On almost pseudo semiconformally symmetric manifolds

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Abstract. The object of the present paper is to study a type of Riemannian manifold, namely, an almost pseudo semiconformally symmetric manifold which is denoted by $A(PSCS)_n$. Several geometric properties of such a manifold are studied under certain curvature conditions. Some results on Ricci symmetric $A(PSCS)_n$ and Ricci-recurrent $A(PSCS)_n$ are obtained. Next, we consider the decomposability of $A(PSCS)_n$. Finally, two non-trivial examples of $A(PSCS)_n$ are constructed.

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1 Introduction

Riemannian symmetric spaces have an important role in differential geometry. They were first classified by Cartan [4] in the late twenties and he also gave a classification of Riemannian symmetric spaces. In 1926, Cartan [4] studied the certain class of Riemannian spaces and introduced the notation of a symmetric space. According to him, an n-dimensional Riemannian manifold M is said to be locally symmetric if its curvature tensor R satisfies $R_{hijk,l} = 0$, where "," represent the covariant differentiation with respect to the metric tensor and R_{hijk} are the components of the curvature tensor of the manifold M. This condition of locally symmetry is equivalent to the fact that the local geodesic symmetry $F(P)$ is an isometry [20] at every point $P \in M$.

After Cartan, the notation of locally symmetric manifolds has been reduced by many authors in several ways to a different extent such as pseudo symmetric manifolds introduced by Chaki [6], recurrent manifolds introduced by Walker [27], conformally symmetric manifolds introduced by Chaki and Gupta [5], conformally recurrent manifolds introduced by Adati and Miyazawa [2], weakly symmetric manifolds introduced by Tamássy and Binh [26], etc.

In 1967, Sen and Chaki [24] obtained an expression for the covariant derivative of the curvature tensor while studying conformally flat space of class one with certain

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curvature restrictions on the curvature tensor, which is as follows:

$$R^h_{ijk,l} = 2\lambda_l R^h_{ijk} + \lambda_i R^h_{ljk} + \lambda_j R^h_{ilk} + \lambda_k R^h_{ijl} + \lambda^h R^l_{ijk}\,, \tag{1.1}$$

where R^h_{ijk} are the components of the curvature tensor R, $R_{lijk} = g_{hl} R^h_{ijk}$, λ_i is a non-zero covariant vector. Later in 1987, Chaki [6] introduced a manifold whose curvature tensor satisfies (1.1) and called it a pseudo symmetric manifold. In the index-free notation this can be defined as:

$$\begin{aligned}(\nabla_E R)(X,Y)W &= 2A(E)R(X,Y)W + A(X)R(E,Y)W \\ &+ A(Y)R(X,E)W + A(W)R(X,Y)E \\ &+ g(R(X,Y)W,E)\rho,\end{aligned} \tag{1.2}$$

where A is a non-zero 1-form called the associate 1-form of the manifold. Here, ρ is a vector field corresponding to 1-form A and is defined by

$$g(E,\rho) = A(E), \tag{1.3}$$

for all vector field E, and ∇ represents the operator of covariant differentiation with respect to the metric tensor g. Taking $A = 0$ in (1.2) the manifold reduces to a symmetric manifold in the sense of Cartan. An n-dimensional pseudo symmetric manifold is denoted by $(PS)_n$. It should be taken into account that the notation of pseudo symmetric manifold studied in particular by Deszez([3],[8],[9],[10]) differ from that of Chaki [6].

In 2008, De and Gazi [11] introduced a type of Riemannian manifold which is a generalization of pseudo symmetric manifolds. Such manifold is called an almost pseudo symmetric manifold and is denoted by $(APS)_n$. A Riemannian manifold $(M_n, g), (n > 2)$ is said to be an almost pseudo symmetric [11] if its curvature tensor R of type $(0,4)$ satisfies the following relation:

$$\begin{aligned}(\nabla_E R)(X,Y,W,V) &= [A(E)+B(E)]R(X,Y,W,V) + A(X)R(E,Y,W,V) \\ &+ A(Y)R(X,E,W,V) + A(W)R(X,Y,E,V) \\ &+ A(V)R(X,Y,W,X),\end{aligned} \tag{1.4}$$

where A, B are non-zero 1-forms given by

$$g(E,\rho) = A(E), g(E,\sigma) = B(E), \tag{1.5}$$

for all vector fields E. In the paper ([12],[13]) it has been mentioned that $(PS)_n$ is a particular case of an $(AP)_n$.

Gray[16] introduced two groups of Riemannian manifolds based on the covariant differentiation of the Ricci tensor. The first group contains all Riemannian manifolds whose Ricci tensor S is a Codazzi tensor, that is,

$$(\nabla_E S)(X,Y) = (\nabla_X S)(E,Y). \tag{1.6}$$

The second group contains all Riemannian manifolds whose Ricci tensor S is cyclic parallel, that is,

$$(\nabla_E S)(X,Y) + (\nabla_X S)(E,Y) + (\nabla_Y S)(E,X) = 0. \tag{1.7}$$

In 1952, Patterson [22] introduced the notion of Ricci-recurrent manifolds. A non-flat Riemannian manifold $(M, g), (n > 2)$ is said to be a Ricci-recurrent manifold [22] if its non-zero Ricci tensor S of type (0,2) satisfies the following condition

$$(\nabla_E S)(X, Y) = \tilde{H}(E)S(X, Y), \tag{1.8}$$

where $\tilde{H}$ is non-zero 1-form called 1-form of recurrence, which is defined by

$$g(E, \mu) = \tilde{H}(E). \tag{1.9}$$

In 2016, Kim [18] introduced a type of curvature tensor which is a combination of conformal and conharmonic curvature tensor, called semiconformal curvature tensor. The semiconformal curvature tensor of type $(1, 3)$ remains invariant under conharmonic transformation [1]. More precisely, the semiconformal curvature tensor $\tilde{P}$ of type $(1, 3)$ on a Riemannian manifold (M_n, g) is defined as follows:

$$\tilde{P}(X, Y)W = -(n-2)b\, C(X, Y)W + [a + (n-2)b]\, H(X, Y)W, \tag{1.10}$$

where a, b are constants not simultaneously zero, $C(X, Y)W$ denotes the conformal curvature tensor of type $(1, 3)$, and $H(X, Y)W$ denotes the conharmonic curvature tensor of type $(1, 3)$. The conformal curvature tensor and the conharmonic curvature tensor[25] are given as follows:

$$\begin{aligned} C(X, Y)W &= R(X, Y)W - \frac{1}{(n-2)}\Big[S(Y, W)X - S(X, W)Y + g(Y, W)LX \\ &- g(X, W)LY\Big] + \frac{r}{(n-1)(n-2)}\Big[g(Y, W)X - g(X, W)Y\Big], \end{aligned} \tag{1.11}$$

and,

$$\begin{aligned} H(X, Y)W &= R(X, Y)W - \frac{1}{(n-2)}\Big[S(Y, W)X - S(X, W)Y + g(Y, W)LX \\ &- g(X, W)LY\Big], \end{aligned} \tag{1.12}$$

where L is the symmetric endomorphism of the tangent space at each point corresponding to the Ricci tensor S, that is, $g(LE, X) = S(E, X)$ and r is the scalar curvature of the manifold. From equations (1.10),(1.11) and (1.12) we obtain an expression for semiconformal curvature tensor $P(X, Y, W, V)$ of type (0,4) as follows:

$$\begin{aligned} P(X, Y, W, V) &= aR(X, Y, W, V) - \frac{a}{(n-2)}\Big[S(Y, W)g(X, V) \\ &- S(X, W)g(Y, V) + S(X, V)g(Y, W) - S(Y, V)g(X, W)\Big] \\ &- \frac{br}{(n-1)}\Big[g(Y, W)g(X, V) - g(X, W)g(Y, V)\Big], \end{aligned} \tag{1.13}$$

where $P(X, Y, W, V) = g(\tilde{P}(X, Y)W, V)$.

