Then for any $X \in \Gamma(TM)$, we have

$$X = QX + PX, \tag{2.14}$$

applying $\bar{J}$ to (2.14), we obtain

$$\bar{J}X = \bar{J}QX + \bar{J}PX, \tag{2.15}$$

and we can write equation (2.15) as

$$\bar{J}X = fX + \omega X, \tag{2.16}$$

where fX and wX are the tangential and transversal components of $\bar{J}X$, respectively. Similarly,

$$\bar{J}V = BV + CV, \tag{2.17}$$

for any $V \in \Gamma(tr(TM))$, where BV and CV are the sections of TM and $tr(TM)$, respectively. Applying $\bar{J}$ to (2.16) and (2.17), we get $f^2 = -I - B\omega$ and $C^2 = -I - \omega B$. Differentiating (2.16) and using (2.6), (2.7) and (2.17), we obtain

$$(\nabla_X f)Y = A_{\omega Y}X + Bh^s(X, Y), \tag{2.18}$$

$$\nabla^s_X \omega Y = \omega \nabla_X Y + Ch^s(X, Y) - h^s(X, fY), \tag{2.19}$$

$$D^l(X, \omega Y) = Ch^l(X, Y) - h^l(X, fY), \tag{2.20}$$

for any $X, Y \in \Gamma(TM)$.
Using Kaehlerian property of $\bar{\nabla}$ with (2.5), we have the following lemma.

Lemma 2.2. *Let M be a SCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then we have*

$$(\nabla_X f)Y = A_{\omega Y}X + Bh(X, Y) \tag{2.21}$$

and

$$(\nabla^t_X \omega)Y = Ch(X, Y) - h(X, fY), \tag{2.22}$$

for any $X, Y \in \Gamma(TM)$, where

$$(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y, \quad (\nabla^t_X \omega)Y = \nabla^t_X \omega Y - \omega \nabla_X Y. \tag{2.23}$$

Now, we will recall the conditions for the integrability of distributions D' and $D^{\perp}$.

Theorem 2.3. *([9]). Let M be a SCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the distribution D' is integrable if and only if*

$$h(X, \bar{J}Y) = h(\bar{J}X, Y),$$

for any $X, Y \in \Gamma(D')$.

Theorem 2.4. *([9]). Let M be a SCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then $D^{\perp}$ is integrable if and only if*

$$A_{\bar{J}V}W = A_{\bar{J}W}V,$$

for any $V, W \in \Gamma(D^{\perp})$.

Definition 2.2. ([8]). A lightlike submanifold (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be totally umbilical in $\bar{M}$, if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that

$$h^l(X, Y) = H^l g(X, Y), \quad h^s(X, Y) = H^s g(X, Y), \quad D^l(X, W) = 0, \tag{2.24}$$

for any $X, Y \in \Gamma(TM)$ and $W \in \Gamma(S(TM^{\perp}))$.

3 Warped product SCR-lightlike submanifolds

Bishop and O'Neill defined warped product manifolds as

Definition 3.1. ([4]). Let B and F be two Riemannian manifolds with Riemannian metrics g_B and g_F, respectively and λ be a positive differentiable function on B. Consider the product manifold $B \times F$ with its projection $\pi : B \times F \to B$ and $\eta : B \times F \to F$. The warped product $M = B \times_{\lambda} F$ is the manifold $B \times F$ equipped with Riemannian metric g such that

$$g = g_B + \lambda^2 g_F.$$

More explicitly, if U is tangent to $M = B \times_{\lambda} F$ at (p, q), then

$$\|U\|^2 = \|\pi_*(U)\|^2 + \lambda^2(\pi(U))\|\eta_*(U)\|^2.$$

Here function λ is called the warping function of the warped product and a warped product manifold is said to be trivial, if λ is constant. For differentiable function λ on M, the gradient $\nabla\lambda$ is defined by

$$g(\nabla\lambda, U) = U\lambda, \forall\ U \in \Gamma(TM).$$

Theorem 3.1. *([4]). Let $M = B \times_{\lambda} F$ be a warped product manifold. If $X, Y \in T(B)$ and $U, V \in T(F)$, then*

