On a class of locally dually flat (α, β)-metric

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Abstract. In this paper, we discuss a class of locally dually flat (α, β)-metrics which are defined as $L = \kappa\alpha + \epsilon\beta$ (κ and ϵ are constants), where α is Riemannian metric and β is 1-form. We classify those with almost isotropic flag curvature.

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1 Introduction

M. Matsumoto [9] introduced the concept of (α, β)-metric on a differentiable manifold M^n, where $\alpha^2 = a_{ij}(x)y^i y^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form. The Matsumoto metric is an interesting (α, β)-metric introduced by using gradient of slope, speed and gravity [8]. This metric formulates the model of a Finsler space. Many authors [8, 1, 11] studied this metric by different perspectives. The notion of dually flat metrics was first introduced by S. I. Amari and H. Nagaoka [2]. Later on, Zhongmin Shen [14] extends the notion of dually flatness to Finsler metrics. In particular, Zhongmin Shen [15] has classified projectively flat Randers metrics with constant flag curvature. In 2009, X. Cheng, Z. Shen and Y. Zhou [4] classified the locally dual flat Randers metrics with almost isotropic flag curvature.

Recently, Q. Xia worked on the dual flatness of Finsler metrics of isotropic flag curvature as well as scalar flag curvature [16]. Further in 2014, S. K. Narasimhamurthy, A. R. kavyashree and Y. K. Mallikarjun [10] discuss characterization of locally dually flat first approximate Matsumoto metric. Locally dually flat Finsler metrics come from information Geometry. Such metrics have very important geometric properties and can play vital role in Finsler Geometry. In this paper, we discussed a class of locally dually flat (α, β)-metrics which are defined as $L = \kappa\alpha + \epsilon\beta$ (κ and ϵ are constants), where α is Riemannian metric and β is 1-form. We classify those with almost isotropic flag curvature.

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2 Preliminaries

For a Finsler Metric $F = F(x, y)$ on a manifold M, the geodesics $c = c(t)$ of F in the local coordinates (x^i) are defined by

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x, \frac{dx}{dt}\right) = 0,$$

where $(x^i(t))$ are the coordinates of $c(t)$ and $G^i = G^i(x, y)$ are defined by

$$G^i = \frac{g^{il}}{4}\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\},$$

where $g_{ij} = \frac{1}{2}[F^2]_{y^i y^j}$ and $(g^{ij}) = (g_{ij})^{-1}$. The local functions $G^i = G^i(x, y)$ define a global vector field $G = y^i \frac{\partial}{\partial x^i} - 2G^i(x, y)\frac{\partial}{\partial y^i}$ on TM. G is called spray of F and G^i are called the spray coefficients.

Definition 2.1. A Finsler metric $F = F(x, y)$ on a manifold is locally dually flat if at every point there is a coordinate system (x^i) in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2}g^{ij}H_{y^j}, \tag{2.1}$$

where $H = H(x, y)$ is a local scalar function. Such a coordinate system is called an adapted coordinate system.

It is known that a Riemannian metric $F = \sqrt{g_{ij}(x)y^i y^j}$ is locally dually flat if and only if in an adapted coordinate system, we have

$$g_{ij}(x) = \frac{\partial^2 \psi}{\partial x^i \partial x^j}(x),$$

where $\psi = \psi(x)$ is a C^∞ function [3, 2].

The First example of non-Riemannian dually flat metric is given in [14] as follows

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2} \pm \frac{\langle x, y\rangle}{1 - |x|^2}. \tag{2.2}$$

This metric is defined on the unit ball $B^n \subset \mathbb{R}^n$.

First, let us introduce our notations. Let $F = \alpha + \frac{\beta^{(n+1)}}{\alpha^n}$ be a Finsler metric on a manifold M.Define $b_{i|j}$ by

$$b_{i|j}\theta^j = db_i - b_j\theta_i^j,$$

where $\theta^i = dx^i$ and $\theta_i^j = \overline{\Gamma}_{ik}^j dx^k$ denote the Levi-Civita connection forms of α.
Let

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}).$$

Clearly, β is closed if and only if $s_{ij} = 0$. We denote $r_{00} = r_{ij}y^i y^j$ and $s_{k0} = s_{km}y^m$.

