Some curves on three-dimensional α-para-Kenmotsu manifolds

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Abstract. In the present paper, we study magnetic biharmonic and biminimal curves on a three-dimensional α-para-Kenmotsu manifold. We obtain necessary and sufficient conditions for biharmonicity and biminimality of a non-null magnetic curve. Also we study the mean curvature vector of a Legendre curve satisfying some recurrent conditions on a three-dimensional α-para-Kenmotsu manifold.

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1 Introduction

The trajectories of charged particles moving on a Riemannian manifold (M, g) under the action of a magnetic field F is known as magnetic curves. In three dimensional oriented Riemannian manifold (M^3, g), a divergence free vector field defined as a magnetic field. A closed 2-form F on M is called the magnetic field. The Lorentz force of a magnetic field F on (M, g) is a (1,1) tensor field Φ is defined by

$$g(\Phi(X), Y) = F(X, Y), \tag{1.1}$$

for any X, Y in $\chi(M)$.
A regular curve γ will be magnetic curve with F, if it satisfies the Lorentz equation (also known as Newton's equation)

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \Phi(\dot{\gamma}), \tag{1.2}$$

where ∇ is the Levi-Civita connection on g. When Lorentz forces vanishes, we have $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$. If $\nabla F = 0$, then a ma magnetic field is known as uniform. The magnetic curves are of constant speed. Specially, unit speed curves are called normal magnetic curves [10].
Let (M, g) and (N, h) be two (pseudo-)Riemannian manifolds and $\Psi : (M, g) \to (N, h)$ a smooth map. The energy functional of Ψ is defined by $E(\Psi) = \frac{1}{2}\int_M |d\Psi|^2 v_g$. Critical points of the energy functional are called harmonic maps and the Euler-Lagrange

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equation for the energy is $\tau(\Psi) = trace\nabla d\Psi = 0$, where ∇ denotes Levi-Civita connection on M. Biharmonic maps which can considered a natural generalization of harmonic maps are defined as critical points of the bienergy functional given by $E(\Psi) = \frac{1}{2}\int_M |\tau(|\Psi)|^2 v_g$. The first variation formula for the bienergy is derived by G. Y. Jiang [14] and it is proved that the Euler-Lagrange equation for bienergy is

$$\tau_2(\Psi) = -J(\tau(\Psi)) = -\nabla\tau(\Psi) - traceR^N(d\Psi, \tau(\Psi))d\Psi = 0,$$

where J is the Jacobi operator, $\nabla = -trace(\nabla^\Psi\nabla^\Psi - \nabla^\Psi_\nabla)$ is the rough Laplacian on the sections of pull bundle $\Psi^{-1}TN$, ∇^Ψ is the pull-back connection [11] and R^N is the curvature operation on N. One can easily see that harmonic maps are always biharmonic. Biharmonic maps which are not harmonic are called proper biharmonic maps.

An immersion $\Psi : (M, g) \to (N, h)$ between Riemannian manifolds is called biminimal if it is a critical point of the bienergy functional for variations normal to the image $\Psi(M) \subset N$, with fixed energy. Equivalently, there exists a constant $\lambda \in R$ such that Ψ is a critical point of the λ-bienergy

$$E_{2,\lambda}(\Psi) = E_2(\Psi) + \lambda E(\Psi),$$

for any smooth variation of the map $\Psi_t : (-\varepsilon, \varepsilon) \times M \to N, \Psi_0 = \Psi$, such that $V = \frac{d\Psi_t}{dt}|_{t=0}$ is normal to $\Psi(M)$ [18]. In [6], magnetic curves in Sasakian 3-manifolds were studied. In [10], magnetic curves in a $(2n+1)$-dimensional Sasakian manifold was studied. In [7], Killing magnetic curves in three-dimensional almost paracontact manifolds were studied. Magnetic curves in cosympletic manifolds were studied in [9]. In the contact manifolds, Legendre curves play an important role, e.g., a diffeomorphism of a contact manifold is a contact transformation if and only if it maps Legendre curves to Legendre curves. Legendre curves on contact manifolds have been studied by C. Baikoussis and D. E. Blair in the paper [1]. Belkhelfa et al [2] have investigated Legendre curves in Riemannian and Lorentzian manifolds. In [8], slant curves, as generalization of Legendre curves, have been studied on three-dimensional Sasakian space forms. Legendre curves on almost contact and contact manifolds have also been studied in the papers [13], [21], [29]. The author has also studied curves on almost contact manifolds in the papers [24], [25], [26].