For $a = 1$ and $b = -\dfrac{1}{(n-2)}$, the semiconformal curvature becomes conformal curvature tensor and for $a = 1$ and $b = 0$, such a tensor reduces to conharmonic curvature tensor. A Riemannian manifold (M_n, g) of dimension $n \geq 4$ is said to be pseudo semiconformally symmetric [17] if its semiconformal curvature tensor P of type $(0,4)$ satisfies the relation

$$\begin{aligned}(\nabla_E P)(X,Y,W,V) &= 2A(E)P(X,Y,W,V) + A(X)P(E,Y,W,V)\\ &+ A(Y)P(X,E,W,V) + A(W)P(X,Y,E,V)\\ &+ A(V)P(X,Y,W,E).\end{aligned} \tag{1.14}$$

The semiconformal curvature tensor is further studied in the recent paper by De and Suh [14]. An almost pseudo symmetric manifold introduced by De and Gazi [11] is an important generalization of symmetric space which is studied by several geometers ([15],[7],[21],[19]), and many others. Motivated by there studies in an almost pseudo symmetric manifold and semiconformal curvature tensor, in the present paper, we introduced a type of non-flat Riemannian manifold $(M_n, g), (n \geq 4)$ whose semiconformal curvature tensor P of type $(0,4)$ satisfies the condition

$$\begin{aligned}(\nabla_E P)(X,Y,W,V) &= [A(E)+B(E)]P(X,Y,W,V) + A(X)P(E,Y,W,V)\\ &+ A(Y)P(X,E,W,V) + A(W)P(X,Y,E,V)\\ &+ A(V)R(X,Y,W,E),\end{aligned} \tag{1.15}$$

where A and B are non-zero 1-forms and are called the associated 1-forms, defined as in (1.5), and ∇ has the meaning previously introduced. The vector fields ρ and σ corresponding to the associated 1-forms A and B respectively shall be called the basic vector fields of the manifold. We shall be calling such a manifold as an almost pseudo semiconformally symmetric manifold and an n-dimensional manifold of this kind shall be denoted by $A(PSCS)_n$. If in (1.15) $A = B$, then the manifold becomes a pseudo semiconformally symmetric manifold defined by (1.14). The manifold $A(PSCS)_n$ includes an almost pseudo conformally symmetric manifold [13] and an almost pseudo conharmonically symmetric manifold [21].

The present paper is organized as follows: After preliminaries, in section 3 we investigated some geometric properties of $A(PSCS)_n$ with non-zero constant scalar curvature and Codazzi type of Ricci tensor. In section 4, Ricci symmetric $A(PSCS)_n$ and Ricci recurrent $A(PSCS)_n$ are studied. Section 5 deals with an Einstein $A(PSCS)_n$. In section 6, it is concerned with the decomposition of $A(PSCS)_n$ and exactly defined each product manifolds of an $A(PSCS)_n$. Finally, we constructed two non-trivial examples of $A(PSCS)_n$.

2 Preliminaries

Let r and S denote the scalar curvature and the Ricci tensor of type (0,2) respectively and L has the meaning already mentioned, that is,

$$g(LE, X) = S(E, X). \tag{2.1}$$

In this section, we will derive some formulas, which we will be using in the study of $A(PSCS)_n$ throughout this paper. Let $\{e_i\}$ be an orthonormal basis of the tangent

space at each point of the manifold where $1 \leq i \leq n$.
Now from equation (1.13), we have

$$\sum_{i=1}^{n} P(X,Y,e_i,e_i) = 0 = \sum_{i=1}^{n} P(e_i,e_i,X,Y), \tag{2.2}$$

and,

$$\sum_{i=1}^{n} P(e_i,Y,W,e_i) = \sum_{i=1}^{n} P(Y,e_i,e_i,W) = -\frac{\{a+(n-2)b\}r}{(n-2)} g(Y,W), \tag{2.3}$$

where, $r = \sum_{i=1}^{n} S(e_i,e_i)$ is the scalar curvature.

Making use of equation (1.13) we obtain the following relations:

$$\begin{aligned}
&(i) && P(X,Y,W,V) = -P(Y,X,W,V),\\
&(ii) && P(X,Y,W,V) = -P(X,Y,V,W),\\
&(iii) && P(X,Y,W,V) = P(W,V,X,Y),\\
&(iv) && P(X,Y,W,V) + P(Y,W,X,V) + P(W,X,Y,V) = 0.
\end{aligned} \tag{2.4}$$

3 An $A(PSCS)_n, (n \geq 4)$ with non-zero constant scalar curvature and Codazzi type of Ricci tensor.

Theorem 3.1. *In $A(PSCS)_n, (n \geq 4)$ the scalar curvature is a non-zero constant if and only if $(4+n)A(E) + nB(E) = 0$, provided $[a+(n-2)b] \neq 0$.*

Proof. Taking covariant derivative of equation (1.13) with respect to E we get,

$$\begin{aligned}
a(\nabla_E R)(X,Y,W,V) &= (\nabla_E P)(X,Y,W,V) + \frac{a}{(n-2)}\Big\{(\nabla_E S)(Y,W)g(X,V)\\
&- (\nabla_E S)(X,W)g(Y,V) + (\nabla_E S)(X,V)g(Y,W)\\
&- (\nabla_E S)(Y,V)g(X,W)\Big\} + \frac{b\,dr(E)}{(n-1)}\Big\{g(Y,W)g(X,V)\\
&- g(X,W)g(Y,V)\Big\}.
\end{aligned} \tag{3.1}$$

Inserting equation (1.15) in equation (3.1) we obtain,

$$\begin{aligned}
a(\nabla_E R)(X,Y,W,V) &= [A(E)+B(E)]P(X,Y,W,V) + A(X)P(E,Y,W,V)\\
&+ A(Y)P(X,E,W,V) + A(W)P(X,Y,E,V)\\
&+ A(V)R(X,Y,W,E) + \frac{a}{(n-2)}\Big\{(\nabla_E S)(Y,W)g(X,V)\\
&- (\nabla_E S)(X,W)g(Y,V) + (\nabla_E S)(X,V)g(Y,W)\\
&- (\nabla_E S)(Y,V)g(X,W)\Big\} + \frac{b\,dr(E)}{(n-1)}\Big\{g(Y,W)g(X,V)\\
&- g(X,W)g(Y,V)\Big\}.
\end{aligned} \tag{3.2}$$

Putting $X = V = e_i, (i = 1, 2, ..., n)$ and $\lambda = \dfrac{\{a + (n-2)b\}r}{(n-2)}$ in equation (3.2), we obtain

$$\begin{aligned}
a(\nabla_E S)(Y, W) &= [A(E) + B(E)]\Big[-\lambda r g(Y, W)\Big] + A(\tilde{P}(E, Y)W) \\
&+ A(Y)\Big[-\lambda r g(E, W)\Big] + A(W)\Big[-\lambda r g(Y, E)\Big] - A(\tilde{P}(W, E)Y) \\
&+ \frac{a}{(n-2)}\Big[n(\nabla_E S)(Y, W) - (\nabla_E S)(W, Y) + dr(E)g(Y, W) \\
&- (\nabla_E S)(Y, W)\Big] + b\, dr(E)g(Y, W). \qquad (3.3)
\end{aligned}$$