The flag curvature in Finsler geometry is the analogue of the sectional curvature in Riemannian geometry. A Finsler metric F on a manifold M is said to be of scalar flag curvature if the flag curvature $K(P, y) = K(x, y)$ is a scalar function on $TM - 0$. It is said to be of almost isotropic flag curvature if $K(P, y) = 3c_{x^m}y^m/F + \sigma$, where $c = c(x)$ and $\sigma = \sigma(x)$ are scalar functions on M. If $c = 0$ and $\sigma = constant$, then F is said to be of constant flag curvature [5, 6, 12].

For a given constant $C \neq 0$, there might be many forms for α satisfying Hamel's projective flatness equation with constant sectional curvature $k_\alpha = -C^2$ and $\beta = \frac{\alpha_{x^m}y^m}{2C\alpha}$. Note that if we take $C = \pm 1$ with

$$\alpha = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2}$$

and

$$\beta = \pm\frac{\langle x, y\rangle}{1 + |x|^2},$$

in this case, F is the Funk metric on the unit ball $B^n \subset \mathbb{R}^n$ given in (2.2).

Definition 2.2. A Finsler Metric on a manifold M is a C^∞ function $F : TM - \{0\} \to [0, \infty)$ satisfying the following conditions

- Regularity: F is C^∞ on $TM - \{0\}$.
- Positive homogeneity: $F(x, \lambda y) = \lambda F(x, y)$, $\lambda > 0$.
- Strong convexity: the fundamental tensor $g_{ij}(x, y)$ is positive definite for all $(x, y) \in TM - \{0\}$, where $g_{ij}(x, y) = \frac{1}{2}[F^2]_{y^i y^j}(x, y)$.

By the homogeneity of F, we have $F(x, y) = \sqrt{g_{ij}(x, y)y^i y^j}$. An important class of Finsler metrics is a class of Riemannian metrics, which are in the form of $F(x, y) = \sqrt{g_{ij}(x)y^i y^j}$. Another important class of Finsler metrics is a class of Minkowski metrics, which are in the form of $F(x, y) = \sqrt{g_{ij}(y)y^i y^j}$.

Dually flat Finsler metrics on an open subset in $\mathbb{R}^n$ can be characterized by a simple PDE.

Lemma 2.1. *[14] A Finsler metric $F = F(x, y)$ on an open subset $U \subset \mathbb{R}^n$ is dually flat if and only if it satisfies the following equations*

$$[F^2]_{x^k y^l}y^k - 2[F^2]_{x^l} = 0. \tag{2.3}$$

In this case, $H = H(x, y)$ in (2.1) is given by $H = \frac{1}{6}[F^2]_{x^m}y^m$.

Definition 2.3. A Finsler metric $F = F(x, y)$ is locally projectively flat if at every point there is a coordinate system (x^i) in which all geodesics are straight line, or equivalently, the spray coefficients are in the following form

$$G^i = Py^i, \tag{2.4}$$

where $P = P(x, y)$ is a local scalar function satisfying $P(x, \lambda y) = \lambda P(x, y)$ for all $\lambda > 0$.

Projectively flat metrics on an open subset in $\mathbb{R}^n$ can be characterized by a simple PDE.

Lemma 2.2. *[7] A Finsler metric $F = F(x, y)$ on an open subset $U \subset \mathbb{R}^n$ is projectively flat if and only if it satisfies the following equations*

$$F_{x^k y^l} y^k - F_{x^l} = 0. \tag{2.5}$$

In this case, local function $P = P(x, y)$ is given by $P = F_{x^m} y^m / 2F$.

