η-Ricci solitons on para-Kenmotsu manifolds

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Abstract. In this paper, we initiate the study of η-Ricci solitons on para-Kenmotsu manifolds. At first we consider η-Ricci solitons on para-Kenmotsu manifolds with harmonic curvature and cyclic parallel Ricci tensor. Beside these we study η-Ricci solitons on para-Kenmotsu manifolds with harmonic Weyl tensor. The next section deals with the study of η-Ricci solitons on para-Kenmotsu manifolds with η-parallel Ricci tensor. Moreover we characterize a para-Kenmotsu manifold satisfying the curvature conditions $P.\phi = 0$ and $Q.P = 0$ respectively whose metric is the η-Ricci soliton. Finally, we have cited an example of a para-Kenmotsu manifold which admits η-Ricci solitons.

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1 Introduction

In 1982, the notion of Ricci flow was introduced by Hamilton[16] to find the canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold M defined as follows:

$$\frac{\partial}{\partial t} g = -2S, \tag{1.1}$$

where S denotes the Ricci tensor. Ricci solitons are special solutions of the Ricci flow equation (1.1) of the form $g = \sigma(t)\psi_t^* g$ with the initial condition $g(0) = g$, where ψ_t are diffeomorphism of M and $\sigma(t)$ is the scaling function. A Ricci soliton is a generalization of an Einstein metric. We recall the notion of Ricci solitons according to [4]. On the manifold M, a Ricci soliton is a triple (g, V, λ) with g a Riemannian metric, V a vector field(called the potential vector field) and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where $\pounds$ is the Lie derivative. Metrics satisfying (1.2) are interesting and useful in physics and are often referred to as quasi-Einstein metrics([6],[7]). Compact Ricci

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solitons are the fixed points of the Ricci flow $\frac{\partial}{\partial t}g = -2S$, projected from the space of metrics onto its quotient modulo diffeomorphisms and scaling and often arise blow-up limits for the Ricci flow and compact manifolds. Theoritical physists have also been looking into the equation of Ricci solitons in relation with string theory. The initial contribution in this direction is due to Fridean[13], who discusses some of its aspects.

Ricci solitons have been studied by many authors, such as ([10],[11],[16],[17]) and many others.

As a generalization of Ricci solitons, the notion of η-Ricci solitons was introduced by Cho and Kimura[8]. This notion has also been studied in [8], for Hopf hypersurfaces in complex space forms. An η-Ricci soliton is a tuple (g, V, λ, μ), where V is a vector field on M, λ and μ are real scalars and g is a Riemannian (or pseudo-Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.3}$$

where S is the Ricci tensor associated to g. In this connection, we mention the works of Blaga ([1],[2],[3]), Prakasha et. al. [22], De and De[9], De et. al.[18], Eyasmin et. al.[12] and many others on η-Ricci solitons. In particular, if $\mu = 0$, then η-Ricci soliton (g, V, λ, μ) reduces to Ricci soliton (g, V, λ). If $\mu \neq 0$, then the η-Ricci soliton is named proper η-Ricci soliton.

Gray[15] introduced the notion of cyclic parallel Ricci tensor and Coddazi type of Ricci tensor. A Riemannian manifold or semi-Riemannian manifold is said to have cyclic parallel Ricci tensor if its Ricci tensor S of type (0,2) is non-zero and satisfies the condition

$$(\nabla_U S)(V, W) + (\nabla_V S)(W, U) + (\nabla_W S)(U, V) = 0. \tag{1.4}$$

Suppose the curvature tensor is harmonic, that is, $divR = 0$, which implies

$$(\nabla_U S)(V, W) = (\nabla_V S)(U, W), \tag{1.5}$$

where '*div*' denotes divergence. This means that the Levi-Civita connection ∇ of such metric is a Yang-Mills connection while keeping the metric on the manifold fixed. Equation (1.5) means that the Ricci tensor S is of Coddazi type.

Also the Weyl tensor C is said to be harmonic if $divC = 0$, where '*div*' denotes divergence.

If the Weyl tensor is harmonic, then we get

$$(\nabla_U S)(V, W) - (\nabla_V S)(U, W) = \frac{1}{2(n-1)}[(Ur)g(V, W) - (Vr)g(U, W)], \tag{1.6}$$

where r is the scalar curvature.

The projective curvature tensor P[24] in a manifold (M, g) is defined by

$$P(U, V)W = R(U, V)W - \frac{1}{(n-1)}[g(V, W)QU - g(U, W)QV], \tag{1.7}$$

where Q is the Ricci operator defined by $S(U, V) = g(QU, V)$ and $U, V, W \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields of M.