Contracting over Y and W in equation (3.3), the above equation reduces to

$$n[a + (n-2)b]\, dr(E) = [a + (n-2)b]r[(4 + n)A(E) + nB(E)]. \qquad (3.4)$$

Assuming $[a + (n-2)b] \neq 0$, then equation (3.4) reduces to

$$n\, dr(E) = r[(4 + n)A(E) + nB(E)]. \qquad (3.5)$$

Clearly if $[(4 + n)A(E) + nB(E)] = 0$ then r is a non-zero constant.
Conversely, if r is a non-zero constant then $[(4 + n)A(E) + nB(E)] = 0$.
This completes the proof. □

Theorem 3.2. *If Ricci tensor in $A(PSCS)_n$ is of Codazzi type then the semiconformal curvature tensor P satisfies Bianchi's second identity.*

Proof. Making use of equation (1.13) we can obtain

$$\begin{aligned}
(\nabla_E P)(X, Y, W, V) &+ (\nabla_X P)(Y, E, W, V) + (\nabla_Y P)(E, X, W, V) \\
&= a\Big[(\nabla_E R)(X, Y, W, V) + (\nabla_X R)(Y, E, W, V) \\
&+ (\nabla_Y R)(E, X, W, V)\Big] - \frac{a}{(n-2)}\Big[(\nabla_E S)(Y, W)g(X, V) \\
&- (\nabla_E S)(X, W)g(Y, V) + (\nabla_E S)(X, V)g(Y, W) \\
&- (\nabla_E S)(Y, V)g(X, W) + (\nabla_X S)(E, W)g(Y, V) \\
&- (\nabla_X S)(Y, W)g(E, V) + (\nabla_X S)(Y, V)g(E, W) \\
&- (\nabla_X S)(E, V)g(Y, W) + (\nabla_Y S)(X, W)g(E, V) \\
&- (\nabla_Y S)(E, W)g(X, V) - (\nabla_Y S)(X, V)g(E, W) \\
&+ (\nabla_Y S)(E, V)g(X, W)\Big] - \frac{b}{(n-1)}\Big[dr(E)\{g(Y, W)g(X, V) \\
&- g(X, W)g(Y, V)\} + dr(X)\{g(E, W)g(Y, V) \\
&- g(Y, W)g(E, V)\} + dr(Y)\{g(X, W)g(E, V) \\
&- g(E, W)g(X, V)\}\Big]. \qquad (3.6)
\end{aligned}$$

Since the Ricci tensor is of Codazzi type, S satisfies the relation:

$$(\nabla_E S)(X, Y) = (\nabla_X S)(E, Y), \qquad (3.7)$$

implies $r =$ constant.
Moreover, inserting equation (3.7) in equation (3.6), we have

$$(\nabla_E P)(X,Y,W,V) + (\nabla_X P)(Y,E,W,V) + (\nabla_Y P)(E,X,W,V) = 0. \tag{3.8}$$

Hence, the theorem is proved. □

Theorem 3.3. *In $A(PSCS)_n$, if the semiconformal curvature tensor P satisfies Bianchi's second identity then $A(PSCS)_n$ reduces to a pseudo semiconformally symmetric manifold, provided $[a+(n-2)b] \neq 0$ and $r \neq 0$.*

Proof. Suppose that the semiconformal tensor P in $A(PSCS)_n$ satisfies Bianchi's second identity. Then making use equation (1.15), we get

$$\begin{aligned}[B(E)-A(E)]P(X,Y,W,V) + [B(X)-A(X)]P(Y,E,W,V)&\\ +[B(Y)-A(Y)]P(E,X,W,V) = 0.&\end{aligned} \tag{3.9}$$

Let $Q(E) = B(E) - A(E)$ and ρ_1 be a basic vector such that

$$g(E,\rho_1) = Q(E), \tag{3.10}$$

for all E. Equation (3.9) with the help of equation (3.10) may be written as

$$Q(E)P(X,Y,W,V) + Q(X)P(Y,E,W,V) + Q(Y)P(E,X,W,V) = 0. \tag{3.11}$$

Putting $X = V = e_i$ in equation (3.11), the above equation reduces to

$$\begin{aligned}Q(E)\Big\{-\frac{[a+(n-2)b]r}{(n-2)}g(Y,W)\Big\} + Q(\tilde{P}(Y,E)W)&\\ -Q(Y)\Big\{-\frac{[a+(n-2)b]r}{(n-2)}g(E,W)\Big\} = 0,&\end{aligned} \tag{3.12}$$

and contracting over Y and W, we infer

$$[a+(n-2)b]rQ(E) = 0. \tag{3.13}$$

Suppose $r \neq 0$ and $[a+(n-2)b] \neq 0$ in above equation implies $Q(E) = 0$.
This completes the proof. □

Theorem 3.4. *If $A(PSCS)_n$ satisfies Bianchi's second identity then the scalar curvature is constant provided $[a+(n-2)b] \neq 0$.*

Proof. Suppose $A(PSCS)_n$ satisfies Bianchi's second identity. Then, from equation (1.13), we obtain

$$\begin{aligned}&\frac{a}{(n-2)}\Big\{(\nabla_E S)(Y,W)g(X,V) - (\nabla_E S)(X,W)g(Y,V) + (\nabla_E S)(X,V)g(Y,W)\\ &\quad -(\nabla_E S)(Y,V)g(X,W) + (\nabla_X S)(E,W)g(Y,V) - (\nabla_X S)(Y,W)g(E,V)\\ &\quad +(\nabla_X S)(Y,V)g(E,W) - (\nabla_X S)(E,V)g(Y,W) + (\nabla_Y S)(X,W)g(E,V)\\ &\quad -(\nabla_Y S)(E,W)g(X,V) - (\nabla_Y S)(X,V)g(E,W) + (\nabla_Y S)(E,V)g(X,W)\Big\}\\ &+\frac{b}{(n-1)}\Big\{dr(E)\{g(Y,W)g(X,V) - g(X,W)g(Y,V)\} + dr(X)\{g(E,W)g(Y,V)\\ &\quad -g(Y,W)g(E,V)\} + dr(Y)\{g(X,W)g(E,V) - g(E,W)g(X,V)\}\Big\} = 0.\end{aligned} \tag{3.14}$$

Contracting equation (3.14) over Y and W, the equation reduces to

$$\frac{a}{(n-2)}\Big[\frac{1}{2}dr(E)g(X,V)+(n-2)(\nabla_E S)(X,V)+(2-n)(\nabla_X S)(E,V)$$
$$-\frac{1}{2}dr(X)g(E,V)-(\nabla_E S)(X,V)+(\nabla_X S)(E,V)\Big]+bg(X,V)dr(E)$$
$$-bg(E,V)dr(X)+\frac{b}{(n-1)}\Big[dr(X)g(E,V)-dr(E)g(X,V)\Big]=0. \tag{3.15}$$

Substituting $X=V=e_i$ in equation (3.15) yields

$$[a+(n-2)b]\,dr(E)=0. \tag{3.16}$$

This completes the proof. □

4 Ricci Symmetric $A(PSCS)_n, (n \geq 4)$ and Ricci-recurrent $A(PSCS)_n, (n \geq 4)$.

Theorem 4.1. *In a Ricci symmetric $A(PSCS)_n, (n \geq 4)$, the Bianchi's second identity holds for semiconformal curvature tensor.*