It is easy to show that any locally projectively flat Finsler metric $F = F(x, y)$ is of scalar flag curvature. Moreover, if $G^i = Py^i$ in a local coordinate system, then the flag curvature is given by

$$K = \frac{P^2 - P_{x^m} y^m}{F^2}. \tag{2.6}$$

Particularly, Beltrami's theorem says that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature.

We have the following

Theorem 2.3. *[15] Let $F = \alpha + \beta$ be a locally projectively flat Randers metric on a manifold. If it is of constant flag curvature, then one of the following holds*

- *F is locally isometric to the Randers metric $F = |y| + by^1$ on $\mathbb{R}^n$, where $0 \leq b < 1$ is a constant.*
- *After normalization, F is locally isometric to the following Randers metric on a unit ball $B^n \subset \mathbb{R}^n$*

  $$F = \frac{\sqrt{|y^2| - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2} \pm \frac{\langle x, y\rangle}{1 - |x|^2} \pm \frac{\langle a, y\rangle}{1 + \langle a, x\rangle}, \tag{2.7}$$

  where $a \in \mathbb{R}^n$ is a constant vector with $|a| < 1$.

A Finsler metric is said to be dually flat and projectively flat on an open subset $U \subset \mathbb{R}^n$ if the spray coefficients G^i follow (2.1) and (2.4) in U. There are Finsler metrics on an open subset in $\mathbb{R}^n$ which are dually flat and projectively flat.

Example: Let $U \subset \mathbb{R}^n$ be a strongly convex domain and there is a Minkowski norm $\phi(y)$ on $\mathbb{R}^n$ such that

$$U = \{y \in \mathbb{R}^n | \phi(y) < 1\}.$$

Define $F = F(x, y) > 0$, $y \neq 0$ by

$$x + \frac{y}{F} \in \partial U, \;\; y \in T_x U = \mathbb{R}^n.$$

It is easy to show that F is a Finsler metric satisfying

$$F_{x^k} = F F_{y^k}. \tag{2.8}$$

Using (2.8), one can easily verify that $F = F(x, y)$ satisfies (2.3) and (2.5). Thus it is dually flat and projectively flat on U. F is called the Funk metric on U.

In fact, every dually flat and projectively flat metric on an open subset in $\mathbb{R}^n$ must be either a Minkowski metric or a Funk metric satisfying (8) after normalization.

3 Locally dually flat (α, β)-metrics

Assume that $F = \kappa\alpha + \epsilon\beta$ is dually flat on an open subset $U \subset \mathbb{R}^n$. Now we prove the following theorem

Theorem 3.1. *If $F = \kappa\alpha + \epsilon\beta$ be a (α, β)-metric on a manifold M. F is a locally dually flat if and only if in an adapted coordinate system, β and α satisfy*

$$r_{00} = \frac{1}{3}(6 - 4\kappa^2)\beta\theta - \frac{1}{3}(2\epsilon^2 + 3)\tau\beta^2 + [\kappa^2\tau + \frac{2}{3}\kappa^2(\epsilon^2\tau b^2 - b_m\theta^m)]\alpha^2, \tag{3.1}$$

$$s_{k0} = -\frac{\theta b_k - \beta\theta_k}{3}, \tag{3.2}$$

$$G^m_\alpha = \frac{1}{3}\kappa^2(2\theta + \epsilon^2\tau\beta)y^m - \frac{1}{3}\kappa^2(\epsilon^2\tau b^m - \theta^m)\alpha^2, \tag{3.3}$$

where $\tau = \tau(x)$ is a scalar function and $\theta = \theta_k y^k$ is a 1-form on M and $\theta^m = a^{im}\theta_i$.