The above mentioned works on η-Ricci soliton motivate us to study η-Ricci soliton in the frame work of para-Kenmotsu manifolds. More precisely, we prove the following theorems:

Theorem 1.1. *A para-Kenmotsu manifold admitting an η-Ricci soliton is of harmonic curvature if and only if the manifold is an Einstein manifold. Also, $\mu = 1$ and $\lambda = n - 2$.*

Theorem 1.2. *A para-Kenmotsu manifold admitting an η-Ricci soliton satisfies cyclic parallel Ricci tensor if and only if the manifold is an Einstein manifold. Moreover, $\mu = 1$ and $\lambda = n - 2$.*

Theorem 1.3. *Let M be a para-Kenmotsu manifold admitting an η-Ricci soliton. The manifold M is of harmonic Weyl tensor if and only if the manifold is an Einstein manifold, provided the scalar curvature r is invariant under the characteristic vector field ξ. Also, $\mu = 1$ and $\lambda = n - 2$.*

As a corollary of the above theorem we have:

Corollary 1.4. *A 3-dimensional para-Kenmotsu manifold admitting an η-Ricci soliton is of harmonic Weyl tensor if and only if the manifold is of constant sectional curvature -1.*

Theorem 1.5. *Let M be a para-Kenmotsu manifold admitting an η-Ricci soliton with η-parallel Ricci tensor. Then $\mu = 1, \lambda = n - 2$ and the manifold is an Einstein manifold.*

Theorem 1.6. *If a para-Kenmotsu manifold admits an η-Ricci soliton and satisfies the curvature condition $P.\phi = 0$, then $\mu = 1, \lambda = n-2$ and the manifold is an Einstein manifold.*

Theorem 1.7. *If a para-Kenmotsu manifold admits an η-Ricci soliton and satisfies the curvature condition $Q.P = 0$, then $\mu = 1, \lambda = n - 2$ and the manifold is an Einstein manifold.*

2 Preliminaries

In this section we gather the formulas and results of para-Kenmotsu manifold which will be required on later sections. To know more fact about paracontact metric geometry, we may refer to ([19],[25]) and references therein. Several years ago, the notion of paracontact metric structures were introduced in [19]. Since the publication of [25], paracontact metric manifolds have been studied by many authors in recent years. The importance of para-Kenmotsu geometry, has been pointed out especially in the last years by several papers highlighting the exchanges with the theory of para-Kähler manifolds and its role in semi-Riemannian geometry and mathematical physics([5],[20],[21]).

Let M be an n-dimensional differentiable manifold of class C^∞ in which there are given a (1,1)-type tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2 U = U - \eta(U)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi U) = 0. \tag{2.1}$$

Then (ϕ, ξ, η) is called an almost paracontact structure and M an almost paracontact manifold. Moreover, if M admits a semi-Riemannian metric g such that

$$g(\xi, U) = \eta(U), \quad g(\phi U, \phi V) = -g(U, V) + \eta(U)\eta(V), \tag{2.2}$$

then (ϕ, ξ, η, g) is called an almost paracontact metric structure and M an almost paracontact metric manifold[23]. An almost paracontact metric manifold (M, ϕ, ξ, η, g) is called an almost α-paracosymplectic manifold, if

$$d\eta = 0, \quad d\Phi = 2\alpha\eta \wedge \Phi, \tag{2.3}$$

where α may be a constant or function on M. For a particular choices of the function $\alpha = 1$ in (2.3) we have almost para-Kenmotsu manifolds. If further the normality condition is fulfilled, then manifolds are called para-Kenmotsu. We may refer to [21] and references therein for more information about para-Kenmotsu manifolds. In a para-Kenmotsu manifold the following relations hold:

$$(\nabla_U \phi)V = g(\phi U, V)\xi - \eta(V)\phi U, \tag{2.4}$$

$$\nabla_U \xi = U - \eta(U)\xi, \tag{2.5}$$

$$(\nabla_U \eta)V = g(U, V) - \eta(U)\eta(V), \tag{2.6}$$

$$\eta(R(U, V)W) = -\eta(U)g(V, W) + \eta(V)g(U, W), \tag{2.7}$$

$$R(U, V)\xi = \eta(U)V - \eta(V)U, \tag{2.8}$$

$$S(U, \xi) = -(n-1)\eta(U), \quad Q\xi = -(n-1)\xi, \tag{2.9}$$

for any vector fields U, V, W where Q is the Ricci operator, that is, $g(\phi U, V) = S(U, V)$ of the manifold.