In the present paper, we study magnetic biharmonic and biminimal curves in a three-dimensional α-para-Kenmostu manifold with α= constant. Also we study Legendre curve satisfying some recurrent conditions in a three-dimensional α-para-Kenmostu manifold.

The present paper is organized as follows:

After the introduction in Section 1, we give some required preliminaries in Section 2. In Section 3, we prove that γ is slant if and only if M is cosymplectic for a contact magnetic curve γ in a three-dimensional α-para-Kenmotsu manifold. Section 4 is devoted to study biharmonic magnetic curves on three-dimensional α-para-Kenmotsu manifold and we obtain necessary and sufficient conditions for biharmonicity of a non-null magnetic curve. In Section 5, we study biminimal curves in three-dimensional α-para-Kenmotsu manifold. In the next Section, we study Legendre curves whose mean curvature vector field satisfies some recurrent conditions with respect to Levi-Civita connection ∇. Last Section we construct an example of a Legendre curve in a three-dimensional α-para-Kenmotsu manifold which is geodesic.

2 Preliminaries

Let $(M, \varphi, \eta, \xi, g)$ be a $(2n+1)-$dimensional smooth manifold, where φ is an $(1,1)$ tensor field, ξ is a vector field, η is an 1-form on M. We say that (φ, η, ξ) is called an almost para contact metric structure on M, if satisfies the conditions

$$\varphi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \tag{2.1}$$

$$\varphi\xi = 0, \quad \eta(\varphi) = 0. \tag{2.2}$$

The tensor field φ induces an almost paracomplex structure on the distribution $\mathcal{D} = \ker\eta$, that is, the eigen distribution $\mathcal{D}^+, \mathcal{D}^-$ corresponding to the eigen values 1, -1 of φ, respectively, have equal dimension n.
M is said to be almost paracontact manifold if it is endowed with an almost paracontact structure [5], [12], [15], [30].

An almost paracontact manifold is called an almost paracontact metric manifold if it is additionally endowed with pseudo-Riemannian metric g of signature $(n+1, n)$ and such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X), \tag{2.4}$$

for any vector fields X and Y on M.
An almost paracontact structure is said to be a contact structure if

$$g(X, \varphi Y) = d\eta(X, Y), \tag{2.5}$$

with the associated metric g [30]. Moreover, we can define a skew-symmetric 2-form Φ by $\Phi(X, Y) = g(X, \varphi Y)$, which is called the fundamental form corresponding to the structure. Note that $\eta \wedge \Phi^n$ up to a constant factor the Riemannian volume element of M.

On an almost paracontact manifold, one defines the (2,1)-tensor field $N^{(1)}$ by

$$N^{(1)}(X, Y) = [\varphi, \varphi](X, Y) - 2d\eta(X, Y)\xi, \tag{2.6}$$

where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]. \tag{2.7}$$

If $N^{(1)}$vanishes identically, then the almost paracontact manifold (structure) is said to be normal. The normality condition say that the almost paracomplex structure J defined on $M \times R$ by

$$J(X, \lambda \tfrac{d}{dt}) = (\varphi X + \lambda \xi, \eta(X) \tfrac{d}{dt}),$$

is integrable.

Here we mention two important result from [28]. For a three-dimensional almost paracontact metric manifold M, the following conditions are equivalent [28]

(i) M is normal,

(ii) there exist functions α, β on M such that

$$(\nabla_X \varphi)Y = \alpha(g(\varphi X, Y)\xi - \eta(Y)\varphi X) + \beta(g(X, Y)\xi - \eta(Y)X), \tag{2.8}$$

(iii) there exist functions α and β on M.

