# Curvature constrained on the base-manifold for $(2, m)$-Einstein warped product manifolds

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**Abstract.** For the studied cases in [11], the author showed that having the $f$-curvature-base $R_{f_B}$ is equal to requiring a flat metric on the base-manifold. In [12] the authors introduced a new kind of Einstein warped product manifold, composed by positive-dimensional manifold and negative-dimensional manifold, the so called PNDP-manifolds. The aim of this paper is to extend the work done in [11] to $m$-dimensional fiber showing if the value of $m$ can influence the result, i.e., finding base-manifolds with non-flat metric for $\dim F \neq 2$, and doing some considerations of the $(2, m)$-PNDP manifolds with $R_{f_B}$. As a result, we find out that the dimension of fiber-manifold does not change the result of [11]. Finally, we add a special remark about the possible use of the $(n, -n)$-PNDPs, a special kind of Einstein warped product manifold, in superconductor Graphene theory.

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## 1 Introduction

In recent years the study of warped product manifolds (WPM) is of great interest both for the mathematicians and physicists. Many works have been published that have studied and introduced new types of WPM, (to name a few reference see [7], [14], [5] and [4]). Aytimur and Zgr in [2] proved some results concerning the *Einstein statistical WPM*, and in [12] Pigazzini et al. introduced a new type of WPM so called PNDP-manifolds, where the fiber is a manifold with negative dimension.

In [11] Pigazzini introduced a simple constraint on the base-manifold called f-curvature-base ($R_{f_B}$) and proposed to use it in order to simplify the equations, trying to constructing a nonRicci-flat metric with non-constant Ricci curvature, on the base-manifold obtaining, as a result for the cases examined, that this is equivalent to the request for a flat metric. This paper is an extension of the works done in [11] and moreover we make also a consideration about [12]. In the third part of the

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paper we reconsider the $(n, -n)$-PNDP-manifolds and we suggest a possible use in the superconductor Graphene theory.

# 2 Main section

**Definition 2.1.** A metric which satisfies the condition $Ric = \lambda g$ for some constant $\lambda$, is said to be an Einstein metric. A manifold which admits an Einstein metric is called an Einstein manifold. (See [13]).

**Definition 2.2.** A warped product manifold is Einstein (see [11], also [9], [3]) if and only if

$$\bar{Ric} = \lambda \bar{g} \iff \begin{cases} Ric - \frac{m}{f}\nabla^2 f = \lambda g \\ \ddot{Ric} = \mu \ddot{g} \\ f\Delta f + (m-1)|\nabla f|^2 + \lambda f^2 = \mu, \end{cases} \tag{2.1}$$

where $\lambda$ and $\mu$ are constants, $m$ is the dimension of $F$, $\nabla^2 f$, $\Delta f$ and $\nabla f$ are, respectively, the Hessian, the Laplacian and the gradient of $f$ for $g$, with $f : (B) \to (0, \infty)$ a smooth positive function. Contracting the first equation of (2.1) we get:

$$R_B f^2 - mf\Delta f = nf^2\lambda, \tag{2.2}$$

where $n$ and $R_B$ is the dimension and the scalar curvature of $B$ respectively. By the third equation, considering $m \neq 0$ and $m \neq 1$, we have:

$$mf\Delta f + m(m-1)|\nabla f|^2 + m\lambda f^2 = m\mu. \tag{2.3}$$

Now from (2.2) and (2.3) we obtain:

$$|\nabla f|^2 + \left[\frac{\lambda(m-n) + R_B}{m(m-1)}\right] f^2 = \frac{\mu}{(m-1)}. \tag{2.4}$$

**Definition 2.3.** Let $(M, \bar{g}) = (B, g) \times_f (F, \ddot{g})$ be an *Einstein warped-product manifold* with $\bar{g} = g + f^2\ddot{g}$. We define the scalar curvature of the base-manifold $(B, g)$ as *$f$-curvature-base* $(R_{f_B})$, if it is a multiple of the warping function $f$, i.e., $R_{f_B} = cf$ for $c$ an arbitrary constant belonging to $\mathbb{R}$ (see [11] as reference).

**Remark 2.4.** Since a warped product manifold (WPM) implies a non-constant warping function $f$ (otherwise it would be a simply Riemannian product-manifold (for more details see [8] and [10]), the results analyzed from now on will be considered from this point of view.

## 2.1 Case 1. Ricci-flat EWP ($\lambda = 0$)

We further consider an non-Ricci-flat fiber-manifold ($\mu \neq 0$).

