# Twisted products Berwald metrics of polar type

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**Abstract.** Let $(\overline{M}, \overline{F})$ be an $(n-1)$-dimensional compact Finslerian manifold with $n > 1$. Let consider a Finslerian metric on $M \equiv (0, \infty) \times \overline{M}$ of the form $F(x, y) = \sqrt{(y^1)^2 + f(x^1, x^2, ..., x^n)\overline{F}^2(x^2, ..., x^n, y^2, ..., y^n)}$ where $f$ is a positive function on $M$ and $(x^1, ..., x^n, y^1, ..., y^n)$ is a local coordinate of a point $(x, y)$ in the tangent bundle of $M$. In this paper, we express the geometry of $(M, F)$ in term of $f$ and the geometry of $(\overline{M}, \overline{F})$. Curvatures are calculated in the Berwald case. An example of a twisted product Berwald metric is given for $n = 3$.

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## 1 Introduction

Twisted product metrics are natural extensions of warped product metrics [1]. These both kinds of metrics play a major role in Differential Geometry as well as in General Relativity. For examples, the warped product metrics are used to construct Riemannian metrics with negative curvature [3] and the twisted product metrics are applied in theory of projective mappings. In general relativity, many basic solutions of the Einstein equation are warped products metrics[6]. In Finslerian realm the twisted product of manifolds was studied for the first time by Kozma, Peter and Shimada [5]. Later, in 2013, Peyghan, Tayebi and Nourmohammadi Far [7] studied locally dually flat twisted product Finsler manifold.

Let $(\widetilde{M}, \widetilde{F})$ and $(\overline{M}, \overline{F})$ be two Finslerian manifolds. Consider $f_i : \widetilde{M} \times \overline{M} \longrightarrow (0, \infty)$ with $i = 1, 2$ two $C^\infty$ maps. Then, on the product manifold $\widetilde{M} \times \overline{M}$, one can define the Finslerian metric

$$F(x_1, x_2, y_1, y_2) = \sqrt{f_1(x_1, x_2)\widetilde{F}^2(x_1, y_1) + f_2(x_1, x_2)\overline{F}^2(x_2, y_2)} \tag{1.1}$$

for any $(x_1, x_2) \in \widetilde{M} \times \overline{M}$ and $(y_1, y_2) \in \mathring{T}\widetilde{M} \times \mathring{T}\overline{M}$ where $\mathring{T}\widetilde{M} \equiv T\widetilde{M}\backslash\{0\}$. The couple $({}_{f_1}\widetilde{M} \times_{f_2} \overline{M}, F)$ is called a doubly twisted product Finslerian manifold. If $f_1(x_1, x_2) = 1$ then the doubly twisted product manifold ${}_{f_1}\widetilde{M} \times_{f_2} \overline{M}$ is called a twisted product manifold [7] and is denoted by $\widetilde{M} \times_{f_2} \overline{M}$.

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In this paper we study the geometry of the twisted product Finslerian manifold $(\widetilde{M}\times_f \overline{M}, F)$ when $F$ is a Berwald metric and $\widetilde{M}\equiv(0,\infty)$. In particular, if the twisted function $f(x_1,x_2)=f(x_1)$ then $(\widetilde{M}\times_f \overline{M}, F)$ becomes a warped product Berwald manifold of polar type.

This work is organised as follows. In Section 2, we give some basic notions on Finslerian manifolds. The Section 3 is devoted to study the Berwald curvatures. The Berwald Ricci and scalar curvatures are evaluated in natural coordinates. Finally, as an example, we show that the application $F: T\big((0,\infty)\times U\big)\longrightarrow\mathbb{R}$ defined by

$$F(x^1,x^2,x^3;y^1,y^2,y^3)=\sqrt{(y^1)^2+fe^{\rho(x^2,x^3)}(y^2)^{2p}(y^3)^{2q}}, \tag{1.2}$$

where $\rho$ is a $C^\infty$ function on $U\subset\mathbb{R}^2$, and $p$ and $q$ are some real numbers, is a twisted product Berwald metric of polar type on $(0,\infty)\times U$.

# 2 Some basic notions on Finslerian manifolds

Let $M$ be an $n-$dimensional manifold. We denote by $T_xM$ the tangent space at $x\in M$ and by $TM:=\bigcup_{x\in M}T_xM$ the tangent bundle of $M$. Set $\mathring{T}M=TM\backslash\{0\}$ and $\pi: TM\longrightarrow M: \pi(x,y)\longmapsto x$ the natural projection. Let $(x^1,...,x^n)$ be a local coordinate on an open subset $U$ of $M$ and $(x^1,...,x^n,y^1,...,y^n)$ be the local coordinate on $\pi^{-1}(U)\subset TM$. The local coordinate system $(x^i)_{i=1,...,n}$ produces the coordinate bases $\{\frac{\partial}{\partial x^i}\}_{i=1,...,n}$ and $\{dx^i\}_{i=1,...,n}$ respectively, for $TM$ and cotangent bundle $T^*M$. We use Einstein summation convention: repeated upper and lower indices will automatically be summed unless otherwise will be noted.

**Definition 2.1.** Let $M$ be an $n-$dimensional manifold. A function $F: TM\longrightarrow[0,\infty)$ is called a *Finslerian metric* on $M$ if :

(i) $F$ is $C^\infty$ on the entire slit tangent bundle $\mathring{T}M$,

(ii) $F$ is positively 1-homogeneous on the fibers of $TM$, that is $\forall c>0,\ F(x,cy)=cF(x,y)$,

(iii) the Hessian matrix $(g_{ij}(x,y))_{1\le i,j\le n}$ with elements

$$g_{ij}(x,y):=\frac{1}{2}\frac{\partial^2F^2(x,y)}{\partial y^i\partial y^j} \tag{2.1}$$

is positive definite at every point $(x,y)$ of $\mathring{T}M$.

Consider the differential map $\pi_*$ of the submersion $\pi:\mathring{T}M\longrightarrow M$. The vertical subspace of $T\mathring{T}M$ is defined by $\mathcal{V}:=ker(\pi_*)$ and is locally spanned by the set $\{F\frac{\partial}{\partial y^i}, 1\le i\le n\}$, on each $\pi^{-1}(U)\subset\mathring{T}M$.