$$\nabla_X \xi = \beta \varphi X + \alpha(X - \eta(X)\xi), \tag{2.9}$$

where ∇ is Levi-Civita connection on M. The functions α, β appearing in the above equations are given by

$$2\alpha = \text{Trace}\{X \to \nabla_X \xi\}, \quad 2\beta = \text{Trace}\{X \to \varphi \nabla_X \xi\}. \tag{2.10}$$

A three-dimensional normal almost paracontact metric manifold is said to be

- paracosymplectic if $\alpha = \beta = 0$,
- quasi-para-Sasakian manifold if and only if $\alpha = 0$ and $\beta \neq 0$,
- β-para-Sasakian manifold if and only if $\alpha = 0$ and β is a non-zero constant, in particular, para-Sasakian if $\beta = -1$,
- α-para-Kenmotsu manifold if and only if $\alpha \neq 0$ and α is a constant and $\beta = 0$.

In a three-dimensional α-para-Kenmotsu manifold, the following result hold [27]

$$\begin{aligned} R(X, Y)Z = (\frac{r}{2} + 2\alpha^2)[g(Y, Z)X - g(X, Z)Y \\ - (\frac{r}{2} + 3\alpha^2)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi \\ + (\frac{r}{2} + 3\alpha^2)[\eta(X)Y - \eta(Y)X]\eta(Z), \end{aligned} \tag{2.11}$$

and

$$S(X, Y) = (\frac{r}{2} + \alpha^2)g(X, Y) - (\frac{r}{2} + 3\alpha^2)\eta(X)\eta(Y)), \tag{2.12}$$

where α is a constant and r is the scalar curvature of the manifold.

$$(\nabla_X \varphi)Y = \alpha[g(\varphi X, Y)\xi - \eta(Y)\varphi X], \tag{2.13}$$

$$\nabla_X \xi = \alpha[X - \eta(X)\xi], \tag{2.14}$$

$$(\nabla_X \eta)Y = \alpha[g(X, Y) - \eta(X)\eta(Y)], \tag{2.15}$$

for all vector fields X, Y and $Z \in \chi(M)$.

A curve γ on M is called Frenet curve with respect to Levi-Civita connection on M if one of the following cases holds:

(a) γ is of osculating order 1, i.e., $\nabla_T T = 0$(geodesic).

(b) γ is of osculating order 2, i.e., there exist two orthonormal vetor fields $T(= \dot{\gamma})$, N and a non-negative function k_1(curvature) along γ such that $\nabla_T T = k_1 N, \nabla_T N = -k_1 T$.

(c) γ is of osculating order 3, i.e., there exist three orthonormal vectors $T(= \dot{\gamma})$, N, B and two non-negative functions k_1(curvature) and k_2(torsion) along γ such that

$$\nabla_T T = k_1 N, \tag{2.16}$$

$$\nabla_T N = -k_1 T + k_2 B, \tag{2.17}$$

$$\nabla_T B = -k_2 N, \tag{2.18}$$

where $k_1 = |\nabla_T T|$ and k_2 are the curvature and torsion of the curve with respect to Levi-Civita connection and $\{T, N, B\}$ is an orthonormal Frenet frame and $T = \dot{\gamma}$.

A Frenet curve of osculating order 3 is called a circle if k_1 is a positive constant and $k_2 = 0$, a Frenet curve is called a helix in M if k_1 and k_2 both are positive constants and the curve is called a generalized helix if $\frac{k_1}{k_2}$ is a constant.
A Frenet curve γ in an almost contact metric manifold is said to be Legendre curves (almost contact curve) if it is an integral curve of the distribution $\mathcal{D} = \ker\eta$, i.e., if $\eta(\dot{\gamma}) = 0$.

The angle between the tangent to γ and the reeb vector field ξ is known as the contact angle θ of γ, that is $\cos\theta(s) = g(\dot{\gamma(s)}, \xi)$, where s denotes the arc length parameter of γ. We call $\gamma(s)$ a slant curve if the contact angle θ is constant. The curve of contact angle $\frac{\pi}{2}$ are traditionally called Legendre curves. A curve of contact angle 0 is called a Reeb flow. For more details we refer [1], [3], [8], [17].