$$\nabla_X Y \in T(B), \tag{3.1}$$

$$\nabla_X V = \nabla_V X = \left(\frac{X\lambda}{\lambda}\right) V, \tag{3.2}$$

$$\nabla_U V = -\frac{g(U, V)}{\lambda} \nabla\lambda. \tag{3.3}$$

Corollary 3.2. *([4]). On a warped product manifold, $M = B \times_\lambda F$,*

- *(i) B is totally geodesic in M.*
- *(ii) F is totally umbilical in M.*

We know that on a CR-submanifold, there exist two orthogonal complementary distributions such that one of them is invariant, while the other is anti-invariant under the action of almost complex structure of the ambient space, (see [3]). Moreover, a warped product manifold has fiber and base. Chen [5], observed this similarity between a warped product manifold and a CR-submanifold and introduced the notion of a CR-warped product submanifold of a Kaehler manifold. Further, O'Neill generalized the concept of Riemannian warped products to semi-Riemannian warped products, (see [15]). In [6], Duggal introduced two classes of warped products of lightlike manifolds. Later on, Sahin [17], to construct a new class of lightlike submanifolds, whose geometry is essentially the same as that of their chosen screen distribution, introduced warped product lightlike submanifolds of semi-Riemannian manifolds as follows.

Definition 3.2. ([17]). Let (M_1, g_1) be a totally lightlike submanifold of dimension r and (M_2, g_2) be a semi-Riemannian submanifold of dimension m of a semi-Riemann manifold $\bar{M}$. Then the product manifold $M = M_1 \times_\lambda M_2$ is said to be a warped product lightlike submanifold of $\bar{M}$ with the degenerate metric g defined by

$$g(U,V) = g_1(\pi_* U, \pi_* V) + (\lambda \circ \pi)^2 g_2(\eta_* U, \eta_* V),$$

for every $U, V \in \Gamma(TM)$ and $*$ is the symbol for the tangent map. Here $\pi_* : M_1 \times M_2 \to M_1$ and $\eta_* : M_1 \times M_2 \to M_2$ denote the projection maps given by $\pi(p,q) = p$ and $\eta(p,q) = q$ for $(p,q) \in M_1 \times M_2$.

Thus, we study SCR-lightlike warped product submanifolds of indefinite Kaehler manifolds similar to the idea of CR-warped product submanifolds given by Chen. Firstly, we will investigate SCR-lightlike submanifolds of an indefinite Kaehler manifold, which are warped products of the type $M_\perp \times_\lambda M_T$, where $M_\perp$ is a totally real submanifold and M_T is a holomorphic submanifold of M.

Theorem 3.3. *Let M be a totally umbilical SCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. If $M = M_\perp \times_\lambda M_T$ is a warped product SCR-lightlike submanifold such that $M_\perp$ is a totally real submanifold and M_T is a holomorphic submanifold of $\bar{M}$, then it is a SCR-lightlike product.*

Proof. Assume that $M = M_\perp \times_\lambda M_T$ be a warped product SCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then for any $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\perp)$, using (3.2), we have

$$\nabla_X Z = \nabla_Z X = (Z ln\lambda)X. \tag{3.4}$$

Now for any $X, Y \in \Gamma(D^\perp)$, from (2.21), we get $f\nabla_X Y = -A_{\omega Y}X - Bh(X,Y)$, then for any $Z \in \Gamma(D)$ and using (2.6) and (3.4), we obtain $g(f\nabla_X Y, Z) = -g(A_{\omega Y}X, Z) = \bar{g}(\bar{\nabla}_X \bar{J}Y, Z) = -\bar{g}(\bar{J}Y, \bar{\nabla}_X Z) = -g(\bar{J}Y, \nabla_X Z) = 0$, then using non-degeneracy of D, we derive $f\nabla_X Y = 0$, which further gives that $\nabla_X Y \in \Gamma(D^\perp)$, this implies $D^\perp$

defines a totally geodesic foliation in M.
Let h^T and A^T, respectively, denote the second fundamental form and shape operator of M_T in M, then for any $X, Y \in \Gamma(D')$ and $Z \in \Gamma(D^\perp)$, we have $g(h^T(X,Y),Z) = g(\nabla_X Y, Z) = \bar{g}(\bar{\nabla}_X Y, Z) = -\bar{g}(Y, \bar{\nabla}_X Z) = -g(Y, \nabla_X Z)$, further using (3.4), we get

$$g(h^T(X,Y),Z) = -(Zln\lambda)g(X,Y). \tag{3.5}$$

Now let $\hat{h}$ be the second fundamental form of M_T in $\bar{M}$, therefore we have

$$\hat{h}(X,Y) = h^T(X,Y) + h^l(X,Y) + h^s(X,Y), \tag{3.6}$$

for any X, Y tangent to M_T. Then for $Z \in \Gamma(D^\perp)$, using (3.6), we obtain

$$g(\hat{h}(X,Y),Z) = g(h^T(X,Y),Z) = -(Zln\lambda)g(X,Y). \tag{3.7}$$

Since M_T is a holomorphic submanifold of $\bar{M}$, therefore

$$\hat{h}(X,\bar{J}Y) = \hat{h}(\bar{J}X,Y) = \bar{J}\hat{h}(X,Y). \tag{3.8}$$

Thus using (3.7) and (3.8), we obtain

$$g(\hat{h}(X,Y),Z) = -g(\hat{h}(\bar{J}X,\bar{J}Y),Z) = (Zln\lambda)g(X,Y). \tag{3.9}$$