Proof. Since $A(PSCS)_n$ is Ricci symmetric, the Ricci tensor S satisfies the condition

$$\nabla S = 0$$

and $dr=0$.
Using this, we have

$$(\nabla_E P)(X,Y,W,V)=a(\nabla_E R)(X,Y,W,V).$$

Hence,

$$(\nabla_E P)(X,Y,W,V)+(\nabla_X P)(Y,E,W,V)+(\nabla_Y P)(E,X,W,V)=$$
$$a[(\nabla_E R)(X,Y,W,V)+(\nabla_X R)(Y,E,W,V)+(\nabla_Y R)(E,X,W,V)], \tag{4.1}$$

implies,

$$(\nabla_E P)(X,Y,W,V)+(\nabla_X P)(Y,E,W,V)+(\nabla_Y P)(E,X,W,V)=0. \tag{4.2}$$

Hence, the theorem is proved. □

Theorem 4.2. *In a Ricci symmetric $A(PSCS)_n, (n \geq 4)$ the vector fields corresponding to the 1-forms A and B are in opposite direction, provided $r \neq 0$ and $[a+(n-2)b] \neq 0$.*

Proof. Contracting equation (1.15) over E, we get

$$(div\tilde{P})(X,Y)W=A(\tilde{P}(X,Y)W)+B(\tilde{P}(X,Y)W)-A(X)\left\{\frac{[a+(n-2)b]r}{(n-2)}\right\}$$
$$g(Y,W)+A(Y)\left\{\frac{[a+(n-2)b]r}{(n-2)}\right\}g(X,W)+A(\tilde{P}(X,Y)W). \tag{4.3}$$

Moreover we have,

$$(div\tilde{P})(X,Y)W = \frac{a(n-3)}{(n-2)}\Big\{(\nabla_X S)(Y,W) - (\nabla_Y S)(X,W)\Big\}$$
$$-\Big\{\frac{[a(n-1)+b(n-2)]}{2(n-1)(n-2)}\Big\}\Big\{dr(X)g(Y,W) - dr(Y)g(X,W)\Big\}. \tag{4.4}$$

Combining equations (4.3) and (4.4), the above equations reduces to

$$A(\tilde{P}(X,Y)W) + B(\tilde{P}(X,Y)W) - A(X)\Big\{\frac{[a+(n-2)b]r}{(n-2)}\Big\}$$
$$g(Y,W) + A(Y)\Big\{\frac{[a+(n-2)b]r}{(n-2)}\Big\}g(X,W) + A(\tilde{P}(X,Y)W)$$
$$= \frac{a(n-3)}{(n-2)}\Big\{(\nabla_X S)(Y,W) - (\nabla_Y S)(X,W)\Big\}$$
$$-\Big\{\frac{[a(n-1)+b(n-2)]}{2(n-1)(n-2)}\Big\}\Big\{dr(X)g(Y,W) - dr(Y)g(X,W)\Big\}. \tag{4.5}$$

Suppose the manifold is Ricci symmetric, then equation (4.5) becomes

$$2A(\tilde{P}(X,Y)W) + B(\tilde{P}(X,Y)W) - A(X)\Big\{\frac{[a+(n-2)b]r}{(n-2)}\Big\}g(Y,W)$$
$$+A(Y)\Big\{\frac{[a+(n-2)b]r}{(n-2)}\Big\}g(X,W) = 0. \tag{4.6}$$

Inserting $Y = W = e_i$ in equation (4.6) and taking summation over $1 \leq i \leq n$, we obtain

$$[a+(n-2)b]r[(n+1)A(X) + B(X)] = 0. \tag{4.7}$$

If $r \neq 0$ and $[a+(n-2)b] \neq 0$, then above equation gives $B(X) = -(n+1)A(X)$. Therefore, this led to the statement of the above theorem. □

Corollary 4.3. *In a Ricci symmetric $A(PSCS)_n, (n \geq 4)$ the scalar curvature vanishes if $[(n+1)A(X) + B(X)] \neq 0$, provided $[a+(n-2)b] \neq 0$.*

Theorem 4.4. *In a Ricci-recurrent $A(PSCS)_n, (n \geq 4)$, if the scalar curvature is non-zero and $[a+(n-2)b] \neq 0$, then $\tilde{H}(E) = 3A(E) + B(E)$, for all E.*

Proof. Equation (1.13) making use of (1.15) results in the following

$$[A(E)+B(E)]P(X,Y,W,V) + A(X)P(E,Y,W,V) + A(Y)P(X,E,W,V)$$
$$+A(W)P(X,Y,E,V) + A(V)R(X,Y,W,E) = a(\nabla_E R)(X,Y,W,V)$$
$$-\frac{a}{(n-2)}\Big\{(\nabla_E S)(Y,W)g(X,V) - (\nabla_E S)(X,W)g(Y,V)$$
$$+(\nabla_E S)(X,V)g(Y,W) - (\nabla_E S)(Y,V)g(X,W)\Big\}$$
$$-\frac{b\,dr(E)}{(n-1)}\Big\{g(Y,W)g(X,V) - g(X,W)g(Y,V)\Big\}. \tag{4.8}$$

Now, contracting above equation yields

$$dr(E) = r\tilde{H}(E). \tag{4.9}$$

The use of equations (1.8) and (4.9) in equation (4.8) gives

$$\begin{aligned}
[A(E) + B(E)]P(X, Y, W, V) + A(X)P(E, Y, W, V) + A(Y)P(X, E, W, V) \\
+A(W)P(X, Y, E, V) + A(V)R(X, Y, W, E) = a(\nabla_E R)(X, Y, W, V) \\
-\frac{a}{(n-2)}\Big\{S(Y, W)g(X, V) - S(X, W)g(Y, V) \\
+S(X, V)g(Y, W) - S(Y, V)g(X, W)\Big\}H(E) \\
-\frac{br\tilde{H}(E)}{(n-1)}\Big\{g(Y, W)g(X, V) - g(X, W)g(Y, V)\Big\}.
\end{aligned} \tag{4.10}$$

Putting $X = V = e_i$ in equation (4.10), we get

$$\begin{aligned}
[A(E) + B(E)]\Big\{-\frac{[a + (n-2)b]r}{(n-2)}\Big\}g(Y, W) + A(\tilde{P}(E, Y)W) \\
-A(Y)\Big\{\frac{[a + (n-2)b]r}{(n-2)}\Big\}g(E, W) - A(W)\Big\{\frac{[a + (n-2)b]r}{(n-2)}\Big\}g(Y, E) \\
-A(\tilde{P}(W, E)Y) = -r\Big\{\frac{[a + (n-2)b]}{(n-2)}\Big\}g(Y, W)\tilde{H}(E).
\end{aligned} \tag{4.11}$$

Moreover, inserting $Y = W = e_i$ in equation (4.11), the above equation becomes

$$[(n + 4)A(E) + nB(E)] = n\tilde{H}(E). \tag{4.12}$$

Similarly, taking $E = Y = e_i$ in equation (4.11) gives,

$$(1 + n)A(W) + B(W) = \tilde{H}(W), \tag{4.13}$$

and replacing $W = E$ in above equation, we get

$$(1 + n)A(E) + B(E) = \tilde{H}(E). \tag{4.14}$$

Again, contracting the equation (4.11) over E and W, we infer

$$(n + 1)A(Y) + B(Y) = \tilde{H}(Y). \tag{4.15}$$

Substituting $Y = E$ in equation (4.15) gives

$$(1 + n)A(E) + B(E) = \tilde{H}(E). \tag{4.16}$$

Combining equations (4.12),(4.14) and (4.16), we obtain

$$\tilde{H}(E) = 3A(E) + B(E). \tag{4.17}$$

Hence, $\tilde{H}(E) = 3A(E) + B(E)$ provided $r \neq 0$ and $[a + (n-2)b] \neq 0$. $\square$