Definition 2.1. A para-Kenmotsu manifold M satisfies the η-parallel Ricci tensor condition [14] if

$$g((\nabla_U Q)V, W) = 0,$$

for arbitrary vector fields U, V, W.

Proposition 2.1. *For an η-Ricci soliton on a para-Kenmotsu manifold, the Ricci tensor S is of the form*

$$S(U, V) = -(1 + \lambda)g(U, V) - (\mu - 1)\eta(U)\eta(V), \tag{2.10}$$

$$QU = -(1 + \lambda)U - (\mu - 1)\eta(U)\xi \tag{2.11}$$

and

$$\lambda + \mu = n - 1. \tag{2.12}$$

Remark 2.2. The above form of the Ricci tensor is also deduced by Blaga[1].

3 Proof of the main theorems

Proof of Theorem 1.1. Taking covariant differentiation of (2.10) with respect to W we obtain

$$(\nabla_W S)(U,V) = -(\mu-1)[(\nabla_W \eta)U\eta(V) + (\nabla_W \eta)V\eta(U)]. \tag{3.1}$$

Using (2.6) in (3.1), we get

$$(\nabla_W S)(U,V) = -(\mu-1)[g(U,W)\eta(V) + g(V,W)\eta(U) - 2\eta(U)\eta(V)\eta(W)]. \tag{3.2}$$

In view of (3.2) it follows that

$$(\nabla_W S)(U,V) - (\nabla_U S)(V,W) = -(\mu-1)[g(V,W)\eta(U) - g(U,V)\eta(W)]. \tag{3.3}$$

By hypothesis, the manifold is of harmonic curvature, that is,

$$(\nabla_W S)(U,V) = (\nabla_U S)(V,W).$$

Hence from (3.3) we get

$$(\mu-1)[g(V,W)\eta(U) - g(U,V)\eta(W)] = 0. \tag{3.4}$$

Putting $W = \xi$ in the above equation yields

$$(\mu-1)g(\phi U, \phi V) = 0. \tag{3.5}$$

It follows that $\mu = 1$. Therefore (2.12) implies $\lambda = n-2$. Hence from (2.10), we get

$$S(U,V) = -(n-1)g(U,V).$$

Conversely, suppose the manifold is an Einstein manifold. Then obviously the Ricci tensor is of Coddazi type. Therefore the manifold is of harmonic curvature.
This completes the proof.

Proof of Theorem 1.2. Using (3.2) in (1.4), we get

$$(\mu-1)[g(U,V)\eta(W) + g(U,W)\eta(V) - 2\eta(U)\eta(V)\eta(W)$$

$$+g(U,V)\eta(W) + g(V,W)\eta(U) - 2\eta(U)\eta(V)\eta(W) \tag{3.6}$$

$$+g(U,W)\eta(V) + g(V,W)\eta(U) - 2\eta(U)\eta(V)\eta(W)] = 0.$$

Putting $W = \xi$ in (3.6) we obtain

$$(\mu-1)[g(U,V) - \eta(U)\eta(V)] = 0.$$

Then it follows that

$$\mu = 1.$$

Hence from (2.12), we get $\lambda = n-2$. Therefore (2.10) implies

$$S(U,V) = -(n-1)g(U,V).$$

Conversely, if the manifold is an Einstein manifold then it can be easily seen that the Ricci tensor is cyclic parallel.
This completes the proof.

Proof of Theorem 1.3. Let the para-Kenmotsu manifold M is of harmonic Weyl tensor. Then (1.6) gives

$$(\nabla_U S)(V,W) - (\nabla_V S)(U,W) = \frac{1}{2(n-1)}[(Ur)g(V,W) - (Vr)g(U,W)]. \tag{3.7}$$

Making use of (3.2) in (3.7) we have

$$-(\mu-1)[g(U,W)\eta(V) - g(V,W)\eta(U)] = \frac{1}{2(n-1)}[(Ur)g(V,W) - (Vr)g(U,W)]. \tag{3.8}$$

Putting $V = \xi$ in the forgoing equation gives

$$-(\mu-1)[g(U,W) - \eta(U)\eta(W)] = \frac{1}{2(n-1)}[(Ur)\eta(W) - (\xi r)g(U,W)]. \tag{3.9}$$

Replacing W by ϕW in (3.9) we obtain

$$(\mu-1)g(U,\phi W) = \frac{1}{2(n-1)}(\xi r)g(U,\phi W). \tag{3.10}$$

Taking $(\xi r) = 0$, then the above equation implies

$$(\mu-1)g(U,\phi W) = 0.$$

It follows that $\mu = 1$. Then from (2.12), we have $\lambda = n-2$. Hence (2.10) implies

$$S(U,V) = -(n-1)g(U,V).$$

Conversely, if the manifold is an Einstein manifold then it is Ricci symmetric ($\nabla S = 0$) and the scalar curvature is constant. Hence the Weyl tensor is harmonic.
This completes the proof.