On adding (3.7) and (3.9), we derive

$$g(\hat{h}(X,Y),Z) = 0. \tag{3.10}$$

Thus from (3.6), (3.8) and (3.10), we have

$$g(h(X,Y),\bar{J}Z) = g(\hat{h}(X,Y),\bar{J}Z) = -g(\hat{h}(X,\bar{J}Y),Z) = 0. \tag{3.11}$$

Thus $g(h(D',D'),\bar{J}D^\perp) = 0$, this yields that $h(D',D')$ has no component in $\bar{J}D^\perp$, which implies that D' defines a totally geodesic foliation in M. Hence, we conclude that $M = M_\perp \times_\lambda M_T$ is a SCR-lightlike product. □

From Theorem (3.3), we conclude that there exist no warped product SCR-lightlike submanifold of the type $M = M_\perp \times_\lambda M_T$ in an indefinite Kaehler manifold $\bar{M}$. Therefore, in the proceeding part of the paper, we consider warped product SCR-lightlike submanifolds of the type $M = M_T \times_\lambda M_\perp$ in an indefinite Kaehler manifold $\bar{M}$. For simplification, we call a warped product SCR-lightlike submanifold of the type $M = M_T \times_\lambda M_\perp$, a SCR-lightlike warped product. Now, we prove a basic lemma for later use.

Lemma 3.4. *Let M be a SCR-lightlike warped product submanifold of an indefinite Kaehler manifold $\bar{M}$, then*

(i) $\bar{g}(h^s(D,D),\bar{J}D^\perp) = 0$,

(ii) $\bar{g}(h^s(X,Z),\bar{J}V) = -\bar{J}X(ln\lambda)g(Z,V)$,

for any $X \in \Gamma(D')$ and $Z, V \in \Gamma(D^\perp)$.

Proof. Since $\bar{M}$ is a Kaehler manifold, therefore for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$, we have $\bar{J}\bar{\nabla}_X Z = \bar{\nabla}_X \bar{J}Z$. Using (2.6) and (2.7), we have $\bar{J}\nabla_X Z + \bar{J}h(X,Z) = -A_{\bar{J}Z}X + D^l(X,\bar{J}Z) + \nabla^s_X \bar{J}Z$, then taking inner product with $\bar{J}Y$, for $Y \in \Gamma(D)$, we have $g(\nabla_X Z, Y) = -g(A_{\bar{J}Z}X, \bar{J}Y)$. On taking into account (2.8) and (3.2), we derive $\bar{g}(h^s(X,\bar{J}Y),\bar{J}Z) = 0$, which proves (i).
Next for any $X \in \Gamma(D')$ and $Z, V \in \Gamma(D^{\perp})$, from (2.6), (2.13) and (3.2), we have

$$\begin{aligned}\bar{g}(h^s(X,Z),\bar{J}V) &= \bar{g}(\bar{\nabla}_Z X, \bar{J}V) = -\bar{g}(\bar{J}\bar{\nabla}_Z X, V)\\ &= -\bar{g}(\bar{\nabla}_Z \bar{J}X, V) = -g(\nabla_Z \bar{J}X, V)\\ &= -\bar{J}X(ln\lambda)g(Z,V),\end{aligned}$$

which proves (ii). □

Lemma 3.5. *Let $M = M_T \times_\lambda M_\perp$ be a SCR-lightlike warped product submanifold of an indefinite Kaehler manifold $\bar{M}$, then*

$$h^l(X,Z) = 0 \quad and \quad h^*(X,Z) = 0,$$

for any $X \in \Gamma(D')$ and $Z \in \Gamma(D^{\perp})$.

Proof. For any $X \in \Gamma(D'), Y \in \Gamma(Rad(TM)) \subset \Gamma(D')$ and $Z \in \Gamma(D^{\perp})$, from (2.6), we have $\bar{g}(h^l(X,Z),Y) = \bar{g}(\bar{\nabla}_X Z, Y)$. Further we have $\bar{g}(h^l(X,Z),Y) = -\bar{g}(Z,\bar{\nabla}_X Y) = -g(Z,\nabla_X Y)$. Since M is a SCR-lightlike warped product submanifold, therefore D' defines a totally geodesic foliation in M and thus we have $\bar{g}(h^l(X,Z),Y) = 0$, which gives that $h^l(X,Z) = 0$. Similarly, we can prove the second part of the assertion. □

Now we are ready to give a characterization theorem for existence of SCR-lightlike warped product submanifolds of indefinite Kaehler manifolds.