5 Einstein $A(PSCS)_n, (n \geq 4)$

Theorem 5.1. *If an Einstein $A(PSCS)_n, (n \geq 4)$ is an $A(PS)_n$ and $2a(n-1) - bn(n-2) \neq 0$ and $3A(E) + B(E) \neq 0$, then its scalar curvature vanishes, provided $a \neq 0$.*

Proof. In Einstein manifold the Ricci tensor is given by

$$S(E, X) = \frac{r}{n} g(E, X), \tag{5.1}$$

implies,

$$dr(E) = 0 \text{ and } (\nabla_E S)(X, Y) = 0. \tag{5.2}$$

Using equations (1.13),(5.1) and (5.2), we obtain

$$\begin{aligned} P(X, Y, W, V) = aR(X, Y, W, V) &- r\left[\frac{2a(n-1) - bn(n-2)}{n(n-1)(n-2)}\right]\Big[g(Y, W)g(X, V) \\ &- g(X, W)g(Y, V)\Big]. \end{aligned} \tag{5.3}$$

The covariant derivative of equation (5.3) gives

$$(\nabla_E P)(X, Y, W, V) = a(\nabla_E R)(X, Y, W, V). \tag{5.4}$$

Now, inserting equation (5.4) in equation (1.13), we obtain

$$\begin{aligned} a(\nabla_E R)(X, Y, W, V) &= [A(E) + B(E)]\Big\{aR(X, Y, W, V) \\ &- r\Big\{\frac{[2a(n-1) - bn(n-2)]}{n(n-1)(n-2)}\Big\}\Big[g(Y, W)g(X, V) \\ &- g(X, W)g(Y, V)\Big]\Big\} + A(X)\Big\{aR(E, Y, W, V) \\ &- r\Big\{\frac{[2a(n-1) - bn(n-2)]}{n(n-1)(n-2)}\Big\}\Big[g(Y, W)g(E, V) \\ &- g(E, W)g(Y, V)\Big]\Big\} + A(Y)\Big\{aR(X, E, W, V) \\ &- r\Big\{\frac{[2a(n-1) - bn(n-2)]}{n(n-1)(n-2)}\Big\}\Big[g(E, U)g(Y, V) \\ &- g(Y, U)g(E, V)\Big]\Big\} + A(W)\Big\{aR(Y, Z, E, V) \\ &- r\Big\{\frac{[2a(n-1) - bn(n-2)]}{n(n-1)(n-2)}\Big\}\Big[g(Y, E)g(X, V) \\ &- g(X, E)g(Y, V)\Big]\Big\} + A(V)\Big\{aR(X, Y, W, E) \\ &- r\Big\{\frac{[2a(n-1) - bn(n-2)]}{n(n-1)(n-2)}\Big\}\Big[g(Y, W)g(X, E) \\ &- g(X, W)g(Y, E)\Big]\Big\}. \end{aligned} \tag{5.5}$$

Assume $a \neq 0$. Suppose that an Einstein $A(PSCS)_n$ is an $A(PS)_n$. Then equation (5.5) becomes

$$\begin{aligned}&\Big[\frac{r\{2a(n-1)-bn(n-2)\}}{n(n-1)(n-2)}\Big]\Big[\{A(E)+B(E)\}\Big[g(Y,W)g(X,V)\\&-g(X,W)g(Y,V)\Big]+A(X)\Big[g(Y,W)g(E,V)-g(E,W)g(Y,V)\Big]\\&+A(Y)\Big[g(E,W)g(X,V)-g(X,W)g(E,V)\Big]+A(W)\Big[g(Y,E)g(X,V)\\&-g(X,E)g(Y,V)\Big]+A(V)\Big[g(Y,W)g(X,E)-g(X,W)g(Y,E)\Big]\Big]=0.\end{aligned} \tag{5.6}$$

Putting $X = V = e_i$ in equation (5.6), the above equation reduces to

$$\begin{aligned}&r\Big[2a(n-1)-bn(n-2)\Big]\Big[\{A(E)+B(E)\}(n-1)g(Y,W)+A(E)g(Y,W)\\&-A(Y)g(E,W)+A(Y)(n-1)g(E,W)+A(W)(n-1)g(Y,E)\\&+A(E)g(Y,W)-A(W)g(Y,E)\Big]=0.\end{aligned} \tag{5.7}$$

Moreover, taking $Y = W = e_i$ in equation (5.7) gives

$$r[2a(n-1)-bn(n-2)][(n+4)A(E)+nB(E)]=0. \tag{5.8}$$

Similarly, contracting equation (5.7) over Y and E we infer

$$r[2a(n-1)-bn(n-2)][(n+1)A(W)+B(W)]=0. \tag{5.9}$$

Substituting $W = E$ in equation (5.9) gives

$$r[2a(n-1)-bn(n-2)][(n+1)A(E)+B(E)]=0. \tag{5.10}$$

Again, putting $W = E = e_i$ in equation (5.7), we get

$$r[2a(n-1)-bn(n-2)][(n+1)A(Y)+B(Y)]=0, \tag{5.11}$$

and substituting $Y = E$ in equation (5.11) gives,

$$r[2a(n-1)-bn(n-2)][(n+1)A(E)+B(E)]=0. \tag{5.12}$$

Combining the equations (5.8),(5.10) and (5.12), we obtain the following result

$$r[2a(n-1)-bn(n-2)][3A(E)+B(E)]=0. \tag{5.13}$$

Hence, the theorem is proved. □

Suppose $r = 0$ in equation (5.5) then Einstein $A(PSCS)_n$ is an $A(PS)_n$, provided $a \neq 0$. Thus, we can state the following:

Theorem 5.2. *If $a \neq 0$ and scalar curvature vanishes in Einstein $A(PSCS)_n, (n \geq 4)$ then such a manifold is an $A(PS)_n$.*

Theorem 5.3. *If the vector field ρ_1 defined by $g(E,\rho_1) = B(E) - A(E)$, for all E, is a parallel vector field in an Einstein $A(PSCS)_n, (n \geq 4)$ with $a \neq 0$ and $||\rho_1||^2 \neq 0$, then it is an $A(PS)_n$.*

Proof. Let us consider that the vector field ρ_1 defined in equation (3.10) is parallel in an Einstein $A(PSCS)_n$. Then, we get

$$\nabla_E \rho_1 = 0, \tag{5.14}$$

for all E.
Which gives,

$$R(E, X, \rho_1, V) = 0.$$

Contracting the above equation we get

$$S(X, \rho_1) = 0.$$

Then, from equation (5.1), we have

$$rg(X, \rho_1) = 0. \tag{5.15}$$

If $||\rho_1||^2 \neq 0$, then above equation follows that $r = 0$.
Therefore, by equation (5.5), Einstein $A(PSCS)_n$ reduces to $A(PS)_n$, provided $a \neq 0$.
Hence, this completes the theorem. □

6 Decomposition of $A(PSCS)_n, (n \geq 4)$

A Riemannian manifold (M^n, g) is said to be decomposable or a product manifold[23] if it can be written as $M_1^p \times M_2^{n-p}$ for $2 \leq p \leq (n-2)$, that is, in some coordinate neighborhood of the Riemannian manifold (M^n, g) the metric can be expressed as

$$ds^2 = g_{ij}dx^i dx^j = \bar{g}_{ab}dx^a dx^b + g^*_{\alpha\beta}dx^\alpha dx^\beta, \tag{6.1}$$

where $\bar{g}_{ab}$ are functions of $x^1, x^2, ..., x^p$ denoted by $\bar{x}$ and $g^*_{\alpha\beta}$ are functions of $x^{p+1}, x^{p+2}, ..., x^n$ denoted by x^* : $a, b, c, ...$run from 1 to p and $\alpha, \beta, \gamma, ...,$ run from $p+1$ to n. In (6.1), $\bar{g}_{ab}$ and $g^*_{\alpha\beta}$ are the matrices of $M_1^p (p \geq 2)$ and $M_2^{n-p} (n-p \geq 2)$ respectively, which are called the components of the decomposable manifold $M^n = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$.