An horizontal subspace $\mathcal{H}$ of $T\mathring{T}M$ is by definition any complementary to $\mathcal{V}$. The bundles $\mathcal{H}$ and $\mathcal{V}$ give a smooth splitting

$$T\mathring{T}M=\mathcal{H}\oplus\mathcal{V}. \tag{2.2}$$

An Ehresmann connection is a selection of a horizontal subspace $\mathcal{H}$ of $T\mathring{T}M$. It is known [4] that $\mathcal{H}$ can be canonically defined from the geodesics equation.

**Definition 2.2.** Let $\pi : \mathring{T}M \longrightarrow M$ be the restricted projection.

(1) An Ehresmann-Finsler connection of $\pi$ is the subbundle $\mathcal{H}$ of $T\mathring{T}M$ given by

$$\mathcal{H} := ker\theta, \tag{2.3}$$

where $\theta : T\mathring{T}M \longrightarrow \pi^*TM$ is the bundle morphism defined by

$$\theta|_{(x,y)} = \frac{\partial}{\partial x^i} \otimes \frac{1}{F}(dy^i + N^i_j dx^j) \tag{2.4}$$

with $N^i_j(x,y) := \frac{\partial G^i(x,y)}{\partial y^j}$ for

$$G^i(x,y) := \frac{1}{4} g^{il}(x,y)\Big[\frac{\partial g_{jl}}{\partial x^k}(x,y) + \frac{\partial g_{kl}}{\partial x^j}(x,y) - \frac{\partial g_{jk}}{\partial x^l}(x,y)\Big] y^j y^k. \tag{2.5}$$

(2) The form $\theta : T\mathring{T}M \longrightarrow \pi^*TM$ induces a linear map

$$\theta|_{(x,y)} : T_{(x,y)}\mathring{T}M \longrightarrow T_xM, \tag{2.6}$$

for each point $(x,y) \in \mathring{T}M$; where $x = \pi(x,y)$.
The vertical lift of a section $\xi$ of $\pi^*TM$ is a unique section $\mathbf{v}(\xi)$ of $T\mathring{T}M$ such that for every $(x,y) \in \mathring{T}M$,

$$\pi_*(\mathbf{v}(\xi))|_{(x,y)} = 0_{(x,y)} \text{ and } \theta(\mathbf{v}(\xi))|_{(x,y)} = \xi_{(x,y)}. \tag{2.7}$$

(3) The differential projection $\pi_* : T\mathring{T}M \longrightarrow \pi^*TM$ induces a linear map

$$\pi_*|_{(x,y)} : T_{(x,y)}\mathring{T}M \longrightarrow T_xM, \tag{2.8}$$

for each point $(x,y) \in \mathring{T}M$; where $x = \pi(x,y)$.
The horizontal lift of a section $\xi$ of $\pi^*TM$ is a unique section $\mathbf{h}(\xi)$ of $T\mathring{T}M$ such that for every $(x,y) \in \mathring{T}M$,

$$\pi_*(\mathbf{h}(\xi))|_{(x,y)} = \xi_{(x,y)} \text{ and } \theta(\mathbf{h}(\xi))|_{(x,y)} = 0_{(x,y)}. \tag{2.9}$$

We have the following.

**Definition 2.3.** A Finslerian tensor field $T$ of type $(q, 0; p_1, p_2)$ on $\mathring{T}M$ is a $C^\infty$ section of the tensor bundle

$$\underbrace{\pi^*T^*M \otimes ... \otimes \pi^*T^*M}_{p_1-times} \otimes \underbrace{T^*\mathring{T}M \otimes ... \otimes T^*\mathring{T}M}_{p_2-times} \otimes \bigotimes^{q} \pi^*TM. \tag{2.10}$$

**Remark 2.4.** In a local chart,

$$T = T^{k_1...k_q}_{i_1...i_{p_1} j_1...j_{p_2}} \partial_{k_1} \otimes ... \otimes \partial_{k_q} \otimes dx^{i_1} \otimes ... \otimes dx^{i_{p_1}} \otimes \varepsilon^{j_1} \otimes ... \otimes \varepsilon^{j_{p_2}}$$

where $(\partial_{k_1} \otimes ... \otimes \partial_{k_q} \otimes dx^{i_1} \otimes ... \otimes dx^{i_{p_1}} \otimes \varepsilon^{j_1} \otimes ... \otimes \varepsilon^{j_{p_2}})_{k \in \{1,...,n\}^q, i \in \{1,...,n\}^{p_1}, j \in \{1,...,n\}^{p_2}}$ is a basis section of this tensor and, the $\partial_{k_r} := \frac{\partial}{\partial x^{k_r}}$ as well as $\varepsilon^{j_s}$ are respectively the basis sections for $\pi^*TM$ and $T^*\mathring{T}M$ dual of $T\mathring{T}M$.

**Example 2.5.** (1) The Hessian matrix $g$, defined in (2.1), is of type $(0,0;2,0)$.

(2) The Ehresmann-Finsler form $\theta$ is of type $(1,0;0,1)$.

The following lemma defines the Chern connection on $\pi^*TM$.

**Lemma 2.1.** *[8] Let $(M,F)$ be a Finslerian manifold and $g$ its fundamental tensor. There exists a unique linear connection $\nabla$ on the vector bundle $\pi^*TM$ such that, for all $X,Y \in \chi(\mathring{T}M)$ and for every $\xi,\eta \in \Gamma(\pi^*TM)$, one has the following properties:*

*(i)* $\nabla_X \pi_* Y - \nabla_Y \pi_* X = \pi_*[X,Y]$,

*(ii)* $X(g(\xi,\eta)) = g(\nabla_X \xi,\eta) + g(\xi,\nabla_X \eta) + 2\mathcal{A}(\theta(X),\xi,\eta)$ *where* $\mathcal{A} := \frac{F}{2}\frac{\partial g_{ij}}{\partial y^k} dx^i \otimes dx^j \otimes dx^k$ *is the Cartan tensor.*