Theorem 3.6. *A proper totally umbilical SCR-lightlike submanifold M of an indefinite Kaehler manifold $\bar{M}$ with totally real distribution $D^{\perp}$ being integrable is locally a SCR-lightlike warped product if and only if*

$$A_{\bar{J}Z}X = -(\bar{J}X)(\mu)Z, \tag{3.12}$$

for each $X \in \Gamma(D'), Z \in \Gamma(D^{\perp})$ and μ is a $C^{\infty}-$ function on M such that $Z\mu = 0$ for each $Z \in \Gamma(D^{\perp})$.

Proof. Let M be a proper totally umbilical SCR-lightlike warped product submanifold of the type $M_T \times_\lambda M_\perp$. As $\bar{M}$ is a Kaehler manifold, therefore for each $X \in \Gamma(D')$ and $Z \in \Gamma(D^{\perp})$, from (2.13), we have $\bar{\nabla}_X \bar{J}Z = \bar{J}\bar{\nabla}_X Z$, which on using (2.5), (2.24) and (3.2) gives that $-A_{\bar{J}Z}X + \nabla^t_X \bar{J}Z = \bar{J}X(ln\lambda)Z$. On equating tangential components on both sides, we derive $A_{\bar{J}Z}X = -\bar{J}X(ln\lambda)Z$. As $\mu = ln\lambda$ is a function on M_T, therefore $Z(\mu) = Z(ln\lambda) = 0$, for all $Z \in \Gamma(D^{\perp})$.

Conversely, let M be a proper totally umbilical SCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ satisfying (3.12). For $X, Y \in \Gamma(D')$ and $Z \in \Gamma(D^{\perp})$, using (3.12), we have $g(A_{\bar{J}Z}X, Y) = -g(((\bar{J}X)\mu)Z, Y) = 0$, then using (2.8), we get $\bar{g}(h^s(X,Y),\bar{J}Z) = 0$. Thus $\bar{g}(h^s(D',D'),\bar{J}Z) = 0$ and also $\bar{g}(h^l(D',D'),\bar{J}Z) = 0$, for any $Z \in \Gamma(D^{\perp})$. Therefore, we have

$$\bar{g}(h(D',D'),\bar{J}Z) = 0,$$

that is, $h(D', D')$ has no component in $\bar{J}D^{\perp}$, which implies that D' defines a totally geodesic foliation in M.
Now taking inner product of (3.12) with $U \in \Gamma(D^{\perp})$ and using hypothesis alongwith (2.6), (2.13), (2.24) and (3.2), we have

$$\begin{aligned} g(((\bar{J}X)\mu)Z, U) &= -g(A_{\bar{J}Z}X, U) = -\bar{g}(\bar{J}Z, \nabla_X U) \\ &= -\bar{g}(\bar{J}Z, \nabla_U X) = \bar{g}(\bar{\nabla}_U \bar{J}Z, X) \\ &= -g(\nabla_U Z, \bar{J}X), \end{aligned} \tag{3.13}$$

where $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. Then using the definition of gradient $g(\nabla\phi, X) = X\phi$ in (3.13), we get

$$g(\nabla_U Z, \bar{J}X) = -g(\nabla\mu, \bar{J}X)g(U, Z). \tag{3.14}$$

Let h' be the second fundamental form of $D^{\perp}$ in M and let ∇' be the induced connection of $D^{\perp}$ in M, then for $U, Z \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$, we have

$$g(h'(U, Z), \bar{J}X) = g(\nabla_U Z - \nabla'_U Z, \bar{J}X) = g(\nabla_U Z, \bar{J}X). \tag{3.15}$$

Then from (3.14) and (3.15), we derive

$$g(h'(U, Z), \bar{J}X) = -g(\nabla\mu, \bar{J}X)g(U, Z). \tag{3.16}$$

Then using non-degeneracy of D, from (3.16), we get

$$h'(U, Z) = -\nabla\mu g(U, Z), \tag{3.17}$$

which implies that the distribution $D^{\perp}$ is totally umbilical in M. By hypothesis, the totally real distribution $D^{\perp}$ is integrable and further, using (3.17) and the condition $Z\mu = 0$ for each $Z \in \Gamma(D^{\perp})$ implies that each leaf of $D^{\perp}$ is an intrinsic sphere in M. Thus by virtue of the result of [11], which states that "*If the tangent bundle of a Riemannian manifold M splits into an orthogonal sum $TM = E_0 \oplus E_1$ of non-trivial vector sub-bundles such that E_1 is spherical and it's orthogonal complement E_0 is auto parallel, then the manifold M is locally isometric to a warped product $M_0 \times_f M_1$*", thus we conclude that M is locally a SCR-lightlike warped product of the type $M_T \times_\lambda M_\perp$ in $\bar{M}$, where $\lambda = e^{\mu}$. Hence the proof is complete. □

From (2.12), we notice that the induced connection ∇ on M is not a metric connection, in general. Therefore, in next theorem, we give one important result on induced connection for SCR-lightlike warped product submanifolds.