Proof. It is straight forward to verify the sufficient condition. Thus we shall only prove necessary condition. Assume that $F = \kappa\alpha + \epsilon\beta$ is dually flat on an open subset $U \subset \mathbb{R}^n$. First we have the following identities

$$\alpha_{x^k} = \frac{y_m\partial G^m_\alpha}{\alpha\partial y^k},\ \beta_{x^k} = b_{m|k}y^m + b_m\frac{\partial G^m_\alpha}{\partial y^k},\ s_{y^k} = \frac{\alpha b_k - s y_k}{\alpha^2}, \tag{3.4}$$

where $s = \frac{\beta}{\alpha}$ and $y_k = a_{jk}y^j$.
In view of equation (3.4), we have

$$[F^2]_{x^k} = 2(\kappa + s\,\epsilon)\big[(\kappa y_m + \alpha\epsilon b_m)\frac{\partial G^m_\alpha}{\partial y^k} + \alpha\epsilon b_{m|k}y^m\big], \tag{3.5}$$

$$\begin{aligned}[F^2]_{x^l y^k}y^l = {} & 2\epsilon\frac{(\alpha b_k s y_k)}{\alpha^2}\big[2(\kappa y_m + \alpha\epsilon b_m)G^m_\alpha + \alpha\epsilon r_{00}\big] \\ & + 2(\kappa + s\,\epsilon)\big[2(\kappa a_{mk} + \frac{y_k}{\alpha}\epsilon b_m)G^m_\alpha \\ & + (\kappa y_m + \alpha\epsilon b_m)\frac{\partial G^m_\alpha}{\partial y^k} + \frac{r_{00}}{\alpha}\epsilon y_k + \alpha\epsilon b_{k|0}\big]\end{aligned} \tag{3.6}$$

Using (2.3) and (3.5) in (3.6), we get

$$\begin{aligned}& \epsilon(\alpha^2 b_k - \beta y_k)\big[2(\kappa y_m + \alpha\epsilon b_m)G^m_\alpha + \alpha\epsilon r_{00}\big] + (\alpha\kappa + \beta\epsilon)\alpha \\ & \times\big[2(\kappa\alpha a_{mk} + y_k\epsilon b_m)G^m_\alpha - (\kappa\alpha y_m + \alpha^2\epsilon b_m)\frac{\partial G^m_\alpha}{\partial y^k} \\ & + r_{00}\epsilon y_k + \alpha^2\epsilon(3s_{k0} - r_{k0})\big] = 0.\end{aligned} \tag{3.7}$$

Rewriting (3.7) as a polynomial in α, we have

$$\begin{aligned}& \big[-\epsilon b_m\frac{\partial G^m_\alpha}{\partial y^k} + \kappa\epsilon(3s_{k0} - r_{k0})\big]\alpha^4 + \big[2\epsilon^2 b_k b_m G^m_\alpha + \epsilon^2 b_k r_{00} \\ & + 2\kappa^2 a_{mk}G^m_\alpha - \kappa^2 y_m\frac{\partial G^m_\alpha}{\partial y^k} - \epsilon^2\beta b_m\frac{\partial G^m_\alpha}{\partial y^k} + \beta\epsilon^2(3s_{k0} - r_{k0})\big]\alpha^3 \\ & + \kappa\epsilon(2b_k y_m G^m_\alpha + 2b_m y_k G^m_\alpha + r_{00}y_k + 2\beta a_{mk}G^m_\alpha - \beta y_m\frac{\partial G^m_\alpha}{\partial y^k})\alpha^2 \\ & - 2\kappa\epsilon\beta y_k y_m G^m_\alpha = 0.\end{aligned} \tag{3.8}$$

From (3.8) we know that the coefficient of α is zero. Hence the coefficients of α^3 must be zero. Thus we have

$$\begin{aligned}(3.9)\qquad & 2\epsilon^2 b_k b_m G^m_\alpha + \epsilon^2 b_k r_{00} + 2\kappa^2 a_{m\kappa} G^m_\alpha - \kappa^2 y_m \frac{\partial G^m_\alpha}{\partial y^k} \\ & - \epsilon^2 \beta b_m \frac{\partial G^m_\alpha}{\partial y^k} + \beta\epsilon^2 (3s_{k0} - r_{k0}) = 0'\end{aligned}$$

$$\begin{aligned}(3.10)\qquad & (-b_m \frac{\partial G^m_\alpha}{\partial y^k} + 3s_{k0} - r_{k0})\alpha^4 + (2b_k y_m G^m_\alpha + 2b_m y_k G^m_\alpha \\ & + r_{00} y_k + 2\beta a_{mk} G^m_\alpha - \beta y_m \frac{\partial G^m_\alpha}{\partial y^k})\alpha^2 - 2\beta y_k y_m G^m_\alpha = 0.\end{aligned}$$