One has $\nabla_{\frac{\delta}{\delta x^j}} \frac{\partial}{\partial x^k} = \Gamma^i_{jk} \frac{\partial}{\partial x^i}$ where

$$\Gamma^i_{jk} \quad := \quad \frac{\partial^2 G^i}{\partial y^j \partial y^k} \tag{2.11}$$

which can be written as

$$\Gamma^i_{jk} \quad = \quad \frac{1}{2} g^{il} \left( \frac{\delta g_{jl}}{\delta x^k} + \frac{\delta g_{lk}}{\delta x^j} - \frac{\delta g_{jk}}{\delta x^l} \right) \tag{2.12}$$

with

$$\left\{ \frac{\delta}{\delta x^i} := \frac{\partial}{\partial x^i} - N^j_i \frac{\partial}{\partial y^j} = \mathbf{h}(\frac{\partial}{\partial x^i}) \right\}_{i=1,\dots,n}. \tag{2.13}$$

**Definition 2.6.** Let $F$ be a Finslerian metric on an $n$-dimensional manifold $M$ and $x \in M$. $F$ is called a Berwald metric if, for a local coordinate $(x^i, y^i)_{i=1,\dots,n}$ in $\mathring{T}M$, the Christoffel symbols $\Gamma^l_{ij}$ of the Chern connection are only functions of the point $x$ in $M$.

**Example 2.7.** All Riemannian metrics and all locally Minkowskian metrics are examples of Berwald metrics. In fact,

(1) for Riemannian metrics, $\Gamma^i_{jk} = \gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{lk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right)$. In particular, the functions $\Gamma^i_{jk}$ are independant of $y$.

(2) for locally Minkowskian metrics, in a neighborhood $V$ of a point $x \in M$, the functions $\Gamma^i_{jk}$ vanish identically. Hence, on $V$, $\Gamma^i_{jk}$ can depend at most on $x$.

# 3 Berwald Ricci and scalar curvatures

Let $(\overline{M}, \overline{F})$ be an $n-1$ dimensional Finslerian manifold and $f$ a positive $C^\infty$ function on $(0,\infty) \times \overline{M}$. One can show that $F : T\big((0,\infty) \times \overline{M}\big) \longrightarrow [0,\infty)$, defined by

$$F(x^1,x^2,...,x^n;y^1,y^2,...,y^n) = \sqrt{(y^1)^2 + f(x^1,x^2,...,x^n)\overline{F}^2(x^2,...,x^n;y^2,...,y^n)}, \tag{3.1}$$

is a Finslerian metric on $M \equiv (0,\infty) \times \overline{M}$. In particular, if $f(x^1, x^2, ..., x^n) = f(x^1)$ then $F$ is warped product Finslerian metric of cylindrical type. If $f(x^1, x^2, ..., x^n) = f(x^2, ..., x^n)$ then $f\overline{F}^2$ can be treated as a conformal metric of $\overline{F}^2$ whose conformal factor is $f$. In this last case, $F$ can be seen as a simple product Finslerian metric.

For the Finsler metric $F(x,y) = \sqrt{(y^1)^2 + f(x)\overline{F}^2(\overline{x},\overline{y})}$ where $(\overline{x}) = (x^2, ..., x^n)$ is a local coordinate in $\overline{M}$ and $(\overline{y}) = (y^2, ..., y^n)$ are vector components in $T_{\overline{x}}\overline{M}$, the fundamental tensor is

$$(g_{ij}(x,y)) = \begin{pmatrix} 1 & 0 \\ 0 & f(x)(\overline{g}_{ij}(\overline{x},\overline{y})) \end{pmatrix} \tag{3.2}$$

where $\overline{g}$ is the fundamental tensor associated with $\overline{F}$.

The inverse $g^{-1}$ of $g$ is given by

$$(g^{ij}(x,y)) = \begin{pmatrix} 1 & 0 \\ 0 & f^{-1}(x)(\overline{g}^{ij}(\overline{x},\overline{y})) \end{pmatrix}. \tag{3.3}$$

**Definition 3.1.** The full curvature associated with the Chern connection $\nabla$ on the vector bundle $\pi^*TM$ over the manifold $\mathring{T}M$ is the application

$$\phi : \begin{array}{ccc} \chi(\mathring{T}M) \times \chi(\mathring{T}M) \times \Gamma(\pi^*TM) & \to & \Gamma(\pi^*TM) \\ (X,Y,\xi) & \mapsto & \phi(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_Y\nabla_X\xi - \nabla_{[X,Y]}\xi. \end{array}$$

By the relation (2.2), we have

$$\nabla_X = \nabla_{\hat{X}} + \nabla_{\check{X}}, \tag{3.4}$$

where $X = \hat{X} + \check{X}$ with $\hat{X} \in \Gamma(\mathcal{H})$ and $\check{X} \in \Gamma(\mathcal{V})$.

Using the metric $F$, one can define the full curvature of $\nabla$ as:

$$\begin{aligned} \Phi(\xi,\eta,X,Y) &= g(\phi(X,Y)\xi,\eta) \\ &= g(\phi(\hat{X},\hat{Y})\xi + \phi(\hat{X},\check{Y})\xi + \phi(\check{X},\hat{Y})\xi + \phi(\check{X},\check{Y})\xi,\eta) \\ &= \mathbf{R}(\xi,\eta,X,Y) + \mathbf{P}(\xi,\eta,X,Y) + \mathbf{Q}(\xi,\eta,X,Y), \end{aligned} \tag{3.5}$$

where $\mathbf{R}(\xi,\eta,X,Y) = g(\phi(\hat{X},\hat{Y})\xi,\eta)$, $\mathbf{P}(\xi,\eta,X,Y) = g(\phi(\hat{X},\check{Y})\xi,\eta) + g(\phi(\check{X},\hat{Y})\xi,\eta)$ and $\mathbf{Q}(\xi,\eta,X,Y) = g(\phi(\check{X},\check{Y})\xi,\eta)$ are respectively the *first (horizontal) curvature*, *mixed curvature* and *vertical* curvature.

In particular, if $\nabla$ is the Chern connection, the $\mathbf{Q}$-curvature vanishes.