Theorem 3.7. *For a proper SCR-lightlike warped product submanifold $M = M_T \times_\lambda M_\perp$ of an indefinite Kaehler manifold $\bar{M}$, the induced connection ∇ can never be a metric connection.*

Proof. If possible, then let ∇ is a metric connection on M, therefore, according to (2.12), we have $h^l = 0$. We know that $\bar{\nabla}$ is a metric connection on $\bar{M}$, therefore for $X \in \Gamma(Rad(TM))$ and $Z, W \in \Gamma(D^{\perp})$, we have $\bar{g}(\bar{\nabla}_Z W, X) = -\bar{g}(W, \bar{\nabla}_Z X)$, further using (2.6) and (3.2), we derive

$$\bar{g}(h^l(Z, W), X) = -X(ln\lambda)g(Z, W). \tag{3.18}$$

Since $h^l = 0$, therefore (3.18) becomes, $X(ln\lambda)g(Z, W) = 0$, which implies that $X(ln\lambda) = 0$ or $g(Z, W) = 0$, but this a contradiction as M is a proper SCR-lightlike warped product submanifold and $D^{\perp}$ is non-degenerate. Hence the proof follows. □

4 SCR-lightlike warped product submanifolds and canonical structures

In this section, we derive some characterizations in terms of the canonical structures f and ω on a SCR-lightlike submanifold of an indefinite Kaehler manifold under which it reduces to a SCR-lightlike warped product submanifold. Before proving the main results, firstly we give a basic lemma.

Lemma 4.1. *Let $M = M_T \times_\lambda M_\perp$ be a SCR-lightlike warped product submanifold of an indefinite Kaehler manifold $\bar{M}$, then*

$$(\nabla_Z f)X = fX(ln\lambda)Z,$$

$$(\nabla_U f)Z = f(\nabla ln\lambda)g(U, Z),$$

for any $U \in \Gamma(TM), X \in \Gamma(D')$ and $Z \in \Gamma(D^\perp)$, where $\nabla(ln\lambda)$ denotes the gradient of $ln\lambda$.

Proof. For $X \in \Gamma(D')$ and $Z \in \Gamma(D^\perp)$, from (2.23) and (3.2), we have $(\nabla_Z f)X = \nabla_Z fX = fX(ln\lambda)Z$.
Again using (2.23), for $U \in \Gamma(TM)$ and $Z \in \Gamma(D^\perp)$, we get $(\nabla_U f)Z = -f\nabla_U Z$, which implies that $(\nabla_U f)Z \in \Gamma(D')$. Then for any $X \in \Gamma(D)$, we have

$$\begin{aligned} g((\nabla_U f)Z, X) &= -g(f\nabla_U Z, X) = g(\nabla_U Z, fX) \\ &= \bar{g}(\bar{\nabla}_U Z, fX) = -\bar{g}(Z, \nabla_U fX) \\ &= -fX(ln\lambda)g(Z, U). \end{aligned} \tag{4.1}$$

Then from definition of gradient of λ and non-degeneracy of D, the result follows. □

Theorem 4.2. *Let M be a SCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with totally real distribution $D^\perp$ being integrable, then M is locally a SCR-lightlike warped product submanifold if and only if*

$$(\nabla_U f)V = ((fV)\mu)PU + g(PU, PV)\bar{J}(\nabla\mu), \tag{4.2}$$

for each $U, V \in \Gamma(TM)$, where μ is a $C^\infty-$ function on M satisfying $Z\mu = 0$ for each $Z \in \Gamma(D^\perp)$.

Proof. Assume that M be a SCR-lightlike warped product submanifold of an indefinite Kaehler manifold $\bar{M}$. Then, for any $U, V \in \Gamma(TM)$, we have

$$(\nabla_U f)V = (\nabla_{QU} f)QV + (\nabla_{PU} f)QV + (\nabla_U f)PV. \tag{4.3}$$

Since D' defines a totally geodesic foliation in M, therefore using (2.21), we have

$$(\nabla_{QU} f)QV = 0. \tag{4.4}$$

Further using Lemma (4.1), we obtain

$$(\nabla_{PU} f)QV = f(QV)(ln\lambda)PU, \tag{4.5}$$

$$(\nabla_U f)PV = g(U, PV)f(\nabla ln\lambda) = g(PU, PV)f(\nabla ln\lambda). \tag{4.6}$$

Then from (4.3) - (4.6), we derive (4.2). Since $\mu = ln\lambda$ is a function on M_T, therefore $Z(\mu) = Z(ln\lambda) = 0$, for all $Z \in \Gamma(D^\perp)$.