**Theorem 2.1.** *Let $(M^{2+m}, \bar{g}) = (B^2, g) \times_f (F^m, \ddot{g})$, be an Einstein warped-product manifold Ricci-flat (i.e., $\bar{Ric} = \lambda\bar{g}$ with $\lambda = 0$), where $(B^2, g)$ is a smooth surface with non-zero $R_{f_B}$, and $(F^m, \ddot{g})$ is a smooth Einstein-surface (i.e., $\ddot{Ric} = \mu\ddot{g}$). Then such a structure $(M^{2+m}, \bar{g})$, cannot exist.*

*Proof.* In our case, we have $n = 2$, $\lambda = 0$ and $R_B = R_{f_B}$ (see [11]). Then (2.2) and (2.3) become:

$$\Delta f - hf^2 = 0, \tag{2.5}$$

with $h = c/m$, and

$$f\Delta f + (m-1)|\nabla f|^2 + = \mu. \tag{2.6}$$

Then (2.4) becomes:

$$(m-1)|\nabla f|^2 + hf^3 = \mu. \tag{2.7}$$

Now, by the initial hypothesis (non-zero $R_{f_B}$), we assume that $h \neq 0$ with $f$ non-constant, and set $p = (m-1)$ and $u = -hf$ for an open set, where $u$ is nonzero. Thus:

$$\Delta u + u^2 = 0, \tag{2.8}$$

$$u\Delta u + p|\nabla u|^2 - u^3 - h^2\mu = 0, \tag{2.9}$$

$$p|\nabla u|^2 - u^3 - h^2\mu = 0. \tag{2.10}$$

For the sake of simplicity, we replace the constant $h^2\mu$ with the constant $A$.

Let $g$ be the metric on $B$ and assume that $u$ is a nonzero (hence necessarily positive) solution, for the above system on a simply-connected open subset $B' \subset B$. The equation (2.10) implies that $\omega_1 = (u^3+A)^{-\frac{1}{2}}p^{\frac{1}{2}}du$, hence we have to assume $(u^3+A)$ nonzero; $\omega_1$ is a 1-form with $g$-norm 1 on $B'$ and hence $g$ can be written as $g = \omega_1^2 + \omega_2^2$ for some $\omega_2$, which is also a unit 1-form. We further fix an orientation by considering $\omega_1 \wedge \omega_2$ as the $g$-area form on $B'$. Then $\star du = (u^3+A)^{\frac{1}{2}}p^{-\frac{1}{2}}\omega_2$, and since $d(\star du) = \Delta u\, \omega_1 \wedge \omega_2$, it follows that:

$$\begin{aligned} &p^{-\frac{1}{2}}\tfrac{3}{2}(u^3+A)^{-\frac{1}{2}}u^2 du \wedge \omega_2 + p^{-\frac{1}{2}}(u^3+A)^{\frac{1}{2}}d\omega_2 \\ &= d[(u^3+A)^{\frac{1}{2}}p^{-\frac{1}{2}}\omega_2] = -u^2\omega_1 \wedge \omega_2 = -u^2(u^3+A)^{-\frac{1}{2}}p^{\frac{1}{2}}du \wedge \omega_2 \\ &= \tfrac{3}{2}(u^3+A)^{-\frac{1}{2}}u^2 du \wedge \omega_2 + pu^2(u^3+A)^{-\frac{1}{2}}du \wedge \omega_2 = -(u^3+A)^{\frac{1}{2}}d\omega_2. \end{aligned}$$

Then $(-\frac{3}{2} - p)(u^3+A)^{-1}u^2 du \wedge \omega_2 = d\omega_2$ and we have $d[(u^3+A)^{\frac{3+2p}{6}}\omega_2] = 0$, i.e., $\omega_2 = (u^3+A)^{\frac{-3-2p}{6}}dv$, so the metric $g = p(u^3+A)^{-1}du^2 + (u^3+A)^{-1-\frac{2}{3}p}dv^2$ has a singularity in $u^3 = -A$. $\square$