3 Contact magnetic curves

Let M be a three -dimensional α-para-Kenmotsu manifold and Φ be the fundamental 2-form of M. Since $\Phi = d\eta$, then magnetic field F_q on M can be defined by

$$q\Phi(X, Y) = F_q(X, Y), \tag{3.1}$$

where q is a real constant and X, $Y \in \chi(M)$. F_q known as the contact magnetic field with the strength q. The contact magnetic field vanishes and magnetic curves are the geodesic on M, if $q = 0$. Now, we assume $q \neq 0$.

The Lorentz force Φ_q associated to the contact magnetic field F_q may be easily determined combining (2.5) and (1.1), that is

$$\Phi_q = -q\varphi. \tag{3.2}$$

Now, the Lorentz equation (1.2) is given by

$$\nabla_{\dot\gamma}\dot\gamma = -q\varphi\dot\gamma, \tag{3.3}$$

where γ is a Frenet curve parametrized by arc length and the solution of the above equation known as the normal magnetic curve or contact magnetic curve.

Let $\gamma : I \subset R \to M$ be differentiable curve parametrized by arclength immersed in a Riemmian manifold (M, g). Then $\tau(\gamma) = \nabla^{\gamma}_{\frac{\partial}{\partial s}} d\gamma(\frac{\partial}{\partial s}) = \nabla_T T$ and the biharmonic equation for γ reduces to $0 = \tau_2(\gamma) = \nabla^3_T T - R(T, \nabla_T T)T$, that is, γ is called a biharmonic curve if it is a solution of this equation [19]. On the other hand, the biminimality equation for γ is given by $0 = \tau_{2,\lambda}(\gamma) = [\tau_2(\gamma)]^{\perp} - \lambda[\tau(\gamma)]^{\perp}$, for a value of $\lambda \in R$, where $[,]^{\perp}$ denotes the normal component of $[,]$, that is, γ is called a biminimal curve if it is a solution of this equation. In particular, γ is called free biminimal if it is biminimal for $\lambda = 0$.

Assume that $\gamma : I \subset R \to M$ is a magnetic curve parametrized by arclength in three-dimensional α-para-Kenmotsu manifold $(M, \varphi, \xi, \eta, g)$ and assume $g(T, T) = \epsilon = \pm 1$, where $T = \dot\gamma$.

4 Biharmonic magnetic curve on three-dimensional α-para-Kenmotsu manifolds

In this section we study biharmonic magnetic curves on three-dimensional α-para-Kenmotsu manifolds.

Differentiating $g(\dot\gamma, \xi)$ along a contact magnetic curve γ in contact α-para-Kenmotsu manifold and using (2.14) and (3.3), we have

$$\begin{aligned} \frac{d}{dt} g(\dot\gamma;\xi) &= g(\nabla_{\dot\gamma}\dot\gamma, \xi) + g(\dot\gamma, \nabla_{\dot\gamma}\xi,) \\ &= \alpha(\epsilon - (\eta(\dot\gamma))^2). \end{aligned} \tag{4.1}$$

Hence we can state the following

Proposition 4.1. *Let γ be a contact magnetic curve in a α-para-Kenmotsu manifold M with $\epsilon - (\eta(\dot\gamma))^2 \neq 0$. Then γ is a slant curve if and only if M is paracosymplectic.*

Differentiating (3.3) along γ and using (2.13), we get

$$\begin{aligned} \nabla^3_T T = &(q\alpha^2\epsilon - q^3 - 2\alpha^2 q(\eta(T))^2)\varphi T \\ &+ (3\alpha q^2(\eta(T))^2 - q^2\alpha\epsilon)\xi \\ &- 2q^2\alpha\eta(T)T. \end{aligned} \tag{4.2}$$

From (2.11) and (3.3), we get

$$R(T, \nabla_T T)T = [(\frac{r}{2} + 3\alpha^2)(\eta(T))^2 - (\frac{r}{2} + 2\alpha^2)\epsilon]\varphi T. \tag{4.3}$$

So we get

$$\begin{aligned}\tau_2(\gamma) = {} & [(\frac{r}{2}+2\alpha^2)\epsilon-(\frac{r}{2}+3\alpha^2)(\eta(T))^2+q\alpha^2\epsilon-q^3-2\alpha^2q(\eta(T))^2]\varphi T\\ & +(3\alpha q^2(\eta(T))^2-q^2\alpha\epsilon)\xi\\ & -2q^2\alpha\eta(T)T. \end{aligned} \tag{4.4}$$