In a local coordinate system, the components of the Chern curvature are:

$$\begin{aligned} \Phi(\partial_i,\partial_j,\hat{\partial}_k + \check{\partial}_k,\hat{\partial}_l + \check{\partial}_l) &= \mathbf{R}(\partial_i,\partial_j,\hat{\partial}_k + \check{\partial}_k,\hat{\partial}_l + \check{\partial}_l) + \mathbf{P}(\partial_i,\partial_j,\hat{\partial}_k + \check{\partial}_k,\hat{\partial}_l + \check{\partial}_l) \\ &= \Big(\frac{\delta\Gamma^s_{il}}{\delta x^k} - \frac{\delta\Gamma^s_{ik}}{\delta x^l}\Big)g_{js} + \Big(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}\Big)g_{jr} - F\frac{\partial\Gamma^s_{ik}}{\partial y^l}g_{js} \end{aligned} \tag{3.6}$$

where $\partial_i := \frac{\partial}{\partial i} \in \pi^*TM$, $\hat{\partial}_k := \frac{\delta}{\delta x^k} \in \mathcal{H}$ and $\check{\partial}_k := F\frac{\partial}{\partial y^k} \in \mathcal{V}$.

**Remark 3.2.** In natural coordinates, the curvatures $\mathbf{R}$ and $\mathbf{P}$ can also be found in [2].

For the Berwald metric $F(x;y) = \sqrt{(y^1)^2 + f(x)\overline{F}^2(\overline{x},\overline{y})}$, by the Definition 2.6, the Christoffel symbols are

$$\Gamma^1_{ij} = 0 \text{ for } i,j \in [1,n], \tag{3.7}$$

$$\Gamma^k_{11} = 0 \text{ for } k \in [1,n], \tag{3.8}$$

$$\Gamma^a_{1b} = \frac{1}{2f}\frac{\partial f}{\partial x^1}\delta^a_b \text{ for } a,b \in [2,n], \tag{3.9}$$

$$\begin{aligned}
\Gamma^a_{bc} &= \frac{1}{2}g^{ad}\left(\frac{\partial g_{bd}}{\partial x^c} + \frac{\partial g_{cd}}{\partial x^b} - \frac{\partial g_{bc}}{\partial x^d}\right) \text{ for } a,b,c,d \in [2,n] \\
&= \frac{1}{2}f^{-1}\overline{g}^{ad}\left[\frac{\partial(f\overline{g}_{bd})}{\partial x^c} + \frac{\partial(f\overline{g}_{cd})}{\partial x^b} - \frac{\partial(f\overline{g}_{bc})}{\partial x^d}\right] \\
&= \overline{\Gamma}^a_{bc} + \frac{1}{2}f^{-1}\left(\frac{\partial f}{\partial x^c}\delta^a_b + \frac{\partial f}{\partial x^b}\delta^a_c - \frac{\partial f}{\partial x^d}\overline{g}^{ad}\overline{g}_{bc}\right).
\end{aligned} \tag{3.10}$$

If $F$ is a Berwald metric then the relation (3.6) becomes

$$\begin{aligned}
\Phi_{ijkl} &= \Big(\frac{\partial\Gamma^s_{il}}{\partial x^k} - \frac{\partial\Gamma^s_{ik}}{\partial x^l}\Big)g_{js} + \Big(\Gamma^s_{ik}\Gamma^r_{ls} - \Gamma^s_{il}\Gamma^r_{ks}\Big)g_{jr} \\
&\overset{(3.9)}{=} \Big(\frac{\partial\Gamma^a_{il}}{\partial x^k} - \frac{\partial\Gamma^a_{ik}}{\partial x^l}\Big)g_{ja} + \Big(\Gamma^a_{ik}\Gamma^d_{la} - \Gamma^a_{il}\Gamma^d_{ka}\Big)g_{jd}
\end{aligned} \tag{3.11}$$

where $\Phi_{ijkl} = \Phi(\partial_i, \partial_j, \hat{\partial}_k + \check{\partial}_k, \hat{\partial}_l + \check{\partial}_l)$. In particular, by the relations (3.7)-(3.10)

$$\Phi_{i1kl} = 0, \tag{3.12}$$

$$\Phi_{1b11} = 0, \tag{3.13}$$

$$\begin{aligned}
\Phi_{1b1c} &= \Big(\frac{\partial\Gamma^a_{1c}}{\partial x^1} - \frac{\partial\Gamma^a_{11}}{\partial x^c}\Big)g_{ba} + \Big(\Gamma^a_{11}\Gamma^d_{ca} - \Gamma^a_{1c}\Gamma^d_{1a}\Big)g_{bd} \\
&= \Big[-\frac{1}{2f}\Big(\frac{\partial f}{\partial x^1}\Big)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial (x^1)^2}\Big]\overline{g}_{bc} - \frac{1}{2f}\frac{\partial f}{\partial x^1}\delta^a_c\Big(\frac{1}{2f}\frac{\partial f}{\partial x^1}\delta^d_a\Big)f\overline{g}_{bd} \\
&= \frac{1}{2}\left[\frac{\partial^2 f}{\partial(x^1)^2} - \frac{3}{2f}\Big(\frac{\partial f}{\partial x^1}\Big)^2\right]\overline{g}_{bc},
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
\Phi_{abcd} &= \Big(\frac{\partial\Gamma^r_{ad}}{\partial x^c} - \frac{\partial\Gamma^r_{ac}}{\partial x^d}\Big)g_{br} + \Big(\Gamma^r_{ac}\Gamma^s_{dr} - \Gamma^r_{ad}\Gamma^s_{cr}\Big)g_{bs} \\
&= \left\{\frac{\partial}{\partial x^c}\left[\overline{\Gamma}^r_{ad} + \frac{1}{2}f^{-1}\left(\frac{\partial f}{\partial x^d}\delta^r_a + \frac{\partial f}{\partial x^a}\delta^r_d - \frac{\partial f}{\partial x^s}\overline{g}^{rs}\overline{g}_{ad}\right)\right]\right. \\
&\quad \left. -\frac{\partial}{\partial x^d}\left[\overline{\Gamma}^r_{ac} + \frac{1}{2}f^{-1}\left(\frac{\partial f}{\partial x^c}\delta^r_a + \frac{\partial f}{\partial x^a}\delta^r_c - \frac{\partial f}{\partial x^s}\overline{g}^{rs}\overline{g}_{ac}\right)\right]\right\} f\overline{g}_{br} \\
&\quad + \left\{\left[\overline{\Gamma}^r_{ac} + \frac{1}{2}f^{-1}\left(\frac{\partial f}{\partial x^c}\delta^r_a + \frac{\partial f}{\partial x^a}\delta^r_c - \frac{\partial f}{\partial x^s}\overline{g}^{rs}\overline{g}_{ac}\right)\right]\right. \\
&\quad \times \left[\overline{\Gamma}^s_{dr} + \frac{1}{2}f^{-1}\left(\frac{\partial f}{\partial x^r}\delta^s_d + \frac{\partial f}{\partial x^d}\delta^s_r - \frac{\partial f}{\partial x^t}\overline{g}^{ts}\overline{g}_{dr}\right)\right] \\
&\quad - \left[\overline{\Gamma}^r_{ad} + \frac{1}{2}f^{-1}\left(\frac{\partial f}{\partial x^d}\delta^r_a + \frac{\partial f}{\partial x^a}\delta^r_d - \frac{\partial f}{\partial x^s}\overline{g}^{rs}\overline{g}_{ad}\right)\right] \\
&\quad \left. \times \left[\overline{\Gamma}^s_{cr} + \frac{1}{2}f^{-1}\left(\frac{\partial f}{\partial x^r}\delta^s_c + \frac{\partial f}{\partial x^c}\delta^s_r - \frac{\partial f}{\partial x^t}\overline{g}^{ts}\overline{g}_{cr}\right)\right]\right\} f\overline{g}_{bs}.
\end{aligned} \tag{3.15}$$