**Remark 2.5.** The analysis of the singularity is substantially the same as in [11].

Consider an open set for which $(u^3+A)$ is nonzero. The Gaussian curvature is given by: $K = -\frac{9}{2}(u^3+A)^{-1}u^4p - \frac{9}{2}(u^3+A)^{-1}u^4p^2 - (u^3+A)^{-1}u^4p^3 + 2up^3 + 3up^2$. In this case, it is easy to verify that for the initial hypothesis, where we have set $R_B = R_{f_B}$ (i.e., $K = -u\frac{m}{2}$), and we observe that $K$ is incompatible with our analysis. In fact we have:

$$-u\frac{m}{2} = -\frac{9}{2}(u^3+A)^{-1}u^4p - \frac{9}{2}(u^3+A)^{-1}u^4p^2 - (u^3+A)^{-1}u^4p^3 + 2up^3 + 3up^2, \tag{2.11}$$

or $m = (u^3 + A)^{-1}u^3(9p + 9p^2 + 2p^3) - 6p^2 - 4p^3$. Now remembering that $m = p + 1$, we have:

$$\begin{aligned} & p + 1 + 6p^2 + 4p^3 = (u^3 + A)^{-1}u^3(9p + 9p^2 + 2p^3) \\ & \Leftrightarrow (p + 1 + 6p^2 + 4p^3)(u^3 + A) = u^3(9p + 9p^2 + 2p^3) \\ & \Leftrightarrow (p + 1 + 6p^2 + 4p^3)u^3 + Ap + A + A6p^2 + A4p^3 = (9p + 9p^2 + 2p^3)u^3 \\ & \Leftrightarrow Ap + A + A6p^2 + A4p^3 = (-2p^3 + 3p^2 + 8p - 1)u^3. \end{aligned}$$

Since $u$ must be non-constant, this implies:
(a) $4p^3 + 6p^2 + p + 1 = 0$ and
(b) $-2p^3 + 3p^2 + 8p - 1 = 0$.
But the polynomials (a) and (b) have different solutions, so (2.11) is satisfied only for the constant $u$ (i.e., $f = constant$), which is not admitted in our initial assumptions, and therefore such $(M^{2+m}, \bar{g})$ cannot exist.

## 2.2 Case 2. Non Ricci-flat EWP ($\lambda \neq 0$)

We assume the case of a Ricci-flat fiber-manifold ($\mu = 0$). The following result holds:

**Theorem 2.2.** *Let $(M^{2+m}, \bar{g}) = (B^2, g) \times_f (F^m, \ddot{g})$ be an Einstein warped-product manifold, where $(B^2, g)$ is a smooth surface with non-zero $R_{f_B}$, and let $(F^m, \ddot{g})$ be a smooth Ricci-flat surface (i.e., $\ddot{Ric} = \mu\ddot{g}$, with $\mu = 0$). Then $(M^{2+m}, \bar{g})$, cannot exist.*

*Proof.* The analysis is essentially the same as seen so far, so we assume $h \neq 0$ and set $u = -hf$, where $f$ is not constant. The equations (2.2) and (2.4) become:

$$hf^2 - \Delta f - l\lambda f = 0, \tag{2.12}$$

$$|\nabla f|^2 + \frac{\lambda}{p} f^2 - \frac{l\lambda}{p} f^2 + \frac{h}{p} f^3 = 0, \tag{2.13}$$

where $h = \frac{c}{m}$, $l = \frac{2}{m}$ and $p = (m - 1)$. Setting $u = -hf$, we obtain:

$$u^2 + \Delta u + Qu = 0, \tag{2.14}$$

$$|\nabla u|^2 - Su^2 + Tu^2 - Du^3 = 0, \tag{2.15}$$

with $Q = \lambda l$, $S = \frac{\lambda l}{p}$, $T = \frac{\lambda}{p}$ and $D = \frac{1}{p}$, where it is easy to see that $D \neq 0$, $T \neq 0$, $S \neq 0$ and $S - T \neq 0$, for $m \neq 2$.
By the same token as in case (1b), we obtain from (2.15) that $du = (u^3D + Su^2 - Tu^2)^{1/2}\omega_1$. This implies that we have to assume $(u^3D + Su^2 - Tu^2)$ to be nonzero. Then $\omega_1 = (u^3D + Su^2 - Tu^2)^{-1/2}du$, so $\star du = (u^3D + Su^2 - Tu^2)^{1/2}\omega_2$. Since $d(\star du) = \Delta u \, \omega_1 \wedge \omega_2$, we obtain

$$d\omega_2 = \left(\frac{-3}{2}u^2D - Su + Tu - u^2 - Qu\right)(u^3D + Su^2 - Tu^2)^{-1}du \wedge \omega_2. \tag{2.16}$$

Thus we can write $\omega_2 = u^{-A}(Du + S - T)^{-B}dv$ for some constant $A$ and $B$ and for some function $v$. Since for $u = \frac{T-S}{D}$ we have a singularity and we have assumed $(u^3D + Su^2 - Tu^2)$ to be nonzero, then we must consider $u \neq \frac{T-S}{D}$.