Conversely, let M be a SCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ satisfying (4.2). Let $U, V \in \Gamma(D')$, then (4.2) implies that $(\nabla_U f)V = 0$, then using (2.21), we have $Bh(U, V) = 0$, this shows that $h(U, V)$ has no component in $\bar{J}D^\perp$, for each $U, V \in \Gamma(D')$, which yields that D' defines a totally geodesic foliation in M.
Now for $U, V \in \Gamma(D^\perp)$, from (4.2), we have

$$(\nabla_U f)V = g(PU, PV)\bar{J}\nabla\mu. \tag{4.7}$$

Taking inner product of (4.7) with $X \in \Gamma(D)$, we obtain

$$g((\nabla_U f)V, X) = g(PU, PV)g(\bar{J}\nabla\mu, X) = -g(PU, PV)g(\nabla\mu, \bar{J}X). \tag{4.8}$$

Then for $U, V \in \Gamma(D^\perp)$ and $X \in \Gamma(D)$, using (2.21), we get

$$\begin{aligned} g((\nabla_U f)V, X) = g(A_{\omega V}U, X) &= -\bar{g}(\bar{\nabla}_U \bar{J}V, X) \\ &= g(\nabla_U V, \bar{J}X). \end{aligned} \tag{4.9}$$

From (4.8) and (4.9), we have

$$g(\nabla_U V, \bar{J}X) = -g(PU, PV)g(\nabla\mu, \bar{J}X). \tag{4.10}$$

Let h' be the second fundamental form of $D^\perp$ in M and let ∇' be the induced connection of $D^\perp$ in M, then for $U, V \in \Gamma(D^\perp)$ and $X \in \Gamma(D)$, we get

$$g(h'(U, V), \bar{J}X) = g(\nabla_U V - \nabla^{'}_U V, \bar{J}X) = g(\nabla_U V, \bar{J}X). \tag{4.11}$$

Now from (4.10) and (4.11), we derive

$$g(h'(U, V), \bar{J}X) = -g(PU, PV)g(\nabla\mu, \bar{J}X), \tag{4.12}$$

then the non-degeneracy of D implies that $h'(U, V) = -\nabla\mu g(PU, PV)$, which shows that the distribution $D^\perp$ is totally umbilical in M. Moreover, by hypothesis, the totally real distribution $D^\perp$ is integrable and in view of condition that $Z\mu = 0$, for each $Z \in \Gamma(D^\perp)$, each leaf of $D^\perp$ is an intrinsic sphere. Thus, by similar argument as in Theorem (3.6), M is locally a SCR-lightlike warped product of the type $M_T \times_\lambda M_\perp$ in $\bar{M}$ with a warping function $\lambda = e^\mu$, which completes the proof. □

Theorem 4.3. *Let M be a SCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with totally real distribution $D^\perp$ being integrable, then M is locally a SCR-lightlike warped product submanifold if and only if*

$$\bar{g}((\nabla^t_U \omega)V, \bar{J}W) = -QV(\mu)g(U, W), \tag{4.13}$$

for any $U, V \in \Gamma(TM)$ and $W \in \Gamma(D^\perp)$, where μ is a $C^\infty-$ function on M satisfying $W\mu = 0$ for each $W \in \Gamma(D^\perp)$.

Proof. Let M be SCR-lightlike warped product submanifold of an indefinite Kaehler manifold $\bar{M}$. Therefore, the distribution D' defines a totally geodesic foliation in M, thus using (2.23) for $U, V \in \Gamma(D')$ and $W \in \Gamma(D^{\perp})$, we have

$$\bar{g}((\nabla^t_U\omega)V, \bar{J}W) = \bar{g}(-\omega\nabla_U V, \bar{J}W) = -g(\nabla_U V, W) = 0. \tag{4.14}$$

For $U, W \in \Gamma(D^{\perp})$ and $V \in \Gamma(D')$, using (2.22) and Lemma (3.4), we obtain

$$\begin{aligned}\bar{g}((\nabla^t_U\omega)V, \bar{J}W) &= -\bar{g}(h^s(U, fV), \bar{J}W) = \bar{J}fV(ln\lambda)g(U, W)\\ &= -QV(ln\lambda)g(U, W).\end{aligned} \tag{4.15}$$

Now for $U \in \Gamma(D')$ and $V \in \Gamma(D^{\perp})$ or $U, V \in \Gamma(D^{\perp})$, using (2.22), we get

$$\bar{g}((\nabla^t_U\omega)V, \bar{J}W) = \bar{g}(Ch(U, V), \bar{J}W) = 0, \tag{4.16}$$

where $W \in \Gamma(D^{\perp})$. Thus from (4.14)-(4.16), we derive (4.13). As $\mu = ln\lambda$ is a function on M_T, therefore $W(\mu) = W(ln\lambda) = 0$, for all $W \in \Gamma(D^{\perp})$.