It follows that

$$
\begin{aligned}
\Phi_{abcd} &= f\overline{\Phi}_{abcd} \\
&+\frac{1}{4f}\left(\frac{\partial f}{\partial x^c}\frac{\partial f}{\partial x^a}\overline{g}_{bd}+\frac{\partial f}{\partial x^c}\frac{\partial f}{\partial x^d}\overline{g}_{ab}-\frac{\partial f}{\partial x^c}\frac{\partial f}{\partial x^b}\overline{g}_{ad}\right.\\
&+\frac{\partial f}{\partial x^c}\frac{\partial f}{\partial x^a}\overline{g}_{bd}+\frac{\partial f}{\partial x^a}\frac{\partial f}{\partial x^d}\overline{g}_{bc}-\frac{\partial f}{\partial x^a}\frac{\partial f}{\partial x^b}\overline{g}_{cd}\\
&\left.-\frac{\partial f}{\partial x^r}\frac{\partial f}{\partial x^s}\overline{g}^{rs}\overline{g}_{ac}\overline{g}_{bd}-\frac{\partial f}{\partial x^s}\frac{\partial f}{\partial x^d}\overline{g}^{rs}\overline{g}_{ac}\overline{g}_{br}+\frac{\partial f}{\partial x^s}\frac{\partial f}{\partial x^b}\overline{g}^{rs}\overline{g}_{ac}\overline{g}_{dr}\right)\\
&+\frac{1}{2}\left(\frac{\partial f}{\partial x^b}\overline{g}_{cr}\overline{\Gamma}^r_{ad}-\frac{\partial f}{\partial x^r}\overline{g}_{bc}\overline{\Gamma}^r_{ad}+\frac{\partial f}{\partial x^t}\overline{g}^{rt}\overline{g}_{ad}\overline{g}_{bs}\overline{\Gamma}^s_{cr}\right)\\
&-\frac{1}{2}\left(\frac{\partial f}{\partial x^b}\overline{g}_{dr}\overline{\Gamma}^r_{ac}-\frac{\partial f}{\partial x^r}\overline{g}_{bd}\overline{\Gamma}^r_{ac}+\frac{\partial f}{\partial x^t}\overline{g}^{rt}\overline{g}_{ac}\overline{g}_{bs}\overline{\Gamma}^s_{dr}\right)\\
&-\frac{1}{4f}\left(\frac{\partial f}{\partial x^d}\frac{\partial f}{\partial x^a}\overline{g}_{bc}+\frac{\partial f}{\partial x^d}\frac{\partial f}{\partial x^c}\overline{g}_{ab}-\frac{\partial f}{\partial x^d}\frac{\partial f}{\partial x^b}\overline{g}_{ac}\right.\\
&+\frac{\partial f}{\partial x^d}\frac{\partial f}{\partial x^a}\overline{g}_{bc}+\frac{\partial f}{\partial x^a}\frac{\partial f}{\partial x^c}\overline{g}_{bd}-\frac{\partial f}{\partial x^a}\frac{\partial f}{\partial x^b}\overline{g}_{cd}\\
&\left.-\frac{\partial f}{\partial x^r}\frac{\partial f}{\partial x^s}\overline{g}^{rs}\overline{g}_{ad}\overline{g}_{bc}-\frac{\partial f}{\partial x^s}\frac{\partial f}{\partial x^c}\overline{g}^{rs}\overline{g}_{ad}\overline{g}_{br}+\frac{\partial f}{\partial x^s}\frac{\partial f}{\partial x^b}\overline{g}^{rs}\overline{g}_{ad}\overline{g}_{cr}\right)\\
&-\frac{1}{2f}\left(\frac{\partial f}{\partial x^c}\frac{\partial f}{\partial x^d}\overline{g}_{ab}+\frac{\partial f}{\partial x^c}\frac{\partial f}{\partial x^a}\overline{g}_{bd}-\frac{\partial f}{\partial x^c}\frac{\partial f}{\partial x^b}\overline{g}_{ad}\right)\\
&+\frac{1}{2}\left[\frac{\partial^2 f}{\partial x^c\partial x^d}\overline{g}_{ab}+\frac{\partial^2 f}{\partial x^c\partial x^a}\overline{g}_{bd}-\frac{\partial^2 f}{\partial x^c\partial x^b}\overline{g}_{ad}-\frac{\partial f}{\partial x^s}\frac{\partial}{\partial x^c}\left(\overline{g}^{rs}\overline{g}_{ad}\right)\overline{g}_{br}\right]\\
&+\frac{1}{2f}\left(\frac{\partial f}{\partial x^d}\frac{\partial f}{\partial x^c}\overline{g}_{ab}+\frac{\partial f}{\partial x^d}\frac{\partial f}{\partial x^a}\overline{g}_{bc}-\frac{\partial f}{\partial x^d}\frac{\partial f}{\partial x^b}\overline{g}_{ac}\right)\\
&-\frac{1}{2}\left[\frac{\partial^2 f}{\partial x^d\partial x^c}\overline{g}_{ab}+\frac{\partial^2 f}{\partial x^d\partial x^a}\overline{g}_{bc}-\frac{\partial^2 f}{\partial x^d\partial x^b}\overline{g}_{ac}-\frac{\partial f}{\partial x^s}\frac{\partial}{\partial x^d}\left(\overline{g}^{rs}\overline{g}_{ac}\right)\overline{g}_{br}\right]\\
&= f\overline{\Phi}_{abcd}+\frac{1}{4f}\frac{\partial f}{\partial x^r}\frac{\partial f}{\partial x^s}\overline{g}^{rs}\left(\overline{g}_{ad}\overline{g}_{bc}-\overline{g}_{ac}\overline{g}_{bd}\right)\\
&+\frac{1}{4f}\left(\frac{\partial f}{\partial x^a}\frac{\partial f}{\partial x^d}\overline{g}_{bc}+\frac{\partial f}{\partial x^b}\frac{\partial f}{\partial x^c}\overline{g}_{ad}-\frac{\partial f}{\partial x^a}\frac{\partial f}{\partial x^c}\overline{g}_{bd}-\frac{\partial f}{\partial x^b}\frac{\partial f}{\partial x^d}\overline{g}_{ac}\right)\\
&+\frac{1}{2}\left(\frac{\partial f}{\partial x^b}\overline{g}_{cr}\overline{\Gamma}^r_{ad}-\frac{\partial f}{\partial x^r}\overline{g}_{bc}\overline{\Gamma}^r_{ad}+\frac{\partial f}{\partial x^t}\overline{g}^{rt}\overline{g}_{ad}\overline{g}_{bs}\overline{\Gamma}^s_{cr}\right)\\
&-\frac{1}{2}\left(\frac{\partial f}{\partial x^b}\overline{g}_{dr}\overline{\Gamma}^r_{ac}-\frac{\partial f}{\partial x^r}\overline{g}_{bd}\overline{\Gamma}^r_{ac}+\frac{\partial f}{\partial x^t}\overline{g}^{rt}\overline{g}_{ac}\overline{g}_{bs}\overline{\Gamma}^s_{dr}\right)\\
&+\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^c\partial x^a}\overline{g}_{bd}-\frac{\partial^2 f}{\partial x^c\partial x^b}\overline{g}_{ad}-\frac{\partial^2 f}{\partial x^d\partial x^a}\overline{g}_{bc}+\frac{\partial^2 f}{\partial x^d\partial x^b}\overline{g}_{ac}\right)\\
&+\frac{1}{2}\left[\frac{\partial f}{\partial x^s}\frac{\partial}{\partial x^d}\left(\overline{g}^{rs}\overline{g}_{ac}\right)\overline{g}_{br}-\frac{\partial f}{\partial x^s}\frac{\partial}{\partial x^c}\left(\overline{g}^{rs}\overline{g}_{ad}\right)\overline{g}_{br}\right].
\end{aligned}
\tag{3.16}
$$