The sufficiency is clear because of (3.9) and (3.10). We just prove the necessity Note that

$$(3.11)\qquad y_m \frac{\partial G^m_\alpha}{\partial y^k} = \frac{\partial (y_m G^m_\alpha)}{\partial y^\kappa} - a_{mk} G^m_\alpha,$$

$$(3.12)\qquad b_m \frac{\partial G^m_\alpha}{\partial y^k} = \frac{\partial (b_m G^m_\alpha)}{\partial y^k}.$$

Contracting (3.9) and (3.10) with b^k and using (3.11), (3.12), we get

$$\begin{aligned}(3.13)\qquad & \kappa^2 \frac{\partial (y_m G^m_\alpha)}{\partial y^k} b^k + \epsilon^2 \beta \frac{\partial (b_m G^m_\alpha)}{\partial y^k} b^k \\ & = (2\epsilon^2 b^2 + 3\kappa^2) b_m G^m_\alpha + \epsilon^2 b^2 r_{00} + \beta\epsilon^2 (3s_0 - r_0)\end{aligned}$$

$$\begin{aligned}(3.14)\qquad & \alpha^4 \frac{\partial (b_m G^m_\alpha)}{\partial y^k} b^k + \beta\alpha^2 \frac{\partial (y_m G^m_\alpha)}{\partial y^k} b^k = (3s_0 - r_0)\alpha^4 \\ & + (2b^2 y_m G^m_\alpha + 5\beta b_m G^m_\alpha + \beta r_{00})\alpha^2 - 2\beta^2 y_m G^m_\alpha.\end{aligned}$$

Using (3.13) $\times\, \alpha^4$-(3.14) $\times\, \beta$ yield

$$\begin{aligned}(3.15)\qquad & \left[\frac{\partial (y_m G^m_\alpha)}{\partial y^k} b^k - 3b_m G^m_\alpha\right]\alpha^2(\kappa^2\alpha^2 - \beta^2) = (2b_m \alpha^2 G^m_\alpha + r_{00}\alpha^2 - 2\beta y_m G^m_\alpha) \\ & \times (b^2\epsilon^2\alpha^2 - \beta^2).\end{aligned}$$

Because $(b^2\epsilon^2\alpha^2 - \beta^2)$ and $(\kappa^2\alpha^2 - \beta^2)$ and α^2 are all irreducible polynomial of (y^i), and $(\kappa^2\alpha^2 - \beta^2)$ and α^2 are relatively prime polynomial of (y^i), we know that there is a function $\tau = \tau(x)$ on M such that

$$(3.16)\qquad 2b_m \alpha^2 G^m_\alpha + r_{00}\alpha^2 - 2\beta y_m G^m_\alpha = \tau\alpha^2(\kappa^2\alpha^2 - \beta^2),$$

$$(3.17)\qquad \frac{\partial (y_m G^m_\alpha)}{\partial y^k} b^k - 3b_m G^m_\alpha = \tau(b^2\epsilon^2\alpha^2 - \beta^2).$$

Equation (3.16) can be written as

$$(3.18)\qquad 2\beta y_m G^m_\alpha = (2b_m G^m_\alpha + r_{00} - \tau\kappa^2\alpha^2 + \tau\beta^2)\alpha^2.$$