Conversely, let M be a SCR-lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$ with totally real distribution $D^{\perp}$ integrable, satisfying (4.13). For any $U, V \in \Gamma(D')$ and $W \in \Gamma(D^{\perp})$, from (4.13), we have $\bar{g}(w\nabla_U V, \bar{J}W) = 0$, then $g(\nabla_U V, W) = 0$, which implies that $\nabla_U V \in \Gamma(D')$, that is, D' defines a totally geodesic foliation in M. Next for $V \in \Gamma(D)$ and $U, W \in \Gamma(D^{\perp})$, from (4.13), we have

$$\begin{aligned}-V(\mu)g(U, W) &= \bar{g}((\nabla^t_U\omega)V, \bar{J}W) = -\bar{g}(\omega\nabla_U V, \bar{J}W)\\ &= -g(\nabla_U V, W) = -\bar{g}(\bar{\nabla}_U V, W)\\ &= g(V, \nabla_U W).\end{aligned} \tag{4.17}$$

Then using the definition of gradient $g(\nabla\phi, V) = V\phi$ in (4.17), we get

$$g(\nabla_U W, V) = -g(\nabla\mu, V)g(U, W). \tag{4.18}$$

Let h' and ∇', respectively, denote the second fundamental form and the induced connection of $D^{\perp}$ in M, then

$$g(h'(U, W), V) = g(\nabla_U W - \nabla'_U W, V) = g(\nabla_U W, V), \tag{4.19}$$

where $U, W \in \Gamma(D^{\perp})$ and $V \in \Gamma(D)$. Then from (4.18) and (4.19), we derive

$$g(h'(U, W), V) = -g(\nabla\mu, V)g(U, W). \tag{4.20}$$

Then using non-degeneracy of D, from (4.20), we get

$$h'(U, W) = -\nabla\mu g(U, W), \tag{4.21}$$

which implies that the distribution $D^{\perp}$ is totally umbilical in M. From hypothesis, the totally real distribution $D^{\perp}$ is integrable and in view of condition that $W\mu = 0$, for each $W \in \Gamma(D^{\perp})$, each leaf of $D^{\perp}$ is an intrinsic sphere. Thus, by similar argument as in Theorem (3.6), M is locally a SCR-lightlike warped product of the type $M_T \times_{\lambda} M_{\perp}$ in $\bar{M}$ with a warping function $\lambda = e^{\mu}$, which completes the proof. □

5 An inequality for SCR-lightlike warped product submanifolds

Now, we construct an inequality for the second fundamental form h of SCR-lightlike warped product submanifolds of indefinite Kaehler manifolds. For this purpose, we make use of formulae and results discussed in the previous sections.

Theorem 5.1. *Let $M = M_T \times_\lambda M_\perp$ be a SCR-lightlike warped product submanifold of an indefinite Kaehler manifold $\bar{M}$. Then we have*

(i) The squared norm of the second fundamental form satisfies

$$\|h\|^2 \geq 2q\|\nabla(ln\lambda)\|^2, \tag{5.1}$$

where $\nabla(ln\lambda)$ is the gradient of $ln\lambda$ and q is the dimension of $M_\perp$.

(ii) If the equality sign in (5.1) holds identically, then M_T is totally geodesic in $\bar{M}$ and $M_\perp$ is totally umbilical in $\bar{M}$.