**Remark 2.6.** Even here, the analysis of the singularity is substantially the same as in the previous case (i.e., (1b), it is sufficient to consider $\frac{T-S}{D} = A$), so both are equivalent to the studied case in [11].

Continuing with the calculations, we have: $E = (u^3D + Su^2 - Tu^2)^{-1}$ and $G = u^{-2A}(Du+S-T)^{-2B}$. So, by Brioschi's formula, we have that the Gaussian curvature is:

$$\begin{aligned} K = \; & (2A^2 + A)(Du + S - T)^3 u^{(4+6A+4B)} + AD(Du + S - T)^2 u^{(6A+4B+5)} \\ & +(2ABD + 4BD)u^{(6A+4B+7)}(Du + S - T)^{(2B+3)} \\ & +(2B^2D^2 + BD^2)u^{(6A+4B+8)}(Du + S - T)^{(2B+2)}. \end{aligned}$$

Also in this case, for the initial hypothesis $2K = R_{f_B} = cf$, we must have $K = -u\frac{m}{2}$, which means:

$$\begin{aligned} -\tfrac{m}{2} = \; & (2A^2 + A)(Du + S - T)^3 u^{(3+6A+4B)} + AD(Du + S - T)^2 u^{(6A+4B+4)} \\ & +(2ABD + 4BD)u^{(6A+4B+6)}(Du + S - T)^{(2B+3)} \\ & +(2B^2D^2 + BD^2)u^{(6A+4B+7)}(Du + S - T)^{(2B+2)}. \end{aligned} \tag{2.17}$$

Now putting in relation the equation (2.16) with $\omega_2$, we obtain:

$$\frac{-\frac{3}{2}u^2D - Su + Tu - u^2 - Qu}{u^3D + Su^2 - Tu^2} = \frac{-ADu - AS + AT - BDu}{Du^2 + Su - Tu},$$

and solving the partial fractions, we have:

$$A = \frac{Q}{Z} + 1 = \frac{m}{2-m}, \tag{2.18}$$

and then $m \neq 2$ (and $m \neq 0$, $m \neq 1$ from *Definition 2*). As well, we similarly infer

$$B = \frac{3D+2}{2D} - A = \frac{m - 2m^2 + 2}{4 - 2m}, \tag{2.19}$$

and then $m \neq 2$ (and $m \neq 0$, $m \neq 1$ from *Definition 2*), where $Z = S - T$.

If (2.17) has a solution, then certainly the coefficients of $u$ with highest degree must vanish. Hence we can consider the right side of (2.17) composed of:

- $P_1(u) = (2A^2 + A)(Du + S - T)^3 u^{(3+6A+4B)}$, highest degree $6A + 4B + 6$;
- $P_2(u) = AD(Du + S - T)^2 u^{(6A+4B+4)}$, highest degree: $6A + 4B + 6$;
- $P_3(u) = (2ABD+4BD)u^{(6A+4B+6)}(Du+S-T)^{(2B+3)}$, highest degree: $6A+6B+9$;
- $P_4(u) = (2B^2D^2+BD^2)u^{(6A+4B+7)}(Du+S-T)^{(2B+2)}$, highest degree: $6A+6B+9$.

It is worth noticing that the highest degree of $P_1(u)$ is equal to that of $P_2(u)$, and the highest degree of $P_3(u)$ is equal to that of $P_4(u)$. But since the constants $A$ and $B$ can be non-integer and negative, we cannot know in advance which of the two degrees is the highest. We have 3 cases:

I) $6A + 6B + 9$ is the highest degree;
II) $6A + 4B + 6$ is the highest degree;
III) $6A + 6B + 9 = 6A + 4B + 6$.

**Case I.** From the coefficients of $P_3(u)$ and $P_4(u)$ if (2.17) is satisfied, we get: $2A + 2B + 5 = 0$, and considering (2.18) and (2.19) we have:

$$-4m^2 - 4m + 24 = 0, \tag{2.20}$$

i.e., $m = 2$, which is not possible for (2.18) and (2.19), and $m = -3$.

Now in the Case I - the highest degree vanish for $m = -3$, we consider the other degrees and they also must vanish for $m = -3$, and hence we proceed to consider the degree of Case II.