Since α^2 does not contain the factor β, we have

$$y_m G^m_\alpha = \theta\alpha^2, \tag{3.19}$$

$$b_m G^m_\alpha = \beta\theta - \frac{1}{2}r_{00} + \frac{\tau}{2}\kappa^2\alpha^2 - \frac{\tau}{2}\beta^2, \tag{3.20}$$

where $\theta = \theta_k y^k$ is a 1-form on M. then we obtain the following

$$\frac{\partial(y_m G^m_\alpha)}{\partial y^k} = \theta_k\alpha^2 + 2\theta y_k, \tag{3.21}$$

$$\frac{\partial(b_m G^m_\alpha)}{\partial y^k} = \beta\theta_k + \theta b_k - r_{k0} + \tau\kappa^2 y_k - \tau\beta b_k. \tag{3.22}$$

In View of equations (3.19), (3.20), (3.21) and (3.22), equations (3.9) and (3.10) become

$$\begin{aligned}&\beta\epsilon^2(3s_{k0} + \theta b_k - \beta\theta_k) + \kappa^2\alpha^2(\tau b_k\epsilon^2 - \theta_k) + 3\kappa^2 a_{m\kappa}G^m_\alpha\\ &- \kappa^2(2\theta + \tau\beta\epsilon^2)y_k = 0,\end{aligned} \tag{3.23}$$

$$\begin{aligned}&\Big[(3s_{k0} + \theta b_k - \beta\theta_k) + (\tau b_k - \theta_k)\beta\Big]\alpha^2\\ &- (2\theta + \tau\beta)\beta y_k + 3\beta a_{mk}G^m_\alpha = 0.\end{aligned} \tag{3.24}$$

Using (3.23) $\times$ β-(3.24) $\times$ κ^2, we get

$$3s_{k0} + b_k\theta - \theta_k\beta = 0, \tag{3.25}$$

Equation (3.25), gives

$$s_{k0} = -\frac{\theta b_k - \beta\theta_k}{3}. \tag{3.26}$$

Contracting (3.26) with a^{lk}, we get

$$3s^l_0 + \theta b^l - \beta\theta^l = 0. \tag{3.27}$$

Contracting (3.23) with a^{lk} and using (3.27), we get

$$G^m_\alpha = \frac{1}{3}\kappa^2(2\theta + \tau\beta\epsilon^2)y^m - \frac{1}{3}\kappa^2(\tau\epsilon^2 b^m - \theta^m)\alpha^2. \tag{3.28}$$

Using (3.28) in (3.20), we get

$$r_{00} = \frac{1}{3}(6 - 4\kappa^2)\beta\theta - \frac{1}{3}(2\epsilon^2 + 3)\tau\beta^2 + \kappa^2\big[\tau + \frac{2}{3}(\tau b^2\epsilon^2 - b_m\theta^m)\big]\alpha^2 \tag{3.29}$$

□

4 Dually flat and projectively flat metrics

In this section, we are going to prove the Lemma for dually flat and projectively flat metrics

Lemma 4.1. *Let $F = \kappa\alpha + \epsilon\beta$ be a locally dually flat Randers metric on a manifold M. Suppose that β satisfies the following equation:*

$$r_{00} = c(\alpha^2 - \beta^2) - 2\beta s_0, \tag{4.1}$$

where $c = c(x)$ is a scalar function on M. Then F is locally projectively flat in adapted coordinate systems with $G^i = (\kappa^2\epsilon^2\tau\beta + \frac{r_{00}+2\beta s_0}{2F})y^i$.