Proof of Corollary 1.1. We consider a 3-dimensional para-Kenmotsu manifold. It is known that [20]

$$\begin{aligned} R(U,V)W &= g(V,W)QU - g(U,W)QV + S(V,W)U - S(U,W)V \\ &\quad - \frac{r}{2}[g(V,W)U - g(U,W)V], \end{aligned} \tag{3.11}$$

$$QU = \frac{1}{2}[(r+2)U - (r+6)\eta(U)\xi], \tag{3.12}$$

for any vector fields U, V, W where r is the scalar curvature and Q is the Ricci operator defined by $S(U,V) = g(QU,V)$, where S is the Ricci tensor.

Also the author of the paper [20] proved that in a 3-dimensional para-Kenmotsu manifold

$$\xi r = -2(r+6). \tag{3.13}$$

Using (3.13) in (3.10), we get

$$r = -2(\mu+2) = constant,$$

which implies

$$\xi r = 0. \tag{3.14}$$

Using (3.14) in (3.13) we obtain $r = -6$. Hence from (3.12), we get

$$QU = -2U.$$

Therefore from (3.11) it follows that the manifold is of constant sectional curvature -1.
Hence the manifold is an Einstein manifold and the scalar curvature is constant. Therefore from (1.6) it follows that the manifold is of harmonic Weyl tensor.
This completes the proof.

Proof of Theorem 1.4. Let the Ricci tensor of a para-Kenmotsu manifold be η-parallel. Then

$$g((\nabla_V Q)U, W) = 0 \tag{3.15}$$

for arbitrary vector fields U, V, W.
Taking covariant derivative of (2.11) with respect to an arbitrary vector field V, we get

$$\begin{aligned}(\nabla_V Q)U &= \nabla_V QU - Q(\nabla_V U)\\ &= \nabla_V[-(1+\lambda)U - (\mu-1)\eta(U)\xi]\\ &\quad +(1+\lambda)\nabla_V U + (\mu-1)\eta(\nabla_V U)\xi\\ &= -(\mu-1)[((\nabla_V\eta)U)\xi + \eta(U)\nabla_V\xi].\end{aligned} \tag{3.16}$$

Using (2.5) and (2.6) in (3.16), we get

$$(\nabla_V Q)U = -(\mu-1)[g(U,V)\xi + \eta(U)V - 2\eta(U)\eta(V)\xi]. \tag{3.17}$$

Using (3.17) in (3.15) we infer

$$(\mu-1)[g(U,V)\eta(W) + \eta(U)g(V,W) - 2\eta(U)\eta(V)\eta(W)] = 0. \tag{3.18}$$

Putting $W = \xi$ in (3.18) yields

$$(\mu-1)[g(U,V) - \eta(U)\eta(V)] = 0.$$

It follows that $\mu = 1$. Hence from (2.12), we get $\lambda = n-2$. Therefore (2.10) gives

$$S(U,V) = -(n-1)g(U,V).$$

Therefore, the theorem is proved.

Proof of Theorem 1.5. We assume that the para-Kenmotsu manifold M admitting an η-Ricci soliton satisfies the curvature condition

$$P.\phi = 0.$$

This implies

$$P(U,V)\phi W - \phi(P(U,V)W) = 0. \tag{3.19}$$

Putting $W = \xi$ in (3.19), we get

$$\phi(P(U,V)\xi) = 0. \tag{3.20}$$

Again putting $W = \xi$ and using (2.11) in (1.7) yields

$$\begin{aligned} P(U,V)\xi &= \eta(U)V - \eta(V)U \\ &\quad - \frac{1}{n-1}[\eta(V)\{-(1+\lambda)U - (\mu-1)\eta(U)\xi\} \\ &\quad - \eta(U)\{-(1+\lambda)V - (\mu-1)\eta(V)\xi\}], \end{aligned} \tag{3.21}$$