Hence we state the following

Theorem 4.2. *Let M be a three-dimensional α-para-Kenmotsu manifold and $\gamma : I \to M$ be non-null magnetic curve. Then γ is a biharmonic if*

(i) $\alpha = 0$, $\quad r = -\frac{2q^3}{(\eta(T))^2-\epsilon}$, *and*

(ii) $\alpha = 0$, $\quad \eta(T) = 0$, $\quad r = \frac{2q^3}{\epsilon}$.

Corollary 4.3. *Let M be three-dimensional paracosympletic manifold and $\gamma : I \to M$ be non-null magnetic curve. Then γ is a biharmonic curve if M is constant scalar curvature $r = \frac{2q^3}{\epsilon}$ and γ is Legendre curve.*

Corollary 4.4. *There does not exist a biharmonic magnetic curve in three-dimensional α-para-Kenmotsu manifold.*

5 Biminimal curves in three-dimensional α-para-Kenmotsu manifolds

In this section we investigate biminimal curves in three-dimensional α-para-Kenmotsu manifolds.
By using tension field of γ and (4.3) in biminimal equation we get

$$\begin{aligned}\tau_{2,\lambda}(\gamma) = {} & [(\frac{r}{2}+2\alpha^2)\epsilon-(\frac{r}{2}+3\alpha^2)(\eta(T))^2+q\alpha^2\epsilon-q^3-2\alpha^2q(\eta(T))^2-\lambda]\varphi T\\ & +(3\alpha q^2(\eta(T))^2-q^2\alpha\epsilon)\xi, \end{aligned} \tag{5.1}$$

for some $\lambda \in R$. Then we have

Theorem 5.1. *Let $\gamma : I \to M$ be a non-null magnetic curve. Then γ is a biminimal curve if and only if*

(i) $\alpha = 0$, $\quad \frac{2(q^3+\lambda)}{\epsilon-(\eta(T))^2}$, *and*

(ii) $\alpha \neq 0$, $\quad (\eta(T))^2 = \frac{\epsilon}{q}$, $\quad r = \frac{4q(\lambda+q^3)+4\alpha^2\epsilon(3-q^2)}{(2q-1)\epsilon}$.

From the Theorem 4.2 and Theorem 5.1, we conclude

Corollary 5.2. *A free biminimal magnetic Legendre curve in three-dimensional cosymplectic manifold is biharmonic.*

6 Legendre curves on three-dimensional $\alpha-$para-Kenmotsu manifolds with the mean curvature vector satisfying some recurrent conditions

In this section we study Legendre curves on three-dimensional $\alpha-$para-Kenmotsu manifolds with mean curvature vector satisfying some recurrent conditions with respect to Levi-Civita connections. Mean Curvature vector of a Legendre curve has been studied in the papers [21], [16]. For the definition of recurrent, 2-recurrent and generalized 2-recurrent tensors we refer [22], [23].

Definition 6.1. The mean curvature vector $H = \nabla_{\dot{\gamma}}\dot{\gamma}$ of Legendre curve on a three-dimensional α-para-Kenmotsu manifold with respect to Levi-Civita connection ∇ will be called

(i) parallel if $\nabla_{\dot{\gamma}} H = 0$,

(ii) recurrent if $\nabla_{\dot{\gamma}} H = \omega(\dot{\gamma})H$,

(iii) 2-recurrent if $\nabla^2_{\dot{\gamma}} H = \omega(\dot{\gamma})H$,

(iv) generalized 2-recurrent if $\nabla^2_{\dot{\gamma}} H = \omega(\dot{\gamma})H + \rho(\dot{\gamma}, \dot{\gamma})H$,

where ω is an 1-form and ρ is a 2-form defined on the tangent space of γ.

In this section we consider $\{\dot{\gamma}, \varphi\dot{\gamma}, \xi\}$ as orthonormal Frenet frame.