That prove the following.

**Proposition 3.1.** *Let $(I \times_f \overline{M}, F)$ be an n-dimensional twisted product Berwald manifold with $I = (0, \infty)$. Then, in natural coordinates, the full curvature coefficients of $(I \times_f \overline{M}, F)$ are given by*

$$\begin{aligned}
\Phi_{abcd} &= f\overline{\Phi}_{abcd} + \frac{1}{4f}\frac{\partial f}{\partial x^r}\frac{\partial f}{\partial x^s}\overline{g}^{rs}\left(\overline{g}_{ad}\overline{g}_{bc} - \overline{g}_{ac}\overline{g}_{bd}\right) \\
&+\frac{1}{4f}\left(\frac{\partial f}{\partial x^a}\frac{\partial f}{\partial x^d}\overline{g}_{bc} + \frac{\partial f}{\partial x^b}\frac{\partial f}{\partial x^c}\overline{g}_{ad} - \frac{\partial f}{\partial x^a}\frac{\partial f}{\partial x^c}\overline{g}_{bd} - \frac{\partial f}{\partial x^b}\frac{\partial f}{\partial x^d}\overline{g}_{ac}\right) \\
&+\frac{1}{2}\left(\frac{\partial f}{\partial x^b}\overline{g}_{cr}\overline{\Gamma}^r_{ad} - \frac{\partial f}{\partial x^r}\overline{g}_{bc}\overline{\Gamma}^r_{ad} + \frac{\partial f}{\partial x^t}\overline{g}^{rt}\overline{g}_{ad}\overline{g}_{bs}\overline{\Gamma}^s_{cr}\right) \\
&-\frac{1}{2}\left(\frac{\partial f}{\partial x^b}\overline{g}_{dr}\overline{\Gamma}^r_{ac} - \frac{\partial f}{\partial x^r}\overline{g}_{bd}\overline{\Gamma}^r_{ac} + \frac{\partial f}{\partial x^t}\overline{g}^{rt}\overline{g}_{ac}\overline{g}_{bs}\overline{\Gamma}^s_{dr}\right) \qquad (3.17)\\
&+\frac{1}{2}\left(\frac{\partial^2 f}{\partial x^c\partial x^a}\overline{g}_{bd} - \frac{\partial^2 f}{\partial x^c\partial x^b}\overline{g}_{ad} - \frac{\partial^2 f}{\partial x^d\partial x^a}\overline{g}_{bc} + \frac{\partial^2 f}{\partial x^d\partial x^b}\overline{g}_{ac}\right) \\
&+\frac{1}{2}\left[\frac{\partial f}{\partial x^s}\frac{\partial}{\partial x^d}\left(\overline{g}^{rs}\overline{g}_{ac}\right)\overline{g}_{br} - \frac{\partial f}{\partial x^s}\frac{\partial}{\partial x^c}\left(\overline{g}^{rs}\overline{g}_{ad}\right)\overline{g}_{br}\right], \\
\Phi_{1b1c} &= \frac{1}{2}\left[\frac{\partial^2 f}{\partial (x^1)^2} - \frac{3}{2f}\left(\frac{\partial f}{\partial x^1}\right)^2\right]\overline{g}_{bc} \text{ and } \Phi_{i1kl} = \Phi_{1j11} = 0
\end{aligned}$$