which implies

$$P(U,V)\xi = (1 - \frac{1+\lambda}{n-1})[\eta(U)V - \eta(V)U]. \tag{3.22}$$

Using (3.22) in (3.20), we get

$$(1 - \frac{1+\lambda}{n-1})[\eta(U)\phi V - \eta(V)\phi U] = 0. \tag{3.23}$$

Replacing U by ϕU in the above equation yields

$$(1 - \frac{1+\lambda}{n-1})\eta(V)\phi^2 U = 0. \tag{3.24}$$

Putting $V = \xi$ in (3.24), we get

$$(1 - \frac{1+\lambda}{n-1})[U - \eta(U)\xi] = 0. \tag{3.25}$$

Again replacing U by ϕU in the foregoing equation we infer

$$(1 - \frac{1+\lambda}{n-1})\phi U = 0. \tag{3.26}$$

Taking inner product of (3.26) with respect to Z, we get

$$(1 - \tfrac{1+\lambda}{n-1})g(\phi U, Z) = 0.$$

It follows that $1 - \frac{1+\lambda}{n-1} = 0$, which implies $\lambda = n-2$. Therefore from (2.12), we get $\mu = 1$. Hence (2.10) implies

$$S(U,V) = -(n-1)g(U,V).$$

This completes the proof.

Proof of Theorem 1.6. We assume that the para-Kenmotsu manifold M admitting an η-Ricci soliton satisfies the curvature condition $Q.P = 0$. Then

$$(Q.P)(U,V)W = 0. \tag{3.27}$$

From (3.27), it follows that

$$Q(P(U,V)W) - P(QU,V)W - P(U,QV)W - P(U,V)QW = 0. \tag{3.28}$$

Using (2.11) in (3.28), we get

$$2(1+\lambda)P(U,V)W - (\mu-1)\eta(P(U,V)W)\xi + (\mu-1)\eta(U)P(\xi,V)W$$

$$+(\mu-1)\eta(V)P(U,\xi)W + (\mu-1)\eta(W)P(U,V)\xi = 0. \tag{3.29}$$

Putting $W = \xi$ in (3.29) and using (3.22), we get

$$(\lambda+\mu)(1 - \frac{1+\lambda}{n-1})[\eta(U)V - \eta(V)U] = 0. \tag{3.30}$$

Again putting $U = \xi$ and taking inner product with Z in the above equation, we get

$$(\lambda+\mu)(1 - \tfrac{1+\lambda}{n-1})g(\phi V, \phi Z) = 0.$$

Which implies either $\lambda + \mu = 0$ or, $1 - \frac{1+\lambda}{n-1} = 0$.
Here $\lambda + \mu = 0$ contradicts with (2.12). So $1 - \frac{1+\lambda}{n-1} = 0$, which implies $\lambda = n-2$. Therefore from (2.12), we get $\mu = 1$. Hence (2.10) implies

$$S(U,V) = -(n-1)g(U,V).$$

This completes the proof.

4 Example

We consider the 3-dimensional manifold

$$M = \{(x,y,z) \in \mathbb{R}^3, z \neq 0\}$$

and the vector fields

$$e_1 = \tfrac{\partial}{\partial x}, \phi e_1 = e_2 = \tfrac{\partial}{\partial y}, \xi = e_3 = (x+2y)\tfrac{\partial}{\partial x} + (2x+y)\tfrac{\partial}{\partial y} + \tfrac{\partial}{\partial z}.$$

The 1-form $\eta = dz$ defines an almost paracontact structure on M with characteristic vector field $\xi = (x+2y)\frac{\partial}{\partial x} + (2x+y)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$.

In[20] the author obtained the expression of the curvature tensor and the Ricci tensor as follows:

$$R(e_1, e_2)e_3 = 0, R(e_2, e_3)e_3 = -e_2, R(e_1, e_3)e_3 = -e_1,$$

$$R(e_1, e_2)e_2 = e_1, R(e_2, e_3)e_2 = -e_3, R(e_1, e_3)e_2 = 0,$$

$$R(e_1, e_2)e_1 = e_2, R(e_2, e_3)e_1 = 0, R(e_1, e_3)e_1 = e_3$$

and

$$S(e_1, e_1) = -2 = -2g(e_1, e_1),$$

$$S(e_2, e_2) = 2 = -2g(e_2, e_2),$$

$$S(e_3, e_3) = -2 = -2g(e_3, e_3).$$

Again from (2.10) we obtain

$$S(e_1, e_1) = -(1 + \lambda), S(e_2, e_2) = (1 + \lambda),$$
$$S(e_3, e_3) = -(\lambda + \mu).$$

Therefore $\lambda = 1$ and $\mu = 1$. The data (g, ξ, λ, μ) defines an η-Ricci soliton on para-Kenmotsu manifolds.

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