Proof. Let $\{X_1, X_2, X_3, ..., X_p, X_{p+1} = \bar{J}X_1, X_{p+2} = \bar{J}X_2, ..., X_{2p} = \bar{J}X_p, X_{2p+1} = \xi_1, X_{2p+2} = \xi_2, ..., X_{2p+r} = \xi_r, X_{2p+r+1} = \bar{J}\xi_1, X_{2p+r+2} = \bar{J}\xi_2, ..., X_{2p+2r} = \bar{J}\xi_r\}$ be a local orthonormal frame of vector fields on M_T and $\{Z_1, Z_2, Z_3, ..., Z_q\}$ a local orthonormal frame of vector fields on $M_\perp$, then we have

$$\|h\|^2 = \|h(D', D')\|^2 + \|h(D^\perp, D^\perp)\|^2 + 2\|h(D', D^\perp)\|^2. \tag{5.2}$$

By virtue of (2.4), (5.2) becomes

$$\|h\|^2 = \|h^s(D', D')\|^2 + \|h^s(D^\perp, D^\perp)\|^2 + 2\|h^s(D', D^\perp)\|^2. \tag{5.3}$$

Further, we have

$$\begin{aligned} \|h\|^2 = & \sum_{i,j=1}^{2p+2r} \bar{g}(h^s(X_i, X_j), h^s(X_i, X_j)) + \sum_{m,n=1}^{q} \bar{g}(h^s(Z_m, Z_n), h^s(Z_m, Z_n)) \\ & + 2\sum_{i=1}^{2p+2r}\sum_{m=1}^{q} \bar{g}(h^s(X_i, Z_m), h^s(X_i, Z_m)). \end{aligned} \tag{5.4}$$

Thus,

$$\|h\|^2 \geq 2\sum_{i=1}^{2p+2r}\sum_{m=1}^{q} \bar{g}(h^s(X_i, Z_m), h^s(X_i, Z_m)). \tag{5.5}$$

Then using Lemma (3.4), (5.5) reduces to

$$\begin{aligned} \|h\|^2 & \geq 2\sum_{i=1}^{2p+2r}\sum_{m=1}^{q} (X_i ln\lambda)^2 g(Z_m, Z_m) \\ & \geq 2q\|\nabla(ln\lambda)\|^2, \end{aligned} \tag{5.6}$$

which proves the assertion (i). Moreover, if the equality sign in (5.1) holds, then we have

$$h^s(D', D') = 0, \quad h^s(D^\perp, D^\perp) = 0 \quad \text{and} \quad h^s(D', D^\perp) \subset \bar{J}D^\perp. \tag{5.7}$$

Since M_T is totally geodesic in M, then from first condition in (5.7), we have M_T is totally geodesic in $\bar{M}$. Moreover, as $M_\perp$ is totally umbilical in M, the second condition in (5.7) implies that $M_\perp$ is totally umbilical in $\bar{M}$, which completes the proof. ☐

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References

[1] M. Barros and A. Romero, *Indefinite Kaehler manifolds*, Math. Ann. 261 (1982), 55–62.

[2] J. K. Beem, P. E. Ehrlich and T. G. Powell, *Warped product manifolds in Relativity*, Selected Studies: Physics-Astrophysics, Mathematics, History of Science, North-Holland Publishing Co.-Amsterdam, New York, 1982; 41–56.

[3] A. Bejancu, *CR-submanifolds of a Kaehler manifold-I*, Proc. Amer. Math. Soc. 9 (1978), 135–142.

[4] R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. 145 (1969), 1–49.

[5] B. Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds*, Monatsh. Math. 133 (2001), 177–195.

[6] K. L. Duggal, *Constant scalar curvature and warped product globally null manifolds*, J. Geom. Phys. 4 (2002), 327–340.

[7] K. L. Duggal and A. Bejancu, *Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications*, Mathematics and its Applications 364, Kluwer Academic Publishers, Dordrecht, 1996.

[8] K. L. Duggal and D. H. Jin, *Totally umbilical lightlike submanifolds*, Kodai Math. J. 26 (2003), 49–68.

[9] K. L. Duggal and B. Sahin, *Screen Cauchy-Riemann lightlike submanifolds*, Acta Math. Hungar. 106 (2005), 137–165.

[10] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973.

[11] S. Hiepko, *Eine Innere Kennzeichnung der verzerrten Produkte*, Math. Ann. 241 (1979), 209–215.

[12] S. Kumar, *Geometry of warped product lightlike submanifolds of indefinite nearly Kaehler manifolds*, J. Geom. 109 (2018), 21.

[13] S. Kumar, *Warped product semi-transversal lightlike submanifolds of indefinite nearly Kaehler manifolds*, Differ. Geom. Dyn. Syst. 20 (2018), 106–118.

[14] M. Meumertzhein, H. Reckziegel and M. Schaaf, *Decomposition of twisted and warped product nets*, Results Math. 36 (1999), 297–312.

[15] B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, San Diego, 1983.

[16] K. Rajaratnam and R. G. McLenaghan, *Killing tensors, warped products and the orthogonal separation of the Hamilton-Jacobi equation*, J. Math. Phys. 55 (2014), Article ID 013505, 27 pages.

[17] B. Sahin, *Warped product lightlike submanifolds*, Sarajevo J. Math. 14 (2005), 251–260.

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