We will assume throughout this section that all objects indicated by a 'bar' belong to M_1 and all objects indicated by a 'star' belongs to M_2.

Let $\bar{E}, \bar{X}, \bar{Y}, \bar{W}, \bar{V} \in \chi(M_1)$ and $E^*, X^*, Y^*, W^*, V^* \in \chi(M_2)$. Then in a decomposable Riemannian manifold $M^n = M_1^p \times M_2^{n-p} (2 \leq p \leq n-2)$, the following relations hold

$$\begin{aligned} R(E^*, \bar{X}, \bar{Y}, \bar{W}) = 0 = R(\bar{E}, X^*, \bar{Y}, W^*) = R(\bar{E}, X^*, Y^*, W^*), \\ (\nabla_{E^*} R)(\bar{X}, \bar{Y}, \bar{W}, \bar{V}) = 0 = (\nabla_{\bar{E}} R)(\bar{X}, Y^*, \bar{W}, V^*) = (\nabla_{E^*} R)(\bar{X}, Y^*, \bar{W}, V^*), \\ R(\bar{E}, \bar{X}, \bar{Y}, \bar{W}) = \bar{R}(\bar{E}, \bar{X}, \bar{Y}, \bar{W}); R(E^*, X^*, Y^*, W^*) = R^*(E^*, X^*, Y^*, W^*), \\ S(\bar{E}, \bar{X}) = \bar{S}(\bar{E}, \bar{X}); S(E^*, X^*) = S^*(E^*, X^*), \\ (\nabla_{\bar{E}} S)(\bar{X}, \bar{Y}) = (\bar{\nabla}_{\bar{E}} S)(\bar{X}, \bar{Y}); (\nabla_{E^*} S)(X^*, Y^*) = (\nabla^*_{E^*} S)(X^*, Y^*), \end{aligned} \tag{6.2}$$

where $\bar{r}$,r^* and r are scalar curvature of M_1,M_2 and M respectively and are related as $r = \bar{r} + r^*$. Also $S(\bar{E}, X^*) = 0$ and $g(\bar{E}, X^*) = 0$.

Theorem 6.1. *Let an $A(PSCS)_n$ be a decomposable space such that $M^n = M_1^p \times M_2^{n-p}$ for $(2 \leq p \leq n-2)$, then the following holds:*
i) In the case of $A = B = 0$ on M_2, the manifold M_2 is Ricci symmetric and scalar curvature r^* is constant in M_2, provided $d\bar{r}(E^*) = 0$ and $\dfrac{a(n-p-2)}{(n-2)} \neq \dfrac{bp(n-p)}{(n-1)}$.
ii) when M_1 is semiconformally flat, then M_1 is an Einstein manifold.

Proof. Let us consider a Riemannian manifold (M^n, g) which is a decomposable $A(PSCS)_n$, then

$$M^n = M_1^p \times M_2^{n-p}(2 \leq p \leq n-2).$$

Now from equation (1.13), we obtain

$$\begin{aligned} P(X^*, \bar{Y}, \bar{W}, \bar{V}) &= 0 = P(\bar{X}, Y^*, W^*, V^*) \\ &= P(\bar{X}, Y^*, \bar{W}, \bar{V}) = P(\bar{X}, \bar{Y}, W^*, \bar{V}); \\ P(X^*, \bar{Y}, \bar{W}, V^*) &= -\frac{a}{(n-2)}\Big[S(\bar{Y}, \bar{W})g(X^*, W^*) + S(X^*, V^*)g(\bar{Y}, \bar{W})\Big] \\ &- \frac{rb}{(n-1)}\Big[g(\bar{Y}, \bar{W})g(X^*, V^*)\Big]; \\ P(X^*, Y^*, \bar{W}, \bar{V}) &= 0 = P(\bar{X}, \bar{Y}, W^*, V^*); \\ P(X^*, \bar{Y}, W*, \bar{V}) &= \frac{a}{(n-2)}\Big[S(\bar{Y}, \bar{V})g(X^*, W^*) + S(X^*, W^*)g(\bar{Y}, \bar{V})\Big] \\ &+ \frac{rb}{(n-1)}\Big[g(\bar{Y}, \bar{V})g(X^*, W^*)\Big]. \end{aligned} \tag{6.3}$$

Further simplifying the above equation, we get

$$\begin{aligned} (\nabla_{\bar{E}}P)(\bar{X}, \bar{Y}, \bar{W}, \bar{V}) &= [A(\bar{E}) + B(\bar{E})]P(\bar{X}, \bar{Y}, \bar{W}, \bar{V}) + A(\bar{X})P(\bar{E}, \bar{Y}, \bar{W}, \bar{V}) \\ &+A(\bar{Y})P(\bar{X}, \bar{E}, \bar{W}, \bar{V}) + A(\bar{W})P(\bar{X}, \bar{Y}, \bar{E}, \bar{V}) + A(\bar{V})P(\bar{X}, \bar{Y}, \bar{W}, \bar{E}) \end{aligned} \tag{6.4}$$

Putting $\bar{X} = X^*$ in equation (6.4) gives

$$A(X^*)P(\bar{E}, \bar{Y}, \bar{W}, \bar{V}) = 0. \tag{6.5}$$

Also, inserting $\bar{E} = E^*$ in equation (6.4), we have

$$[A(E^*) + B(E^*)]P(\bar{X}, \bar{Y}, \bar{W}, \bar{V}) = 0. \tag{6.6}$$

Similarly inserting $\bar{E} = E^*$ and $\bar{X} = X^*$ in equation (6.4), we infer

$$A(\bar{W})P(X^*, \bar{Y}, E^*, \bar{V}) + A(\bar{V})P(X^*, \bar{Y}, \bar{W}, E^*) = 0. \tag{6.7}$$

Putting $\bar{E} = E^*$ and $\bar{W} = W^*$ in equation (6.4), we get

$$A(\bar{X})P(E^*, \bar{Y}, W^*, \bar{V}) + A(\bar{Y})P(\bar{X}, E^*, W^*, \bar{V}) = 0. \tag{6.8}$$

And, taking $\bar{X} = X^*, \bar{Y} = Y^*$ and $\bar{W} = W^*$ in equation (6.4) results in

$$A(X^*)P(\bar{E}, Y^*, W^*, \bar{V}) + A(Y^*)P(X^*, \bar{E}, W^*, \bar{V}) = 0. \tag{6.9}$$

Substituting $\bar{Y} = Y^*, \bar{W} = W^*$ and $\bar{V} = V^*$ in equation (6.4), we have

$$A(W^*)P(\bar{X}, Y^*, \bar{E}, V^*) + A(V^*)P(\bar{X}, Y^*, W^*, \bar{E}) = 0. \tag{6.10}$$