*for $a, b, c, d \in [2, n]$ and for $i, j, k, l \in [1, n]$.*

**Theorem 3.2.** *Let $(I\times_f \overline{M}, F)$ be an n-dimensional twisted product Berwald manifold with $I = (0, \infty)$ and $\overline{F}$ a local Minkowskian metric on $\overline{M}$. Then $(I \times_f \overline{M}, F)$ is locally Minkowskian manifold if and only if the twisted function $f$ satisfies the following equations:*

$$\begin{aligned}
&\frac{\partial f}{\partial x^r}\frac{\partial f}{\partial x^s}\overline{g}^{rs}\overline{g}_{ad}\overline{g}_{bc} + \frac{\partial f}{\partial x^a}\frac{\partial f}{\partial x^d}\overline{g}_{bc} + \frac{\partial f}{\partial x^b}\frac{\partial f}{\partial x^c}\overline{g}_{ad} + 2f\frac{\partial^2 f}{\partial x^c\partial x^a}\overline{g}_{bd} + 2f\frac{\partial^2 f}{\partial x^d\partial x^b}\overline{g}_{ac} \\
= &\frac{\partial f}{\partial x^r}\frac{\partial f}{\partial x^s}\overline{g}^{rs}\overline{g}_{ac}\overline{g}_{bd} + \frac{\partial f}{\partial x^a}\frac{\partial f}{\partial x^c}\overline{g}_{bd} + \frac{\partial f}{\partial x^b}\frac{\partial f}{\partial x^d}\overline{g}_{ac} + 2f\frac{\partial^2 f}{\partial x^c\partial x^b}\overline{g}_{ad} + 2f\frac{\partial^2 f}{\partial x^d\partial x^a}\overline{g}_{bc}
\end{aligned}$$

(3.18)

*for $a, b, c, d \in [2, n]$ and*

$$\frac{2f}{3}\frac{\partial^2 f}{\partial (x^1)^2} = \left(\frac{\partial f}{\partial x^1}\right)^2. \qquad (3.19)$$

**Theorem 3.3.** *Let $(I\times_f \overline{M}, F)$ be an n-dimensional warped product Berwald manifold with $I = (0, \infty)$ and $\overline{F}$ a Riemannian metric on $\overline{M}$. Then $(I \times_f \overline{M}, F)$ is Riemannian manifold if and only if the warping function $f$ and the full curvatures satisfy the following equations:*

$$\begin{cases}
\Phi_{abcd} = f\overline{\Phi}_{abcd}, \text{ for } a, b, c, d \in [2, n] \\
\Phi_{1b1c} = \frac{1}{2}\left[\frac{\partial^2 f}{\partial (x^1)^2} - \frac{3}{2f}\left(\frac{\partial f}{\partial x^1}\right)^2\right]\overline{g}_{bc}, \text{ for } b, c \in \{2, ..., n\} \\
\Phi_{i1kl} = \Phi_{1j11} = 0, \text{ for } i, j, k \in [1, n].
\end{cases} \qquad (3.20)$$

With respect to the Chern connection, we have the following.

**Definition 3.3.** (1) The Berwald Ricci tensor **Ric** of $(M, F)$ is defined by

$$\mathbf{Ric}(\xi, X) \quad := \quad trace_g\Big[\eta \longmapsto R\big(X, \mathbf{h}(\eta) + \mathbf{v}(\eta)\big)\xi\Big]. \tag{3.21}$$

Locally, we have

$$\mathbf{Ric}(\partial_i, \hat{\partial}_k + \check{\partial}_k) \quad = \quad \frac{\partial \Gamma^l_{il}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial x^l} + \Gamma^s_{ik}\Gamma^l_{ls} - \Gamma^s_{il}\Gamma^l_{ks} \tag{3.22}$$

(2) The Berwald scalar curvature **Scal** of $(M, F)$ is defined by

$$\mathbf{Scal} \quad := \quad trace_{\underline{g}}\Big(\mathbf{Ric}\Big), \quad \underline{g} := \pi^* g. \tag{3.23}$$

Locally, we have

$$\mathbf{Scal} \quad = \quad \Big(\frac{\partial \Gamma^l_{il}}{\partial x^k} - \frac{\partial \Gamma^l_{ik}}{\partial x^l} + \Gamma^s_{ik}\Gamma^l_{ls} - \Gamma^s_{il}\Gamma^l_{ks}\Big) g^{ik}. \tag{3.24}$$

**Proposition 3.4.** *Let $(I \times_f \overline{M}, F)$ be an $n$-dimensional twisted Finslerian manifold with $I = (0, \infty)$ and $\overline{F}$ a Finslerian metric on $\overline{M}$. Then $(I \times_f \overline{M}, F)$ is a local Berwald manifold if and only if the twisted function $f$ and the Ricci curvatures satisfy the following equations:*

$$\begin{aligned}
\mathbf{Ric}_{ac} \quad = \quad & \overline{\mathbf{Ric}}_{ac} - \frac{n-2}{4f^2}\left(\frac{\partial f}{\partial x^a}\frac{\partial f}{\partial x^c} + \frac{\partial f}{\partial x^r}\frac{\partial f}{\partial x^s}\overline{g}^{rs}\overline{g}_{ac}\right) \\
& + \frac{1}{2f}\left((n-3)\frac{\partial f}{\partial x^s}\overline{\Gamma}^s_{ac} + \frac{\partial f}{\partial x^r}\overline{g}^{rs}\overline{g}_{ct}\overline{\Gamma}^t_{as} + \frac{\partial f}{\partial x^t}\overline{g}^{rt}\overline{g}_{as}\overline{\Gamma}^s_{cr} - \frac{\partial f}{\partial x^t}\overline{g}^{rt}\overline{g}_{ac}\overline{\Gamma}^s_{rs}\right) \\
& + \frac{1}{2f}\left[(n-3)\frac{\partial^2 f}{\partial x^a \partial x^c} + \frac{\partial^2 f}{\partial x^r \partial x^s}\overline{g}^{rs}\overline{g}_{ac}\right] \\
& + \frac{1}{2f}\left[\frac{\partial f}{\partial x^r}\frac{\partial}{\partial x^s}\Big(\overline{g}^{rs}\overline{g}_{ac}\Big) - \frac{\partial f}{\partial x^s}\frac{\partial}{\partial x^c}\Big(\overline{g}^{rs}\overline{g}_{ar}\Big)\right]
\end{aligned} \tag{3.25}$$