Proposition 6.1. *The mean curvature vector of a Legendre curve on a three-dimensional α-para-Kenmotsu manifold with respect to Levi-Civita connection is parallel if and only if the curvature k_1 of the curve is zero.*

Proof. By definition of H and Serret-Frenet formula, we get

$$\begin{aligned} \nabla_{\dot{\gamma}} H &= \nabla\dot{\gamma}(k_1\varphi\dot{\gamma}) \\ &= k_1'\varphi\dot{\gamma} + k_1\nabla(\varphi\dot{\gamma}) \\ &= k_1'\varphi\dot{\gamma} + k_1(\nabla\varphi)\dot{\gamma} + k'\varphi(\nabla_{\dot{\gamma}}\dot{\gamma}). \end{aligned} \tag{6.1}$$

Consider H is parallel, then $\nabla_{\dot{\gamma}} H = 0$. Using (2.1), (2.13) and Serret-Frenet formula, we get from above equation

$$k_1^2\dot{\gamma} + k_1'\varphi\dot{\gamma} = 0. \tag{6.2}$$

Taking inner product with $\dot{\gamma}$ in both sides of the above equation we get $k_1 = 0$. The converse is trivial.

Proposition 6.2. *With respect to Levi-Civita connection, the mean curvature vector of a Legendre curve on a three-dimensional α-para-Kenmotsu manifold is recurrent if and only if the curvature k_1 of the curve is zero.*

Proof. Let us consider H is recurrent with respect to Levi-Civita connection. So from definition 6.1., we get

$$k_1^2\dot{\gamma} + (k_1' - k_1\omega(\dot{\gamma}))\varphi\dot{\gamma} = 0. \tag{6.3}$$

Taking inner product with $\dot{\gamma}$ in both sides of the above equation we get $k_1 = 0$. The converse is trivial.

Proposition 6.3. *With respect to levi-Civita connection, the mean curvature vector of a Legendre curve on a three-dimensional α-para-Kenmotsu manifold is 2-recurrent if and only if the curvature k_1 of the curve is zero.*

Proof. Suppose H is 2-recurrent with respect to Levi-Civita connection. So $\nabla^2_{\dot\gamma}\dot\gamma = \omega(\dot\gamma)H$. Using Serret-Frenet formula, we get from above

$$3k_1k_1'\dot\gamma + (k_1'' - k_1\omega(\dot\gamma) + k_1^3)\varphi\dot\gamma = 0. \tag{6.4}$$

Taking inner product with $\dot\gamma$ both sides of the above equation we get $k_1k_1' = 0$. Hence either $k_1 = 0$ or $k_1' = 0$. Taking $k_1' = 0$. Taking inner product with $\varphi\dot\gamma$ both sides of the above equation we get $k_1'' - k_1\omega(\dot\gamma) + k_1^3 = 0$. Hence we get $k_1 = 0$. The converse is trivial.

Proposition 6.4. *With respect to Levi-Civita connection, the mean curvature vector of a Legendre curve on a three-dimensional α-para-Kenmotsu manifold is generalized 2-recurrent if and only if the curvature k_1 of the curve is zero.*

Suppose H is generalized 2-recurrent with respect to Levi-Civita connection. So $\nabla^2_{\dot\gamma}\dot\gamma = \omega(\dot\gamma)H + \rho(\dot\gamma,\dot\gamma)H$. Using Serret-Frenet formula, we get from above after stright forward calculation

$$3k_1k_1'\dot\gamma + (k_1'' - k_1\omega(\dot\gamma) + k_1^3 - \rho(\dot\gamma,\dot\gamma)k_1)\varphi\dot\gamma = 0. \tag{6.5}$$

As before, here we get $k_1 = 0$. The converse is trivial. Hence we can state the following

Theorem 6.5. *For a Legendre curve γ on a three-dimensional α-para-Kenmotsu manifold with respect to Levi-Civita connection the following conditions are equivalent:*

(a) the mean curvature vector of γ is parallel,

(b)the mean curvature vector of γ is recurrent,

(c)the mean curvature vector of γ is 2-recurrent,

(d)the mean curvature vector of γ is generalized 2-recurrent,

(e) γ is geodesic.