**Case II.** If (2.17) is satisfied, by considering coefficients of $P_1(u)$ and $P_2(u)$, we get $(2A^2 + A)D^3 + AD^3 = 0$, and since $D$ is nonzero, we can divide for $D^3$; then:

$$A = -1. \tag{2.21}$$

Considering (2.18), we get $-2 = 0$ which is not possible, regardless of the value of $m$.

**Case III.** The equality in Case III implies $B = -\frac{3}{2}$, and for (2.19) this means:

$$-2m^2 - 2m + 8 = 0, \tag{2.22}$$

which has no solution for the integer values of $m$.

We showed that (2.17) can be satisfied only for some constant value of the function $u$ (i.e., for $f$ constant), which is not admitted in our initial assumptions. Then, also in this case, $(M^{2+m}, \bar{g})$ cannot exist. $\square$

## 2.3 Case 3. $(2, m)$-PNDP-manifolds with $R_{f_B}$

**Remark 2.7.** The $(2, m)$-PNDP manifolds with $R_{f_B}$ do not exist.

From the PNDP-manifold definition (see [12]), for the $(2, m)$ case, we know that $dim\widetilde{B} = dimB' = 1$, hence it immediately follows that such 1-dimensional manifolds are Ricci-flat. From [3], we know that for 1-dimensional base with Ricci-flat fiber (i.e., for $\mu = 0$), there exists an Einstein warped product manifold with $\lambda = m$ and $f = e^t$.

Now if we consider $R_B = R_{f_B}$, we should have $R_B = ce^t$, but also being well known that for a product manifold the Ricci curvature of the product equals the sum of the Ricci curvatures of each manifolds of the product (see [1]), we obtain that such $(2, m)$-PNDP manifolds are Ricci-flat, so the scalar curvature of the base-manifold cannot be $ce^t$.

As known, the PNDP-manifolds are born from the study of the Einstein-warped product manifolds, and for this reason the following section will illustrate an important application of the latter.

# 3 Special remarks about $(n, -n)$-PNDP manifolds in superconductors Graphene mode

First of all we recall and highlight that the purpose of the PNDP-manifolds is precisely to present the point-like manifolds from a mathematical point of view, and introduce a type of manifold with a new kind of hidden dimensions.

In [6], Capozziello et al. introduced the concept of the "point-like manifold" building superconductors with graphen. In particular, they argue that the superconductor Graphene can be produced by molecules organized in point-like structures where sheets are constituted by $(N+1)$-dimensional manifolds. Particles like electrons, photons and "effective gravitons" are string modes moving on this manifold. In fact, according to string theory, bosonic and fermionic fields like electrons, photons and gravitons are particular "states" or "modes" of strings. In their important work, they show that at the beginning, there are point-like polygonal manifolds (with zero spatial dimension) in space with strings attaching them, where all interactions between strings on one manifold are the same, and are concentrated at one point where the manifold is located. They also to show that by joining these manifolds, 1-dimensional polygonal manifolds emerge such that gauge fields and gravitons live, and hence these manifolds are glued to each other to build higher dimensional polygonal manifolds with various orders of gauge fields and curvatures.

In this context, that the $(n,-n)$-PNDP manifolds play an important role. In fact $(n,-n)$-PNDP appears as a point (point-like), because in general, from our interpretation (see [12]), it is a point (positive and negative dimensions hide each other out and and the total dimension equals zero), but in special it is composed of two manifolds, $B$ and $F$ with nonzero dimensions. So for the first time we have an object that looks like a point (is point-like), but has a geometric structure which allows to make calculations.

Coming back to the Graphene superconductors model, our $(n,-n)$-PNDP manifold consists of two manifolds with nonzero dimensions (one with $n$-dimension and one with $-n$-dimension), where these two manifolds can be thought as a result of intersection of other manifolds. Then we can consider these two manifolds as contained in a "$p$-dimensional BULK", but their warped product (which generates the $(n,-n)$-PNDP) creates the point-like polygonal manifold, a point-like space-time as supposed in [6].

The $(n,-n)$-PNDPs can be considered hence as possible mathematical interpretation of point-like manifolds, because they render, for the first time, this abstract concept as a coherent mathematical object.

## 4 Conclusions

We have observed that the dimension of fiber-manifolds does not influence the results obtained in [11]. Not even the construction of a PNDP-manifold is made possible for a 2-dimensional base-manifold case with $R_{f_B}$. In conclusion, we point out a possible important application for $(n;-n)$-PNDP manifolds, in the context of superconductor Graphene theory.

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