*for $a, b \in [2, n]$ and*

$$\mathbf{Ric}_{11} \quad = \quad \frac{n-1}{2}\left[\frac{\partial^2 f}{\partial (x^1)^2} - \frac{3}{2f}\Big(\frac{\partial f}{\partial x^1}\Big)^2\right]. \tag{3.26}$$

**Proposition 3.5.** *Let $(I \times_f \overline{M}, F)$ be an $n$-dimensional twisted Finslerian manifold with $I = (0, \infty)$ and $\overline{F}$ a Finslerian metric on $\overline{M}$. Then $(I \times_f \overline{M}, F)$ is Berwald manifold if and only if the twisted function $f$ and the scalar curvatures satisfy the following equations:*

$$\begin{aligned}
\mathbf{Scal} \quad = \quad & \frac{\overline{\mathbf{Scal}}}{f} - \frac{1}{2f}\left[\frac{1}{2f}\left(\frac{\partial f}{\partial x^1}\right)^2 - \frac{\partial^2 f}{\partial (x^1)^2}\right] - \frac{n(n-2)}{4f^3}\frac{\partial f}{\partial x^r}\frac{\partial f}{\partial x^s}\overline{g}^{rs} \\
& + \frac{n-3}{f^2}\frac{\partial^2 f}{\partial x^r \partial x^s}\overline{g}^{rs} + \frac{(n-3)}{2f^2}\left(\frac{\partial f}{\partial x^r}\overline{\Gamma}^r_{st}\overline{g}^{st} - \frac{\partial f}{\partial x^r}\overline{g}^{rs}\overline{\Gamma}^t_{st}\right) \\
& + \frac{1}{2f^2}\left[\frac{\partial f}{\partial x^r}\frac{\partial}{\partial x^s}\Big(\overline{g}^{rs}\overline{g}_{ac}\Big) - \frac{\partial f}{\partial x^s}\frac{\partial}{\partial x^c}\Big(\overline{g}^{rs}\overline{g}_{ar}\Big)\right]\overline{g}^{ac} \text{ for } a, c, r, s \in [2, n].
\end{aligned} \tag{3.27}$$

*Proof.* By the relations (3.7)-(3.10) in (3.24). □

**Corollary 3.6.** *Let* $(I \times_f \overline{M}, F)$ *be an* $n$*-dimensional twisted product Berwald manifold with* $I = (0, \infty)$*. If* $f(x^1, x^2, ..., x^n) = f(x^1)$ *then the scalar curvatures* **Scal** *of* $(I \times_f \overline{M}, F)$ *and* $\overline{\textbf{Scal}}$ *of* $(\overline{M}, \overline{F})$ *are related by the followin equation*

$$\textbf{Scal} = \frac{\overline{\textbf{Scal}}}{f} - \frac{1}{2f}\left[\frac{1}{2f}\left(\frac{\partial f}{\partial x^1}\right)^2 - \frac{\partial^2 f}{\partial (x^1)^2}\right]. \tag{3.28}$$

## 4 Example of a twisted product Berwald metric of polar type

Let $U = \{\overline{x} = (x^2, x^3) \in \mathbb{R}^2\}$, $\overline{y} = (y^2, y^3) \in T_{\overline{x}}U$ with $y^2, y^3 > 0$ and $f : (0, \infty) \times U \longrightarrow (0, \infty)$ a $C^\infty$ positive function. Consider $p, q \in \mathbb{R} \setminus \{0\}$ such that $p + q = 1$ and $pq < 0$. Then the application $F : T\big((0, \infty) \times U\big) \longrightarrow \mathbb{R}$ defined by

$$F(x^1, x^2, x^3; y^1, y^2, y^3) = \sqrt{(y^1)^2 + f e^{\rho(x^2, x^3)} (y^2)^{2p} (y^3)^{2q}}, \tag{4.1}$$

where $\rho$ is a $C^\infty$ function on $U$ is a Berwald metric on $(0, \infty) \times U$.

By direct calculations, using the relation (2.1), (3.2) and (3.3), we obtain

$$\big(g_{ij}(x, y)\big) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & p(2p-1) f e^{\rho} (y^2)^{2p-2} (y^3)^{2q} & 2pq f e^{\rho} (y^2)^{2p-1} (y^3)^{2q-1}) \\ 0 & 2pq f e^{\rho} (y^2)^{2p-1} (y^3)^{2q-1} & q(2q-1) f e^{\rho} (y^2)^{2p} (y^3)^{2q-2} \end{pmatrix}$$

and

$$\big(g^{ij}(x, y)\big) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{2q-1}{p} f^{-1} e^{-\rho} (y^2)^{-2p+2} (y^3)^{-2q} & 2 f^{-1} e^{-\rho} (y^2)^{-2p+1} (y^3)^{-2q+1} \\ 0 & 2 f^{-1} e^{-\rho} (y^2)^{-2p+1} (y^3)^{-2q+1} & -\frac{2p-1}{q} f^{-1} e^{-\rho} (y^2)^{-2p} (y^3)^{-2q+2} \end{pmatrix}$$

Hence, since $g_{(x,y)}(v, v) = g_{ij}(x, y) v^i v^j > 0$ for every $v \in T_x M \setminus \{0\}$, $F$ is a Finslerian metric on $(0, \infty) \times U$.

Further, the 27 functions $\Gamma^k_{ij}$ are independent of $y$ and satisfy to the relations (3.7), (3.8),(3.9) and (3.10). Hence, $F$ defined in (4.1), is a twisted Berwald metric of polar form.

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