7 Example of three-dimensional α-para-Kenmotsu 3-manifold

We consider the three-dimensional manifold $M = \{(x,y,z) \in R^3, z \neq 0\}$, where (x,y,z) are the standard coordinates in R^3. The vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

are linearly independent at each point of M and

$$[e_1,e_2] = 0, [e_1,e_3] = e_1, [e_2,e_3] = e_2.$$

Let g be the Riemannian metric defined by

$$g(e_1,e_3)=g(e_2,e_3)=g(e_1,e_2)=0, \qquad g(e_1,e_1)=g(e_3,e_3)=1, \quad g(e_2,e_2)=-1.$$

Let η be the 1-form defined by $\eta(Z)=g(Z,e_3)$ for any $Z\in\chi(M)$.

Let φ be the (1,1) tensor field defined by $\varphi(e_1)=e_2$, $\varphi(e_2)=e_1$, $\varphi(e_3)=0$. Then using the linearity of φ and g we have

$$\eta(e_3)=1, \qquad \varphi^2(Z)=Z-\eta(Z)e_3, \qquad g(\varphi Z,\varphi W)=-g(Z,W)+\eta(Z)\eta(W),$$

for any $Z,W\in\chi(M)$. Thus for $e_3=\xi$, (φ,ξ,η,g) defines an almost contact metric structure on M.

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is

$$2g(\nabla_X Y,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g(X,[Y,Z])-g(Y,[X,Z])+g(Z,[X,Y]).$$

Using Koszul's formula we get the following

$$\begin{array}{lll} \nabla_{e_1}e_3=e_1, & \nabla_{e_1}e_2=0, & \nabla_{e_1}e_1=-e_3, \\ \nabla_{e_2}e_3=e_2, & \nabla_{e_2}e_2=e_3, & \nabla_{e_2}e_1=0, \\ \nabla_{e_3}e_3=0, & \nabla_{e_3}e_2=0, & \nabla_{e_3}e_1=0. \end{array}$$

From above we see that the manifold satisfies $\nabla_X\xi=\alpha(X-\eta(X)\xi)$, for $e_3=\xi$ and $\alpha=1$.

Therefore the structure $M(\varphi,\xi,\eta,g)$ is a α-para-Kenmotsu 3-manifold.

With the help of the above results it can be verified that

$$\begin{array}{lll} R(e_1,e_2)e_3=0, & R(e_2,e_3)e_3=-e_2, & R(e_1,e_3)e_3=-e_1, \\ R(e_1,e_2)e_2=e_1, & R(e_2,e_3)e_2=-e_3, & R(e_1,e_3)e_2=0, \\ R(e_1,e_2)e_1=e_2, & R(e_2,e_3)e_1=0, & R(e_1,e_3)e_1=e_3. \end{array}$$

Consider a curve $\gamma: I\to M$ defined by $\gamma(s)=(\sqrt{\frac{1}{2}}s,\sqrt{\frac{1}{2}}s,1)$. Hence $\dot\gamma_1=\sqrt{\frac{1}{2}}$, $\dot\gamma_2=\sqrt{\frac{1}{2}}$ and $\dot\gamma_3=0$. Where $\gamma(s)=(\gamma_1(s),\gamma_2(s),\gamma_3(s))$. Now

$$\eta(\dot\gamma)=g(\dot\gamma,e_3)=g(\dot\gamma_1e_1+\dot\gamma_2e_2+\dot\gamma_3e_3,e_3)=0.$$

$$\begin{aligned} g(\dot\gamma,\dot\gamma) &= g(\dot\gamma_1e_1+\dot\gamma_2e_2+\dot\gamma_3e_3,\dot\gamma_1e_1+\dot\gamma_2e_2+\dot\gamma_3e_3) \\ &= \dot\gamma_1^2+\dot\gamma_2^2+\dot\gamma_3^2 \\ &= \dot\gamma_1^2+\dot\gamma_2^2 \\ &= \tfrac{1}{2}+\tfrac{1}{2} \\ &= 1. \end{aligned}$$

Hence the curve is Legendre curve. For this curve $\nabla_{\dot\gamma}\dot\gamma=0$. Hence this curve is geodesic with respect to Levi-Civita connections. Thus Theorem 6.5 is verified.

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