Moreover, using equation (1.13) gives

$$\begin{aligned}(\nabla_{E^*}P)(X^*, Y^*, W^*, V^*) &= [A(E^*) + B(E^*)]P(X^*, Y^*, W^*, V^*)\\ &+A(X^*)P(E^*, Y^*, W^*, V^*) + A(Y^*)P(X^*, E^*, W^*, V^*)\\ &+A(W^*)P(X^*, Y^*, E^*, V^*) + A(V^*)P(X^*, Y^*, W^*, E^*).\end{aligned} \tag{6.11}$$

From equation (6.11), we obtain

$$[A(\bar{E} + B(\bar{E})]P(X^*, Y^*, W^*, V^*) = 0, \tag{6.12}$$

and,

$$A(\bar{X})P(E^*, Y^*, W^*, V^*) = 0. \tag{6.13}$$

Putting $\bar{E} = E^*, \bar{X} = X^*$ and $\bar{V} = V^*$ in equation (6.4) gives

$$\begin{aligned}(\nabla_{E^*}P)(X^*, \bar{Y}, \bar{W}, V^*) &= [A(E^*) + B(E^*)]P(X^*, \bar{Y}, \bar{W}, V^*)\\ &+A(X^*)P(E^*, \bar{Y}, \bar{W}, V^*) + A(V^*)P(X^*, \bar{Y}, \bar{W}, E^*).\end{aligned} \tag{6.14}$$

Similarly, putting $E^* = \bar{E}, X^* = \bar{X}$ and $V^* = \bar{V}$ in equation (6.11) gives

$$\begin{aligned}(\nabla_{\bar{E}}P)(\bar{X}, Y^*, W^*, \bar{V}) &= [A(\bar{E}) + B(\bar{E})]P(\bar{X}, Y^*, W^*, \bar{V})\\ &+A(\bar{X})P(\bar{E}, Y^*, W^*, \bar{V}) + A(\bar{V})P(\bar{X}, Y^*, W^*, \bar{E}).\end{aligned} \tag{6.15}$$

In regard of equations (6.5) and (6.6), we have the following two cases:

$i)$ $A = B = 0$ on M_2.
$ii)$ M_1 is semiconformally flat.

First, we consider the case (i). Then, equation (6.14) becomes

$$(\nabla_{E^*}P)(X^*, \bar{Y}, \bar{W}, V^*) = 0, \tag{6.16}$$

implies,

$$\begin{aligned}a(\nabla_{E^*}R)(X^*, \bar{Y}, \bar{W}, V^*) - \frac{a}{(n-2)}(\nabla_{E^*}S)(X^*, V^*)g(\bar{Y}, \bar{W})\\ -\frac{b\,dr(E^*)}{(n-1)}g(\bar{Y}, \bar{W})g(X^*, V^*) = 0.\end{aligned} \tag{6.17}$$

Now, Putting $\bar{Y} = \bar{W} = \bar{e}_\alpha, 1 \leq \alpha \leq p$ in equation (6.17), we get

$$\frac{a(n-p-2)}{(n-2)}(\nabla_{E^*}S)(X^*, V^*) - \frac{b\,dr(E^*)}{(n-1)}pg(X^*, V^*) = 0. \tag{6.18}$$

Also, taking $X^* = V^* = e_i^*, p+1 \leq i \leq n$ in equation (6.18) gives

$$\frac{a(n-p-2)}{(n-2)}\, dr^*(E^*) - \frac{bp(n-p)}{(n-1)}\, dr(E^*) = 0. \tag{6.19}$$

If possible let $d\bar{r}(E^*) = 0$. The equation (6.19) becomes

$$\left[\frac{a(n-p-2)}{(n-2)} - \frac{bp(n-p)}{(n-1)}\right] dr^*(E^*) = 0. \tag{6.20}$$

Thus r^* is constant in M_2 provided, $\dfrac{a(n-p-2)}{(n-2)} \neq \dfrac{bp(n-p)}{(n-1)}$. Then from equation (6.18), we get

$$(\nabla_{E^*} S)(X^*, V^*) = 0.$$

Therefore, M_2 is Ricci symmetric.
Secondly, we will consider the case (ii). Since M_1 is semiconformally flat, we get

$$\begin{aligned} aR(\bar{X},\bar{Y},\bar{W},\bar{V}) - \frac{a}{(n-2)}\Big[&S(\bar{Y},\bar{W})g(\bar{X},\bar{V}) - S(\bar{X},\bar{W})g(\bar{Y},\bar{V}) \\ &+S(\bar{X},\bar{V})g(\bar{Y},\bar{W}) - S(\bar{Y},\bar{V})g(\bar{X},\bar{W})\Big] \\ -\frac{br}{(n-1)}\Big[&g(\bar{Y},\bar{W})g(\bar{X},\bar{V}) - g(\bar{X},\bar{W})g(\bar{Y},\bar{V})\Big] = 0. \end{aligned} \tag{6.21}$$

Putting $\bar{X} = \bar{V} = \bar{e}_\alpha$ in equation (6.21), the above equation becomes

$$S(\bar{Y},\bar{W}) = \left[\frac{a\bar{r}(n-1) + br(p-1)(n-2)}{a(n-p-2)}\right] g(\bar{Y},\bar{W}). \tag{6.22}$$

Therefore, M_1 is an Einstein manifold.
Hence, the theorem is proved. □

Theorem 6.2. *Let an $A(PSCS)_n$ be a decomposable space such that $M^n = M_1^p \times M_2^{n-p}$ for $(2 \leq p \leq n-2)$, then the following holds:*
i) In the case of $A = B = 0$ on M_1, the manifold M_1 is Ricci symmetric and scalar curvature $\bar{r}$ is constant in M_1, provided $dr^*(\bar{E}) = 0$ and $\dfrac{a(p-2)}{(n-2)} \neq \dfrac{bp(n-p)}{(n-1)}$.
ii) when M_2 is semiconformally flat, then M_2 is an Einstein manifold.

Proof. Making use of equations (6.12) and (6.13), we get the following two cases:

$$\begin{aligned} &i) \quad A = B = 0 \,\text{on}\, M_1. \\ &ii) \quad M_2 \,\text{is semiconformally flat.} \end{aligned}$$

Proceeding in a similar manner as in Theorem 6.1,
Hence, we will obtain the required result. □

Corollary 6.3. *If $A(PSCS)_n$ is a decomposable space such that $M^n = M_1^p \times M_2^{n-p}$ for $(2 \leq p \leq n-2)$, then one of the decomposed manifold is semiconformally flat while on other manifold both the associate 1-form A and B vanishes.*

7 Examples of $A(PSCS)_4$

In this section, we have constructed two examples of an $A(PSCS)_4$ on coordinate space $\mathbb{R}^4$ (with coordinates(x^1, x^2, x^3, x^4)) and obtain all the non-vanishing components of the curvature tensor, the Ricci tensor, the scalar curvature and the semiconformal curvature tensor along with its covariant derivatives. Then we verified the relation (1.15).