Proof. The formula for the spray coefficient G^i of F,

$$G^i = G^i_\alpha + \frac{r_{00} + 2\beta s_0}{2F} y^i - s_0 y^i + \alpha s^i_0, \tag{4.2}$$

where G^i_α denote the spray coefficients of α. We shall prove that α is projectively flat in the adopted coordinate system, i.e., $G^i_\alpha = P_\alpha y^i$ and β is closed, i.e., $s_{ij} = 0$. By (3.29) and (4.1), we have

$$\begin{aligned} &\left[c - \tau\kappa^2 - \frac{2}{3}\kappa^2(\tau b^2\epsilon^2 - b_m\theta^m)\right]\alpha^2 \\ &= \left[2s_0 + \frac{1}{3}(6 - 4\kappa^2)\theta + (c - \frac{1}{3}(2\epsilon^2 + 3)\tau)\beta\right]\beta. \end{aligned} \tag{4.3}$$

Since α^2 is irreducible polynomial of y^i, we conclude that

$$c - \tau\kappa^2 - \frac{2}{3}\kappa^2(\tau b^2\epsilon^2 - b_m\theta^m) = 0, \tag{4.4}$$

$$s_0 = \frac{1}{2}\left[(\frac{1}{3}(2\epsilon^2 + 3)\tau - c)\beta\right] - \frac{1}{6}(6 - 4\kappa^2)\theta. \tag{4.5}$$

From (3.27), we have

$$s_0 = -\frac{1}{3}(\theta b^2 - \beta b_m\theta^m). \tag{4.6}$$

From (4.5) and (4.6), we get

$$\begin{aligned} &\frac{2}{3}\left[\frac{1}{2}(6 - 4\kappa^2) - b^2\right]\theta = \frac{2}{3}\left[\frac{1}{2}(6 - 4\kappa^2) - b^2\right]\tau\beta\epsilon^2 \\ &+ \left[\tau\kappa^2 - c + \frac{2}{3}\kappa^2(\tau b^2\epsilon^2 - b_m\theta^m)\right]\beta. \end{aligned} \tag{4.7}$$

In view of (4.4) and (4.7), we have

$$\theta = \tau\beta\epsilon^2.$$

By (4.4) we see that $c = \tau\kappa^2$. From (3.26) and $\theta = \tau\beta\epsilon^2$, we get $s_{ij} = 0$. Thus β is closed. Then

$$r_{00} = c(\alpha^2 - \beta^2).$$

Plugging $\theta = \tau\beta\epsilon^2$ into (3.28), we get

$$G^i_\alpha = \kappa^2\epsilon^2\tau\beta y^i.$$

Thus α is projectively flat in the adopted coordinate system. By (4.2), we get

$$G^i = (\kappa^2\epsilon^2\tau\beta + \frac{r_{00} + 2\beta s_0}{2F})y^i. \tag{4.8}$$

Therefore $F = \kappa\alpha + \epsilon\beta$ is projectively flat in adopted coordinate systems. □

Remark 4.1. The S curvature is an important non-Riemannian quantity in Finsler geometry ([5, 4, 12]). A Finsler metric is said to be of isotropic S-curvature if $S = (n+1)c(x)F$. It is shown that metric $F = \kappa\alpha + \epsilon\beta$ is of isotropic S-curvature $S = (n+1)c(x)F$ if and only if it satisfies (4.1).

Lemma 4.2. *Let $F = \kappa\alpha + \epsilon\beta$ be a locally dually flat Rander metric on a manifold M. If it is of almost isotropic flag curvature, $K = \frac{3\tilde{c}_{x^m}(x)y^m}{F+\sigma(x)}$, then it is locally projectively flat in adopted coordinate system with $G^i = (\kappa^2\epsilon^2\tau\beta + \frac{r_{00}+2\beta s_0}{2F})y^i$, where $c = c(x)$ is a scalar function such that $c(x) - \tilde{c}(x) = constant$.*

Proof. Assume that $F = \kappa\alpha + \epsilon\beta$ is of almost isotropic flag curvature $K = \frac{3\tilde{c}_{x^m}(x)y^m}{F+\sigma(x)}$. According to Theorem (1.2) in [13], F Must be of isotropic S-curvature, i.e., β satisfies (4.1) for a scalar function $c = c(x)$ such that $c(x) - \tilde{c}(x) = constant$. Further, because F is locally dually flat, by Lemma 4.1, F is locally projectively flat in adopted coordinate systems with spray coefficients given by (4.8) □

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