Example 7.1. Let us consider a Riemannian metric g defined on 4-dimensional manifold $M^4 = \{(x^1, x^2, x^3, x^4) \in \mathbb{R}^4 : x^1 \neq -1\}$ given by

$$ds^2 = (x^1 + 1)(x^4)^2(dx^1)^2 + 2dx^1dx^2 + (dx^3)^2 + (dx^4)^2. \tag{7.1}$$

A similar Riemannian metric g is given by De and Gazi[13].
Then the covariant and contravariant components of the metric are as follows

$$\begin{gathered} g_{11} = (x^1 + 1)(x^4)^2, g_{12} = g_{21} = 1, g_{33} = g_{44} = 1 \\ g^{11} = 0, g^{12} = g^{21} = 1, g^{33} = g^{44} = 1, g^{22} = -(x^1 + 1)(x^4)^2 \end{gathered} \tag{7.2}$$

All non-vanishing components of the Christoffel symbols and the curvature tensor in the considered metric are as follows:

$$\begin{aligned} \Gamma^4_{11} = -(x^1 + 1)(x^4), \Gamma^2_{11} &= \frac{1}{2}(x^4)^2, \Gamma^2_{14} = (x^1 + 1)(x^4) \\ R_{1441} &= (x^1 + 1) \end{aligned} \tag{7.3}$$

From equations (7.2) and (7.3), the non-vanishing components of Ricci tensor are

$$S_{11} = x^1 + 1. \tag{7.4}$$

The scalar curvature of metric considered is given by,

$$r = 0. \tag{7.5}$$

The only non-vanishing components of the semiconformal curvature tensor are

$$P_{1441} = \frac{a}{2}(x^1 + 1) \neq 0. \tag{7.6}$$

Clearly, the only non-vanishing term of $\nabla_l P_{hijk}$ are

$$\nabla_1 P_{1441} = \frac{a}{2} \neq 0. \tag{7.7}$$

In term of the local coordinate system, let us define the components of the 1-form A and B as

$$A_i = \begin{cases} \dfrac{1}{6(x^1 + 1)} \text{ for } i = 1 \\ 0, \text{ otherwise} \end{cases}$$

and,

$$B_i = \begin{cases} \dfrac{1}{2(x^1 + 1)} \text{ for } i = 1 \\ 0 \text{ , otherwise} \end{cases} \tag{7.8}$$

at any point in M^4.
In (M^4, g) the considered 1-form reduces the equation (1.15) in the following equations

$$\nabla_1 P_{1441} = (3A_1 + B_1)P_{1441} + A_4 P_{1141} + A_4 P_{1411}. \tag{7.9}$$

$$\nabla_4 P_{1141} = [A_4 + B_4]P_{1141} + A_1 P_{4141} + A_1 P_{1441} + A_4 P_{1141} + A_1 P_{1144}. \tag{7.10}$$

$$\nabla_4 P_{1411} = [A_4 + B_4]P_{1411} + A_1 P_{4411} + A_4 P_{1411} + A_1 P_{1441} + A_1 P_{1414}. \tag{7.11}$$

In all other cases excluding (7.9),(7.10), and (7.11), the relation (1.15) either holds trivially or the components of each term vanishes identically.
By (7.8), we get

$$\begin{aligned} \text{RHS of (7.9)} &= (3A_1 + B_1)P_{1441} + A_4 P_{1141} + A_4 P_{1411} \\ &= \left[\frac{3}{6(x^1+1)} + \frac{1}{2(x^1+1)}\right]\frac{a}{2}(x^1+1) \\ &= \frac{a}{4} + \frac{a}{4} \\ &= \frac{a}{2} \\ &= \nabla_1 P_{1441} \\ &= \text{LHS of (7.9)}. \end{aligned} \tag{7.12}$$

By proceeding in a similar manner, it can be shown that the equations (7.10) and (7.11) are also true.
Thus, (M^4, g) is an $A(PSCS)_4$.

Example 7.2. Let us consider a Riemannian metric g defined on 4-dimensional manifold $M^4 = (x^1, x^2, x^3, x^4) \in \mathbb{R}^4$ given by

$$ds^2 = (1 + 2q)[(dx^1)^2 + (dx^2)^2] + (dx^3)^2 + (dx^4)^2, \tag{7.13}$$

where $q = \frac{e^{x^1}}{k^2}$, where k is non-zero constant.
Then the covariant and contravariant components of the metric are as follows:

$$\begin{aligned} &g_{11} = g_{22} = 1 + 2q,\ g_{33} = g_{44} = 1 \\ &g^{11} = g^{22} = \frac{1}{1+2q},\ g^{33} = g^{44} = 1 \end{aligned} \tag{7.14}$$

All the non-vanishing components of the Christoffel symbols and the curvature tensor in the considered metric are

$$\begin{aligned} \Gamma^1_{11} = \Gamma^2_{12} = \frac{q}{1+2q},\ \Gamma^1_{22} &= -\frac{q}{1+2q} \\ R_{1221} &= \frac{q}{1+2q} \end{aligned} \tag{7.15}$$

By (7.14) and (7.15), the non-vanishing components of Ricci tensor are

$$S_{11} = \frac{q}{(1+2q)^2}. \tag{7.16}$$

The Scalar curvature is given by

$$r = g^{ij}S_{ij} = g^{11}S_{11} + g^{22}S_{22} + g^{33}S_{33} + g^{44}S_{44} = \frac{q}{(1+2q)^3}. \tag{7.17}$$

The only non-vanishing components of semiconformal curvature tensors are

$$P_{1221} = \frac{q}{1+2q}\left\{\frac{a}{2} - \frac{b}{3}\right\}. \tag{7.18}$$

From equation (7.18), it can be shown that only non-zero term of $\nabla_l P_{hijk}$ are

$$\nabla_1 P_{1221} = \frac{1}{(1+2q)^2}\left\{\frac{a}{2} - \frac{b}{3}\right\}, \tag{7.19}$$

and all other components of $\nabla_l P_{hijk}$ vanishes identically.
In term of the local coordinate system, let us consider the components of the 1-form A and B as

$$A_i = \begin{cases} \dfrac{1}{6q(1+2q)} \text{ for } i = 1 \\ 0, \text{ otherwise} \end{cases}$$

and,

$$B_i = \begin{cases} \dfrac{1}{2q(1+2q)} \text{ for } i = 1 \\ 0 \text{ , otherwise} \end{cases} \tag{7.20}$$

at any point in M^4.
In (M^4, g), the considered 1-form reduces equation (1.15) into the following equations

$$\nabla_1 P_{1221} = (3A_1 + B_1)P_{1221} + A_2 P_{1121} + A_2 P_{1211}. \tag{7.21}$$

$$\nabla_2 P_{1121} = (A_2 + B_2)P_{1121} + A_1 P_{2121} + A_1 P_{1221} + A_2 P_{1121} + A_1 P_{1122}. \tag{7.22}$$

$$\nabla_2 P_{1211} = [A_2 + B_2]P_{1211} + A_1 P_{2211} + A_2 P_{1211} + A_1 P_{1221} + A_1 P_{1212}. \tag{7.23}$$

The relation (1.15) either holds trivially or the components of each term vanishes identically excluding the above cases.
By (7.21) we get

$$\begin{aligned} RHS \text{ of } (7.21) &= (3A_1 + B_1)P_{1221} + A_2 P_{1121} + A_2 P_{1211}. \\ &= \left[\frac{3}{6q(1+2q)} + \frac{1}{2q(1+2q)}\right]\frac{q}{(1+2q)}\left\{\frac{a}{2} - \frac{b}{3}\right\} \\ &= \frac{1}{(1+2q)^2}\left\{\frac{a}{2} - \frac{b}{3}\right\} \\ &= \nabla_1 P_{1221} \\ &= LHS \text{ of } (7.21). \end{aligned} \tag{7.24}$$

By proceeding similarly it can be shown that the equations (7.22) and (7.23) also holds.
Thus, (M^4, g) is an $A(PSCS)_4$.

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